Recurrent sequences coming from Shimura curves

Frits Beukers

On the occasion of Cam Stewart’s 60th birthday
Recall

\[(n + 1)^2 u_{n+1} = (11n^2 + 11n + 3)u_n + n^2 u_{n-1}\]

Let \(a_n\) be the solution with starting values \(a_0 = 0, a_1 = 5, \ldots\) and \(b_n\) the solution with \(b_0 = 1, b_1 = 3, b_2 = 19, b_3 = 147, \ldots\). Then \(a_n/b_n \to \zeta(2)\) as \(n \to \infty\) fast enough to prove irrationality.
Recurrences and ODE’s

Consider the generating function

\[ u(z) = \sum_{n \geq 0} b_n z^n. \]

Then \( u(z) = 1 + 3z + 19z^2 + 147z^3 + \cdots \) satisfies

\[ z(z^2 + 11z - 1)u'' + (3z^2 + 22z - 1)u' + (z + 3)u = 0. \]

This is a linear second order differential equation with a \( G \)-function solution (i.e. coefficients have denominators of at most exponential growth).
The modular connection

Basis of solutions of
\[ z(z^2 + 11z - 1)u'' + (3z^2 + 22z - 1)u' + (z + 3)u = 0 \]
is
\[ y_1 = u(z), \quad y_2 = u(z) \log(z) + v(z) \]
where \( v(z) = 5z + 75z^2/2 + 5565z^3/18 + \cdots \)
The map \( z \mapsto \frac{1}{2\pi i}y_2/y_1 \) maps \( \mathbb{P}^1 \) to complex upper half plane \( \mathcal{H} \).
Its inverse is the map
\[ \mathcal{H} \to \mathcal{H}/\Gamma_1(5) \leftrightarrow \mathbb{P}^1 \]
where \( \Gamma_1(5) \subset SL(2, \mathbb{Z}) \) is congruence subgroup modulo 5.
The challenge

Find recurrences of the form

\[ P(n)u_{n+1} = Q(n)u_n + R(n)u_{n-1} \]

where \( P, Q, R \) are polynomials of degree 2, which allow a solution \( u_n \) whose coefficients are at most exponential in \( n \).

Alternatively, one can try to find second order linear differential equations of the form

\[ z(z^2 + a_1 z + a_0)y'' + (b_2 z^2 + b_1 z + b_0)y' + (c_1 z + c_0)y = 0 \]

which have a Siegel \( G \)-function solution.
An idea

Start with congruence subgroup $\Gamma$ of $SL(2, \mathbb{Z})$ with four cusps and $X(\Gamma)$ genus zero. The map

$$\mathcal{H} \to \mathcal{H}/\Gamma$$

gives rise to a second order linear differential equation of the desired kind.

This gives us 5 more cases.
Chudnovsky’s idea

Start with an arithmetic quaternion group $\Gamma \subset SL(2, \mathbb{R})$ and then consider $\mathcal{H} \rightarrow \mathcal{H}/\Gamma$.

Example of Lamé equation from Chudnovsky’s *Theta functions*, 1989

$$P(z)u'' + \frac{1}{2}P'(z)u' + \left(-\frac{3}{128} - \frac{3}{64}z\right)u = 0$$

where $P(z) = z(z - 1)(z - 1/2)$.

Recurrence

$$(n+1)(n+1/2)u_{n+1} = (n^2+3/64)u_n - ((n-1)(2n-1) - 3/32)u_{n-1}.$$
Let $B$ be a quaternion algebra over totally real number field $F$. More concrete, take $a, b \in F^*$ and define $B = F \oplus Fi \oplus Fj \oplus Fk$ with

$$i^2 = a, \quad j^2 = b, \quad k = ij = -ji.$$ 

Let $\mathcal{O}$ be a maximal order of $B$ and $\mathcal{O}^\times$ its units. Any embedding $\iota : F \hookrightarrow \mathbb{R}$ induces an embedding of $B$ into either $M_2(\mathbb{R})$ ($2 \times 2$ real matrices) or $\mathbb{H}$ (Hamilton’s quaternions). Suppose that $B \hookrightarrow M_2(\mathbb{R})$ for exactly one place $\iota : F \hookrightarrow \mathbb{R}$. Then we call $\mathcal{O}^\times$, embedded in $M_2(\mathbb{R})$, an arithmetic quaternion group.

More generally, any subgroup $\Gamma \subset B$ commensurable with $\mathcal{O}^\times$ is called an arithmetic quaternion group.

Commensurable means that $\Gamma \cap \mathcal{O}^\times$ has finite index in both $\Gamma$ and $\mathcal{O}^\times$. 
A discrete subgroup $\Gamma \subset SL(2, \mathbb{R})$ is said to be of type $(1; e)$ if $E_\Gamma := \mathcal{H}/\Gamma$ has genus one and the projection $\mathcal{H} \to \mathcal{H}/\Gamma$ ramifies above exactly one point of order $e$.

Such groups are generated by two elements $A, B$ with the single relation $[A, B]^e = -\text{Id}$. The group is determined by the traces of $A, B$ and $AB$.

**Theorem (Takeuchi, 1983)**

There exist, up to conjugation, precisely 71 arithmetic quaternion groups of type $(1; e)$. 
The problem

Let $\Gamma$ be an arithmetic group of type $(1; e)$. The problem is twofold,

1. Determine a Weierstrass equation for $\mathcal{H}/\Gamma$ of the form $y^2 = P(x)$, ($P$ cubic and monic).

2. Determine the constant $C$ (accessory parameter) so that the covering $\mathcal{H} \to \mathcal{H}/\Gamma/\text{inv}$ is determined by

$$P(z)y'' + \frac{1}{2}P'(z)y' + (C - n(n + 1)z/4)y = 0$$

with $n = (-1 + 1/e)/2$. 
Sijsling’s thesis

In the recent PhD-thesis of Jeroen Sijsling he tackled the first problem and found almost all $j$-invariants in Takeuchi’s list.

Techniques used:

1. If $\Gamma$ is commensurable with a triangle group there exists Belyi map $E_\Gamma$ to $\mathbb{P}^1$.
2. According to Shimura-Deligne theory there exists a canonical model of $E_\Gamma$, defined over the narrow classfield of $F$, with good reduction outside a known set of primes.
3. Using explicit calculation of Hecke operators $T_p$ on $H_1(E_\Gamma, \mathbb{Z})$ and the Eichler-Shimura theorem one determines the zeta-function of $E_\Gamma$ at $p$ for a large set of primes $p$.
4. To select a $j$-invariant in an isogeny class one determines the reduction mod $p$ of $E_\Gamma$ at the primes $p$ of multiplicative reduction using a refinement of Cerednik-Drin’feld by Boutot-Zink.
5. Prove correctness for the candidate $j$-invariants.
A sample $j$-invariant

There are three arithmetic quaternion groups of type $(1; 7)$ not commensurable with a triangle group. The $j$-invariants of the Shimura curve $E(\Gamma)$ are the conjugates of

\[-1448892\alpha^2 - 1930931\alpha + 1318350\]
\[7 \cdot 13^2\]

where $\alpha$ is a zero of $x^3 - x^2 - 2x + 1$.

The corresponding quaternion algebra is defined over the field $\mathbb{Q}(\alpha)$ and the discriminant is $\varphi_7 \varphi_{13} \infty_1 \infty_2$. Discriminant of $\mathbb{Q}(\alpha)$ is 49.
Recall, we must determine $P(z)$ and $C$ in

$$P(z)y'' + \frac{1}{2}P'(z)y' + (C - n(n + 1)z/4)y = 0$$

with $n = (-1 + 1/e)/2$. We know $P(z)$ from the $j$-invariant computation. As yet there is no systematic method to compute $C$. Numerically, given $P(z)$ and $n$ and $C$, one can compute generators of the monodromy group and their traces. By interpolation determine $C$ as precise as possible to obtain the desired traces given by the quaternion group. Then guess an algebraic value of $C$. 

We take the two \((1; 4)\)-groups \(\Gamma\) from Takeuchi’s list corresponding to the quaternion algebra over \(\mathbb{Q}(\sqrt{2})\) of discriminant \(\wp_7\infty\). The curves \(\mathcal{H}/\Gamma\) correspond to the conjugates of

\[
y^2 = P(x) = x(x - 1)(x - (3 - 2\sqrt{2})/4).
\]

Numerical approximation (50 decimal places) indicates that \(C = (2 - \sqrt{2})/2^4\) in

\[
P(z)y'' + \frac{1}{2}P'(z)y' + (C + 15z/256)y = 0
\]