Notes on differential equations and hypergeometric functions

(**NOT for publication**)

To be used with care because of possible errors

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June 4, 2012

Contents

Chapter 1

Ordinary linear differential equations

1.1 Differential equations and systems of equations

A differential field K is a field equipped with a derivation, that is, a map $\partial : K \to K$ which has the following properties,

For all $a, b \in K$ we have $\partial(a+b) = \partial a + \partial b$.

For all $a, b \in K$ we have $\partial(ab) = a\partial b + b\partial a$.

The subset $C := \{a \in K | \partial a = 0\}$ is a subfield of K and is called the field of constants. We shall assume that C is *algebraically closed and has characteristic zero*. We shall also assume that ∂ is *non-trivial* that is, there exist $a \in K$ such that $\partial a \neq 0$.
Standard examples which will be used in later chapters are $\mathbb{C}(z)$, $\mathbb{C}((z))$, $\mathbb{C}((z))_{\mathrm{an}}$. They are the field of rational functions, formal Laurent series at $z = 0$ and Laurent series which converge in a punctured disk $0 < |z| < \rho$ for some $\rho > 0$. As derivation in these examples we have ordinary differentiation with respect to z and the field of constants is $\mathbb{C}$.
An *ordinary differential equation* over K is an equation of the form

$$\partial^n y + p_1 \partial^{n-1} y + \cdots + p_{n-1} \partial y + p_n y = 0, \quad p_1, \ldots, p_n \in K.$$

A *system of n first order equations* over K has the form

$$\partial \mathbf{y} = A\mathbf{y}$$

in the unknown column vector $\mathbf{y} = (y_1, \ldots, y_n)^t$ and where A is an $n \times n$-matrix with entries in K.
Note that if we replace $\mathbf{y}$ by $S\mathbf{y}$ in the system, where $S \in GL(n, K)$, we obtain a new system for the new $\mathbf{y}$,

$$\partial \mathbf{y} = (S^{-1}AS + S^{-1}\partial S)\mathbf{y}.$$

Two $n \times n$-systems with coefficient matrices A, B are called *equivalent over K* if there exists $S \in GL(n, K)$ such that $B = S^{-1}AS + S^{-1}\partial S$.

It is well known that a differential equation can be rewritten as a system by putting $y_1 = y, y_2 = \partial y, \ldots, y_n = \partial^{n-1}y$. We then note that $\partial y_1 = y_2, \partial y_2 = y_3, \ldots, \partial y_{n-1} = y_n$ and finally, $\partial y_n = -p_1 y_n - p_2 y_{n-1} - \ldots, -p_n y_1$. This can be rewritten as

$$\partial \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

There is also a converse statement.

Theorem 1.1.1 (Cyclic vector Lemma) *Any system of linear first order differential equations over K is equivalent over K to a system which comes from a differential equation.*

Proof. Let $\partial \mathbf{y} = A\mathbf{y}$ be our $n \times n$ system. Consider the linear form $y = r_1 y_1 + \cdots + r_n y_n$ with $r_1, \ldots, r_n \in K$. Using the differential system for the y_i we see that $\partial y = s_1 y_1 + \cdots + s_n y_n$, where the s_i are obtained via

$$(s_1, \ldots, s_n) = \partial(r_1, \ldots, r_n) + (r_1, \ldots, r_n)A.$$

By repeated application of ∂ we find for each i elements $r_{i1}, \ldots, r_{in} \in K$ such that $\partial^i y = r_{i1} y_1 + \cdots + r_{in} y_n$. Denote the matrix $(r_{ij})_{i=0,\ldots,n-1; j=1,\ldots,n}$ by R. If R is invertible, then $(r_{n1}, \ldots, r_{nn})$ is a K-linear combination of the $(r_{i1}, \ldots, r_{in})$ for $i = 0, 1, \ldots, n-1$. Hence $\partial^n y$ is a K-linear combination of the $\partial^i y$ $(i = 0, \ldots, n-1)$. Moreover, when R is invertible, our system is equivalent, via R, to a system coming from a differential equation. So it suffices to show that there exist $r_1, \ldots, r_n$ such that the corresponding matrix R is invertible. Since ∂ is non-trivial we can find $x \in K$ such that $\partial x \neq 0$. Note that the new derivation $\partial := (x/\partial x)\partial$ has the property that $\partial x = x$, which we may now assume without loss of generality. In case we work in in $\mathbb{C}(z)$ our operator would be $\partial = z\frac{d}{dz}$. Let μ be the smallest index such that the matrix $(r_{ij})_{i=0,\ldots,\mu; j=1,\ldots,n}$ has K-linear dependent rows for every choice of $r_1, \ldots, r_n$. We must show that $\mu = n$.

Suppose that $\mu < n$. For every n-tuple $\mathbf{r} \in K^n$ we denote the derived n-tuples by $\mathbf{r}_i = (r_{i1}, \ldots, r_{in})$. Choose $\mathbf{s} \in K^n$ such that $\mathbf{s}_0, \ldots, \mathbf{s}_{\mu-1}$ are independent. Let $\mathbf{t} \in K^n$ be arbitrary. By $\mathbf{r}_0 \wedge \cdots \wedge \mathbf{r}_\mu$ we denote the vector consisting of the determinants of all $(\mu+1) \times (\mu+1)$ submatrices of the matrix with rows $\mathbf{r}_0, \ldots, \mathbf{r}_\mu$. For any $\lambda \in C$ we have now

$$(\mathbf{s}_0 + \lambda \mathbf{t}_0) \wedge \cdots \wedge (\mathbf{s}_\mu + \lambda \mathbf{t}_\mu) = 0.$$

Expand this with respect to powers of λ. Since we have infinitely many choices for λ the coefficient of every power of λ must be zero. In particular the coefficient of λ. Hence

$$\sum_{i=0}^{\mu} \mathbf{s}_0 \wedge \cdots \wedge \mathbf{t}_i \wedge \cdots \wedge \mathbf{s}_\mu = 0. \tag{1.1}$$

Now put $\mathbf{t} = x^m\mathbf{u}$ with $m \in \mathbb{Z}$ and $\mathbf{u} \in K^n$. Notice that $\mathbf{t}_i = x^m \sum_{j=0}^{i} \binom{m}{j} m^{i-j}\mathbf{u}_j$. Substitute this in (1.1), divide by x^m, and collect equal powers of m. Since m can be chosen in infinitely many ways the coefficient of each power of m must be zero. In particular the coefficient of m^μ is zero. Hence

$$\mathbf{s}_0 \wedge \cdots \wedge \mathbf{s}_{\mu-1} \wedge \mathbf{u}_0 = 0.$$

Since $\mathbf{u}_0$ can be chosen arbitrarily this implies $\mathbf{s}_0 \wedge \cdots \wedge \mathbf{s}_{\mu-1} = 0$ which contradicts the minimality of μ. □

We must also say a few words about the solutions of differential equations. It must be pointed out that in general the solutions lie in a bigger field than K. To this end we shall consider differential field extensions L of K with the property that the field of constants is the same as that of K. A fundamental lemma is the following one.

Lemma 1.1.2 (Wronski) *Let $f_1, \ldots, f_m \in K$. There exists a C-linear relation between these function if and only if $W(f_1, \ldots, f_m) = 0$, where*

$$W(f_1, \ldots, f_m) = \begin{vmatrix} f_1 & \ldots & f_m \\ \partial f_1 & \ldots & \partial f_m \\ \vdots & & \vdots \\ \partial^{m-1} f_1 & \ldots & \partial^{m-1} f_m \end{vmatrix}$$

is the Wronskian determinant of $f_1, \ldots, f_m$.

Proof. If the f_i are C-linear dependent, then the same holds for the columns of $W(f_1, \ldots, f_m)$. Hence this determinant vanishes.

Before we prove the converse statement we need some observations. First notice that

$$W(vu_1, \ldots, vu_m) = v^m W(u_1, \ldots, u_m)$$

for any $v, u_i \in K$. In particular, if we take $v = 1/u_m$ (assuming $u_m \neq 0$) we find

$$W(u_1, \ldots, u_r)/u_m^m = W(u_1/u_m, \ldots, u_{m-1}/u_m, 1) = (-1)^{m-1} W(\partial(u_1/u_m), \ldots, \partial(u_{m-1}/u_m)).$$

Now suppose that $W(f_1, \ldots, f_m)$ vanishes. By induction on m we show that $f_1, \ldots, f_m$ are C-linear dependent. For $m = 1$ the statement is obvious. So assume $m > 1$. If $f_m = 0$ we are done, so we can now assume that f_m is not zero. By the remarks made above, the vanishing of $W(f_1, \ldots, f_m)$ implies the vanishing of $W(\partial(f_1/f_m), \ldots, \partial(f_{m-1}/f_m))$. Hence, by the induction hypothesis, there exist $a_1, \ldots, a_{m-1}$ such that $a_1\partial(f_1/f_m) + \cdots + a_{m-1}\partial(f_{m-1}/f_m) = 0$. After taking primitives and multiplication by f_m on both sides we obtain a linear dependence relation between $f_1, \ldots, f_m$. □

Lemma 1.1.3 *Let L be a differential extension of K with C as field of constants. Then the solution space in L of a linear equation of order n is a C-vector space of dimension at most n.*

Proof. It is clear that the solutions form a C-vector space. Consider $n+1$ solutions $y_1, \ldots, y_{n+1}$ and let W be their Wronskian determinant. Note that the columns of this determinant all satisfy the same K-linear relation given by the differential equation. Hence $W \equiv 0$. According to Wronski's lemma this implies that $y_1, \ldots, y_{n+1}$ are C-linear dependent. $\square$

Combination of this Lemma with the Cyclic vector Lemma yields

Lemma 1.1.4 *Let L be a differential extension of K with C as field of constants. Then the solution space in L^n of an $n \times n$-system of equation $\partial \mathbf{y} = A\mathbf{y}$ is a C-vector space of dimension at most n.*

It is allways possible to find differential extensions which have a maximal set of solutions. Without proof we quote the following theorem.

Theorem 1.1.5 (Picard-Vessiot) *To any $n \times n$-system of linear differential equations over K there exists a differential extension L of K with the following properties*

1. *The field of constants of L is C.*
2. *There is an n-dimensional C-vector space of solutions to the system in L^n.*

Moreover, if L is minimal with respect to these properties then it is uniquely determined up to differential isomorphism.

Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be an independent set of solutions to an $n \times n$-system $\partial \mathbf{y} = A\mathbf{y}$. The matrix Y obtained by concatenation of all columns $\mathbf{y}_i$ is called a *fundamental solution matrix*. The Wronskian lemma together with the Cyclic vector Lemma imply that $\det(Y) \neq 0$.

Exercise 1.1.6 *Let Y be the fundamental solution matrix of an $n \times n$-system $\partial \mathbf{y} = A\mathbf{y}$. Prove that $\det(Y)$ satisfies the first order differential equation $\partial y = \mathrm{trace}(A) y$. Show also that the columns of Y^{-1} satisfy the system of equations $\partial \mathbf{y} = -A^t \mathbf{y}$, where A^t denotes the transpose of A.*

1.2 Local theory

In this section our differential field will be $\mathbb{C}((z))$. We shall denote the derivation $\frac{d}{dz}$ by D (sometimes) and the derivation $z\frac{d}{dz}$ by θ.

Exercise 1.2.1 *Prove by induction on r the following operator identity. For any $r \in \mathbb{N}$*

$$z^r \frac{d}{dz}^r = \theta(\theta - 1) \cdots (\theta - r + 1).$$

Prove for any m,

$$\theta(z^m f(z)) = z^m (\theta + m) f(z).$$

Consider the linear differential equation of order n,

$$\frac{d^n y}{dz^n} + p_1(z)\frac{d^{n-1}y}{dz^{n-1}} + \cdots + p_{n-1}(z)\frac{dy}{dz} + p_n(z)y = 0, \tag{1.2}$$

with $p_i \in \mathbb{C}((z))$. If $z = 0$ is not a pole of any p_i it is called a *regular point* of (1.2), otherwise it is called a *singular point* of (1.2). The point $z = 0$ is called a *regular singularity* if p_i has a pole of order at most i for $i = 1, \ldots, n$.
Another way of characterising a regular singularity is by rewriting (1.2) with respect to the derivation θ. Multiply (1.2) with z^n and use $z^r \frac{d}{dz}^r = \theta(\theta-1)(\theta-r+1)$ to obtain an equation of the form

$$\theta^n y + q_1(z)\theta^{n-1}y + \cdots + q_{n-1}(z)\theta y + q_n(z)y = 0. \tag{1.3}$$

The condition for $z = 0$ to be a regular singularity comes down to $q_i \in \mathbb{C}[[z]]$ for all i.
Similarly we can consider a system of first order equations over $\mathbb{C}((z))$, $\frac{d}{dz}\mathbf{y} = A\mathbf{y}$ where A has now entries in $\mathbb{C}((z))$. Again we call the point $z = 0$ *regular* if all entries of A are in $\mathbb{C}[[z]]$ and *singular* otherwise. We call $z = 0$ a regular singularity if the entries of A have a pole of order at most one. Again, when we write the system with respect to the operator θ, i.e. $\theta\mathbf{y} = zA\mathbf{y}$, the condition that $z = 0$ is a regular singularity comes down to zA having entries in $\mathbb{C}[[z]]$.
One also verifies that a differential equation with a regular point can be rewritten as a system with a regular point and that an equation with a regular singularity can be written as a system with a regular singularity by starting from (1.3).
Finally we remark that for systems the concept regular and regular singularity are not invariant under $\mathbb{C}((z))$-equivalence of systems. For example, the system $\frac{d}{dz}\mathbf{y} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{y}$ is regular at $z = 0$. But if we replace $\mathbf{y}$ by $\begin{pmatrix} 1 & 0 \\ 0 & 1/z \end{pmatrix}\mathbf{u}$ we get $\frac{d}{dz}\mathbf{u} = \begin{pmatrix} 1 & z \\ 0 & 1 + 1/z \end{pmatrix}\mathbf{u}$ which is not regular at $z = 0$ according to our definition.

Theorem 1.2.2 (Cauchy) *Suppose* 0 *is a regular point of* (1.2). *Then there exist* n $\mathbb{C}$-*linear independent Taylor series solutions* $f_1, \ldots, f_n \in \mathbb{C}[[z]]$. *Moreover, any Taylor series solution of* (1.2) *is a* $\mathbb{C}$-*linear combination of* $f_1, \ldots, f_n$. *Moreover, if the coefficients of* (1.2) *all have positive radius of convergence, the same holds for* $f_1, \ldots, f_n$.

This theorem is a consequence of the following statement

Theorem 1.2.3 (Cauchy) *Consider the system of equations* $\frac{d}{dz}\mathbf{y} = A\mathbf{y}$ *and suppose that the entries of* A *are in* $\mathbb{C}[[z]]$. *Then the system has a fundamental solution matrix* Y *with entries in* $\mathbb{C}[[z]]$ *and* $Y(0) = \mathrm{Id}$. *Here Id is the* $n \times n$ *identity matrix. Moreover, if the entries of* A *have positive radius of convergence, the same holds for the entries of* Y.

Clearly the columns of Y form an independent set of n vector solutions of the system. Since the dimension of the solution space is at most n this means that the columns of Y form a basis of solutions in $\mathbb{C}[[z]]$.
There is also a converse statement.

Theorem 1.2.4 *Suppose that the $n \times n$ matrix A has entries in $\mathbb{C}((z))$. Suppose there is a fundamental solution matrix $Y \in GL(n, \mathbb{C}[[z]])$ of $\frac{d}{dz}Y = AY$ and suppose that $Y(0)$ is invertible. Then $z = 0$ is a regular point of the system.*

Proof. The proof consists of the observation that $\frac{d}{dz}Y \cdot Y^{-1}$ has entries in $\mathbb{C}[[z]]$. □

For differential equations this theorem implies that if we have a basis of solutions of the form $f_i = z^i(1 + O(z)),\ i = 0, \ldots, n-1$ then $z = 0$ is a regular point. The extra condition on the shape of the f_i is really necessary since the mere existence a basis of holomorphic solutions does not always imply that $z = 0$ is regular. For example, the equation $\frac{d^2y}{dz^2} - \frac{1}{z}\frac{dy}{dz} = 0$ has $1, z^2$ as basis of solutions, but $z = 0$ is not a regular point.
Note that in the case of systems the condition $Y(0)$ invertible is essential. For example, the system $\frac{d}{dz}\mathbf{y} = \frac{1}{z^2}\begin{pmatrix} z-1 & 1 \\ -1 & 1+z \end{pmatrix}\mathbf{y}$ has $\begin{pmatrix} 1 & 1+z \\ 1-z & 1 \end{pmatrix}$ as fundamental solution matrix.
If a differential equation or a system of equations with a singular point at $z = 0$ has a basis of solutions with components in $\mathbb{C}[[z]]$ we call $z = 0$ an *apparent singularity*.
The proof of Cauchy's theorems follows from the following lemma.

Lemma 1.2.5 *Consider the system $\theta\mathbf{y} = A\mathbf{y}$ where A is an $n \times n$-matrix with entries in $\mathbb{C}[[z]]$. So $z = 0$ is a regular singularity. Let ρ be an eigenvalue of $A(0)$ such that none of $\rho + 1, \rho + 2, \ldots$ is eigenvalue of $A(0)$. Let $(g_0, \ldots, g_n)^t$ be an eigenvector of $A(0)$ with eigenvalue ρ. Then the system has a solution of the form $z^\rho(G_1, \ldots, G_n)^t$ with $G_i \in \mathbb{C}[[z]]$ and $G_i(0) = g_i$ for all i. Moreover, if the entries of A have positive radius of convergence, the same holds for the $G_i(z)$.*

Proof. Write $A = \sum_{i \ge 0} A_i z^i$. We look for a solution $\mathbf{y}$ of the form $\mathbf{y} = z^\rho \sum_{i \ge 0} \mathbf{y}_i z^i$, where the $\mathbf{y}_i$ have constant entries and $\mathbf{y}_0 \ne \mathbf{0}$. Substitution of $\mathbf{y}$ in the differential equation yields the recursion

$$(k + \rho)\mathbf{y}_k - A_0\mathbf{y}_k = A_1\mathbf{y}_{k-1} + \cdots + A_k\mathbf{y}_0$$

for $k = 0, 1, 2, \ldots$. When $k = 0$ we see that the recursion implies that $\mathbf{y}_0$ is an eigenvector of A_0 with eigenvalue ρ. Choose $\mathbf{y}_0$ to be such an eigenvector. Since $\rho + k$ is not an eigenvalue of A_0 for $k = 1, 2, \ldots$, The matrix $k + \rho - A_0$ is invertible for all $k \ge 1$ and our recursion gives the $\mathbf{y}_k$.
Now suppose that the entries of A have positive radius of convergence. This means that there exist $C, \sigma \in \mathbb{R}_{>1}$ such that $||A_i|| \le C\sigma^i$. Here $||B||$ denotes the norm of an $n \times n$ matrix B defined by the supremum of all $|B\mathbf{v}|$ as $\mathbf{v}$ runs over all vectors in $\mathbb{C}^n$ of length 1. It is not hard to show that there exist $k_0 \in \mathbb{Z}_{\ge 0}$ and $\lambda \in \mathbb{R}_{>0}$ such that $||(k + \rho - A_0)^{-1}|| \le (k - \lambda)^{-1}$ whenever $k > k_0$. For future use we also see to it that $k_0 \ge \lambda + 2C$. Let M be the maximum of $|\mathbf{y}_i|$ for $i = 0, \ldots, k_0$. Then, by using the recursion and induction on k one can show that $|\mathbf{y}_k| \le M(2\sigma)^k$ for all $k \ge 0$. □

If we have a system where $z = 0$ is a regular point, this means that $A(0)$ of Lemma 1.2.5 is identically zero. Hence $\rho = 0$ and any non-trivial vector is an eigenvector. So we take the standard basis in $\mathbb{C}^n$ and obtain Cauchy's theorems.
In the following theorem we shall consider expressions of the form z^A where A is a constant $n \times n$ matrix. This is short hand for

$$z^A = \exp(A \log z) = \sum_{k \geq 0} \frac{1}{k!} A^k (\log z)^k.$$

In particular z^A is an $n \times n$ matrix of multivalued functions around $z = 0$. Examples are,

$$z^{\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}} = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}, \qquad z^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & \log z \\ 0 & 1 \end{pmatrix}.$$

Theorem 1.2.6 (Fuchs) *Suppose that the $n \times n$ matrix A has entries in $\mathbb{C}[[z]]$. Then the system of equations $\theta \mathbf{y} = A\mathbf{y}$ has a fundamental matrix solution of the form $S \cdot z^B$, where S is an $n \times n$ matrix with entries in $\mathbb{C}[[z]]$ and B is a constant upper triangular matrix. Any eigenvalue ρ of B is the minimum of all eigenvalues of $A(0)$ of the form $\rho, \rho+1, \rho+2, \ldots$. In particular, if the eigenvalues of $A(0)$ are all distinct modulo 1, the eigenvalues of B and $A(0)$ coincide and $S(0)$ is invertible.*
Moreover, if the entries of A have positive radius of convergence, the same holds for the entries of S.

Notice that the existence of the fundamental solution matrix $S \cdot z^B$ implies that the system is equivalent over $\mathbb{C}((z))$ to $\theta \mathbf{y} = B\mathbf{y}$, which has z^B as fundamental solution matrix.

Proof. We shall prove our theorem by induction on n. When $n = 1$, Lemma 1.2.5 gives a solution of the form $z^\rho G(z)$, as desired.
Suppose now that $n > 1$. Let ρ be an eigenvalue of $A(0)$ such that none of $\rho+1, \rho+2, \ldots$ is an eigenvalue of $A(0)$. Then there exists a solution of the form $z^\rho \mathbf{g}$ where $\mathbf{g}$ has entries in $\mathbb{C}[[z]]$ and at least one of the entries has a non-zero constant term. Without loss of generality we can assume $g_1(0) \neq 0$. Replace $\mathbf{y}$ by

$$\begin{pmatrix} g_1(z) & 0 & \cdots & 0 \\ g_2(z) & 1 & \ldots & 0 \\ \vdots & & & \vdots \\ g_n(z) & 0 & \cdots & 1 \end{pmatrix} \mathbf{y}.$$

Our new equation will have $z^\rho (1, 0, \ldots, 0)^t$ as solution hence it has the form

$$\theta \mathbf{y} = \begin{pmatrix} \rho & l_2 & \cdots & l_n \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix} \mathbf{y}$$

With $l_2, \ldots, l_n \in \mathbb{C}[[z]]$. According to our induction hypothesis the $(n-1) \times (n-1)$ system with coefficient matrix $(a_{ij})_{i,j=2,\ldots,n}$ is equivalent to a system with a constant upper triangular coefficient matrix, say C We can use this to bring our $n \times n$-system in the form with coefficient matrix

$$\begin{pmatrix} \rho & l_2 & \cdots & l_n \\ 0 & & & \\ \vdots & & C & \\ 0 & & & \end{pmatrix}$$

Now replace y_1 by $y_1 + m_2 y_2 + \cdots + m_n y_n$. One verifies that we obtain a new system of the same form as above, except that the l_i have changed into $\tilde{l}_2, \ldots, \tilde{l}_n$ where

$$(\tilde{l}_2, \ldots, \tilde{l}_n) = (l_2, \ldots, l_n) - \theta(m_2, \ldots, m_n) + (m_2, \ldots, m_n)(\rho - C).$$

When $\rho - r$ is not an eigenvalue of C we see that the equation

$$0 = (\lambda_2, \ldots, \lambda_n) z^r - \theta(m_2, \ldots, m_n) + (m_2, \ldots, m_n)(\rho - C)$$

has $(\lambda_2, \ldots, \lambda_n)(r - \rho + C)^{-1} z^r$ as a solution. We can apply this principle to the terms of the power series expansion of $(l_2, \ldots, l_n)$. When none of the numbers $\rho, \rho - 1, \rho - 2, \ldots$ is an eigenvalue of C we can thus find $m_2, \ldots, m_n \in \mathbb{C}[[z]]$ such that the $\tilde{l}_i$ all become zero. Our theorem is proved in this case.
Suppose now that C has an eigenvalue of the form $\rho - k$ for some $k \in \mathbb{Z}_{\geq 0}$. By our induction hypothesis there is only one such k. Using our remarks above we can now choose $m_2, \ldots, m_n \in \mathbb{C}[[z]]$ such that $\tilde{l}_i = \lambda_i z^k$ with $\lambda_i \in \mathbb{C}$ for $i = 2, \ldots, n$. Now replace y_1 by $z^k y_1$. The top row of the coefficient matrix of the new equation now reads $(\rho - k, \lambda_2, \ldots, \lambda_n)$, while the other rows stay the same. Consequently the coefficient matrix now contains only elements from $\mathbb{C}$ and it is upper triangular. This proves our theorem. $\square$

The converse of Theorem 1.2.6 need not hold, so a system with a fundamental matrix solution of the form $S \cdot z^B$ need not have a regular singularity at $z = 0$. Consider for example the fundamental solution matrix $Y = \begin{pmatrix} 1 & 1+z \\ 1-z & 1 \end{pmatrix}$. However, for differential equations we do have an equivalence statement.

Theorem 1.2.7 *The differential equation* (1.2) *has a regular singularity at* $z = 0$ *if and only there exists a basis of solutions of the form*

$$(g_1(z), \ldots, g_n(z)) z^B, \quad g_1(z), \ldots, g_n(z) \in \mathbb{C}[[z]]$$

where B *is an upper triangular constant matrix.*

Proof. Rewrite equation (1.2) in the form (1.3) and apply Theorem 1.2.6 to the corresponding system. Then the first row of the matrix Sz^B contains the desired solutions of the differential equations.

Let us suppose we have a basis of solutions $f_1, \ldots, f_n$ of the form $(f_1, \ldots, f_n) = (g_1, \ldots, g_n)z^B$ with $g_i \in \mathbb{Z}[[z]]$. By induction on n we show that the n-th order equation satisfied by $f_1, \ldots, f_n$ has a regular singularity at $z = 0$. More particularly we show that the equation has the form (1.3).
When $n = 1$ we have $f_1 = z^b g_1(z)$ for some exponent b and f_1 satisfies $\theta f - (b + \theta g_1/g_1)f = 0$. Note that $b + \theta g_1/g_1 \in \mathbb{C}[[z]]$. Let $n > 1$ and suppose our statement holds for any $n-1$-th order equation. Consider the n-th order equation with solutions $f_1, \ldots, f_n$. Then $(f_1, \ldots, f_{n-1}) = (g_1, \ldots, g_{n-1})z^{\tilde{B}}$ where $\tilde{B}$ is the square matrix obtained from B by deleting the last row and column. Our induction hypothesis shows that there exists a monic operator $L \in \mathbb{C}[[z]][\theta]$ of order $n-1$ such that $L(f_i) = 0$ for $i = 1, \ldots, n-1$. Now consider $L(f_n)$. Application of local monodromy shows that f_n changes into a multiple of itself plus a possible linear combination of $f_1, \ldots, f_{n-1}$. Since the latter are annihilated by L we conclude that $L(f_n)$ changes only by constant factors under local monodromy. Hence it has the form $z^b F(z)$ for some $F \in \mathbb{C}[[z]]$. Now note that the composite operator $(\theta - b - \theta F/F) \circ L$ annihilates all $f_1, \ldots, f_n$. Moreover, the composite operator is monic in θ and has coefficients in $\mathbb{C}[[z]]$.

□

Corollary 1.2.8 *Suppose that the coefficients of* (1.2) *converge in a region* $R = \{z | 0 < |z| < \sigma\}$. *Suppose that in every sector of* R *with* 0 *as vertex we have a basis of solutions* $f_1, \ldots, f_n$ *of* (1.2) *and* $\lambda \in \mathbb{R}$ *such that* $z^\lambda f_i(z)$ *tends to* 0 *as* $z \to 0$. *Then* $z = 0$ *is a regular singularity of* (1.2)

The condition that in every sector the solutions are polynomially bounded by $1/|z|$ is called the condition of *moderate growth*.

Proof. Choose a sector S in R with a basis of solutions $f_1, \ldots, f_n$. Let γ be a simple closed path which goes around zero once. Now continue $f_1, \ldots, f_n$ analytically along γ until we return to our sector S. The continuations $\tilde{f}_1, \ldots, \tilde{f}_n$ are still solutions of (1.2). Hence there exists a constant matrix M such that $(\tilde{f}_1, \ldots, \tilde{f}_n) = (f_1, \ldots, f_n)M$ in S. We call M the monodromy matrix. corresponding to the f_i and γ. Choose a constant matrix B such that $e^{2\pi\sqrt{-1}B} = M$. Then the n-tuple of functions $(f_1, \ldots, f_n)z^{-B}$ has trivial monodromy around $z = 0$, and hence these functions can be continued to the punctured disc D. The moderate growth condition now implies that the entries of $(f_1, \ldots, f_n)z^{-B}$ are in fact meromorphic functions. We can now apply our previous theorem. □

Let A be as in Theorem 1.2.6. The eigenvalues of $A(0)$ are called the *local exponents* at $z = 0$ of the system. If we have a differential equation where $z = 0$ is a regular singularity, we first write it in the form (1.3) and then as a system. One verifies that the local exponents of the system are the solutions of the equation $x^n + q_1(0)x^{n-1} + \cdots + q_{n-1}(0)x + q_n(0) = 0$. We call this equation the *indicial equation* and its solutions the *local exponents* of the equation at $z = 0$. Note that if we choose a different local parameter t via $t = c_1 z + c_2 z^2 + \cdots,\ c_1 \neq 0$ and rewrite our equation or system with respect to t, then the local exponents at $t = 0$ are the same as the original exponents. This is worked out in the following exercise.

Exercise 1.2.9 *Let $\theta_z = z\frac{d}{dz}$ and $\theta_t = t\frac{d}{dt}$. Show that there exists a powerseries $g(t)$ in t with constant coefficient 1, such that $\theta_z = g(t)\theta_t$. Now show by induction on n that for every n there exist powerseries $g_1, \ldots, g_{n-1}$ in t with vanishing constant term such that $\theta_z^n = g(t)^n\theta_t^n + g_1(t)\theta_t^{n-1} + \cdots + g_{n-1}(t)\theta_t$. Rewrite equation (1.3) in terms of θ_t and denote the coefficients by $\tilde{q}_i(t)$. Show that $\tilde{q}_i(0) = q_i(0)$ for $i = 1, \ldots, n$.*

Remark 1.2.10 *Notice that if we replace y by $z^\mu w$, the differential equation for w reads*

$$(\theta+\mu)^n w + q_1(z)(\theta+\mu)^{n-1}w + \cdots + q_{n-1}(z)(\theta+\mu)w + q_n(z)w = 0.$$

In particular, the local exponents have all decreased by μ.

Exercise 1.2.11 *Show that the local exponents at a regular point read $0, 1, \ldots, n-1$.*

Exercise 1.2.12 *Consider the linear differential equation*

$$(z^3 + 11z^2 - z)y'' + (3z^2 + 22z - 1)y' + (z+3)y = 0.$$

Show that the local exponents at $z = 0$ are $0, 0$ and determine the recursion relation for the holomorphic solution near $z = 0$. Determine also the first few terms of the expansions of a basis of solutions near $z = 0$.

1.3 Fuchsian equations

In this section our differential field will be $\mathbb{C}(z)$, the field of rational functions in z and we shall consider our differential equations and $n \times n$-systems over this field.
Consider the linear differential equation

$$y^{(n)} + p_1(z)y^{(n-1)} + \cdots + p_{n-1}(z)y' + p_n(z)y = 0, \qquad p_i(z) \in \mathbb{C}(z) \tag{1.4}$$

To study this differential equation near any point $P \in \mathbb{P}^1$ we choose a local parameter $t \in \mathbb{C}(z)$ at this point (usually $t = z - P$ if $P \in \mathbb{C}$ and $t = 1/z$ if $P = \infty$), and rewrite the equation with respect to the new variable t. We call the point P a regular point or a regular singularity if this is so for the equation in t at $t = 0$. It is not difficult to verify that a point $P \in \mathbb{C}$ is regular if and only if the p_i have no pole at P. It is a regular singularity if and only if $\lim_{z\to P}(z-P)^i p_i(z)$ exists for $i = 1, \ldots, n$. The point ∞ is regular or a regular singularity if and only if $\lim_{z\to\infty} z^i p_i(z)$ exists for $i = 1, \ldots, n$.

Definition 1.3.1 *A differential equation over $\mathbb{C}(z)$ or a system of first order equations over $\mathbb{C}(z)$ is called Fuchsian if all points on $\mathbb{P}^1$ are regular or a regular singularity.*

The form of Fuchsian systems is particularly simple. Let our $n \times n$-system be given by

$$\frac{d}{dz}\mathbf{y} = A\mathbf{y}$$

where the entries of A are in $\mathbb{C}(z)$. Let $S = \{p_1, \ldots, p_r\}$ be the set of finite singular points. If we have a Fuchsian system of equations then there exist constant matrices $A_1, \ldots, A_r$ such that

$$A(z) = \frac{A_1}{z - p_1} + \cdots + \frac{A_r}{z - p_r}.$$

The point ∞ is regular if and only if $\sum_{i=1}^r A_i = 0$.
Let $P \in \mathbb{P}^1$ be any point which is regular or a regular singularity. Let t be a local parameter around this point and rewrite the equation (1.4) with respect to the variable t. The corresponding indicial equation will be called the indicial equation of (1.4) at P. The roots of the indicial equation at P are called the *local exponents* of (1.4) at P.
This procedure can be cumbersome and as a shortcut we use the following lemma to compute indicial equations.

Lemma 1.3.2 *Let $P \in \mathbb{C}$ be a regular point or regular singularity of* (1.4)*. Let $a_i = \lim_{z \to P}(z - P)^i p_i(z)$ for $i = 1, \ldots, n$. The indicial equation at P is given by*

$$X(X-1)\cdots(X-n+1) + a_1 X(X-1)\cdots(X-n+2) + \cdots + a_{n-1}X + a_n = 0.$$

When ∞ is regular or a regular singularity, let $a_i = \lim_{z\to\infty} z^i p_i(z)$ for $i = 1, \ldots, n$. The indicial equation at ∞ is given by

$$\begin{aligned} &X(X+1)\cdots(X+n-1) - a_1 X(X+1)\cdots(X+n-2) + \cdots \\ &+(-1)^{n-1}a_{n-1}X + (-1)^n a_n = 0. \end{aligned}$$

Proof. Exercise

Theorem 1.3.3 (Fuchs' relation) *Suppose* (1.4) *is a Fuchsian equation. Let $\rho_1(P), \ldots, \rho_n(P)$ the set of local exponents at any $P \in \mathbb{P}^1$. Then,*

$$\sum_{P \in \mathbb{P}^1} (\rho_1(P) + \cdots + \rho_n(P) - \binom{n}{2}) = -2\binom{n}{2}$$

Since the local exponents at a regular point are always $0, 1, \ldots, n-1$ the terms in the summation are zero when P is a regular point. So, in fact, the summation in this theorem is a finite sum.

Proof. From the explicit shape of the indicial equations, given in the Lemma above, we infer that for $P \in \mathbb{C}$,

$$\rho_1(P) + \cdots + \rho_n(P) = \binom{n}{2} - \mathrm{res}_P(p_1(z)dz)$$

and

$$\rho_1(\infty) + \cdots + \rho_n(\infty) = -\binom{n}{2} - \mathrm{res}_\infty(p_1(z)dz).$$

Substract $\binom{n}{2}$ on both sides and add over all $P \in \mathbb{P}^1$. Using the fact that $\sum_{P\in\mathbb{P}^1} \mathrm{res}_P(p_1(z)dz) = 0$ yields our theorem. □

Exercise 1.3.4 *Let $a, b, c \in \mathbb{C}$. Determine all singularities and their local exponents of the so-called hypergeometric differential equation*

$$z(z-1)F'' + ((a+b+1)z - c)F' + abF = 0.$$

For non-integral c write the recurrence relation for the coefficients of the power series expansions of the solutions around $z = 0$.

From Cauchy's theorem of the previous section follows automatically

Theorem 1.3.5 (Cauchy) *Suppose $P \in \mathbb{C}$ is a regular point of the system of equations $\frac{d}{dz}\mathbf{y} = A\mathbf{y}$. Then there exist n $\mathbb{C}$-linear independent vector solutions $\mathbf{y}_1, \ldots, \mathbf{y}_n$ with Taylor series entries in $z - P$ with positive radius of convergence. Moreover, any Taylor series solution of the system is a $\mathbb{C}$-linear combination of $\mathbf{y}_1, \ldots, \mathbf{y}_n$.*

Corollary 1.3.6 *Any analytic solution of $\frac{d}{dz}\mathbf{y} = A\mathbf{y}$ near a regular point can be continued analytically along any path in $\mathbb{C}$ not meeting any singularity.*

Let S be the set of singularities of $\frac{d}{dz}\mathbf{y} = A\mathbf{y}$ and let $z_0 \in \mathbb{P}^1 \setminus S$. Let $vy_1, \ldots, \mathbf{y}_n$ be an independent set of analytic solutions around z_0. They are the columns of the fundamental solution matrix Y. Let $\gamma \in \pi_1(\mathbb{P}^1 \setminus S, z_0)$. After analytic continuation of Y along γ we obtain a new fundamental solution matrix $\tilde{Y}$. Hence there exists a square matrix $M(\gamma) \in GL(n, \mathbb{C})$ such that $\tilde{Y} = Y \cdot M$. The map $\rho : \pi_1(\mathbb{P}^1 \setminus S) \to GL(n, \mathbb{C})$ given by $\rho : \gamma \mapsto M(\gamma)$ is a group homomorphism and its image is called the *monodromy group* of the system.

1.4 Riemann-Hilbert correspondence

Suppose we are given an $n \times n$ Fuchsian system of first order equations, $\frac{d}{dz}\mathbf{y} = A\mathbf{y}$, where A has entries in $\mathbb{C}(z)$. Let $S \subset \mathbb{P}^1$ be the set of singular points and write $S = \{s_1, s_2, \ldots, s_m\}$. Without loss of generality we can assume that $\infty \notin S$. Fix a base point $z_0 \notin S$ and let γ_i be a simple closed beginning and ending in z_0 and which contains only the point s_i as singularity in its interior. Choose a fundamental solution matrix of the system near z_0. Corresponding to this choice we can associate to each loop γ_i the monodromy matrix M_i. The monodromy representation is determined by these matrices. If we order the s_i such that $\gamma_1 \cdots \gamma_m = 1$ in $\pi_1(\mathbb{P}^1 \setminus S, z_0)$ we have moreover that $M_1 \cdots M_m = \mathrm{Id}$.
We have the following question.

Question 1.4.1 (Riemann-Hilbert problem) *Suppose $S = \{s_1, \ldots, s_m\}$ is a finite subset of $\mathbb{P}^1$. Again we assign a simple loop γ_i to each s_i and order them in such a way that $\gamma_1 \cdots \gamma_m = 1$ in $\pi_1(\mathbb{P}^1 \setminus S, z_0)$. To each $s_i \in S$ we assign a matrix $M_i \in GL(n, \mathbb{C})$ such that $M_1 \cdots M_m = \mathrm{Id}$. Does there exist a Fuchsian system with singularities only in S whose monodromy representation ρ is (up to conjugation) given by $\rho : \gamma_i \mapsto M_i$ for $i = 1, \ldots, n$?*

The answer is as follows,

Theorem 1.4.2 (Plemelj, 1906) *Let notations be as in the above problem. Then there exists an $n \times n$ Fuchsian system $\frac{d}{dz}\mathbf{y} = A\mathbf{y}$ whose monodromy representation is given by the matrices M_i. Moreover, the system can be chosen in such a way that the singular set is given by $S \cup \{a\}$, where a is an arbitrary point in $\mathbb{P}^1 \setminus S$ and forms an apparent singularity of the system.*

So we see that the answer to our question is almost affirmative. It may be necessary to have an extra singularity in the Fuchsian system. It was recently shown by Bolibruch that there are examples of representations of the fundamental group $\pi_1(\mathbb{P}^1 \setminus S)$ where any corresponding Fuchsian system requires an extra singularity. It is also known that we do not need an extra singularity if the representation is irreducible or if one of the matrices M_i is semi-simple.
To prove Plemelj's theorem we shall use the following theorem.

Theorem 1.4.3 (Birkhoff, Grothendieck) *Any holomorphic vector bundle on $\mathbb{P}^1$ of rank n is of the form $\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$ where the m_i are integers and where $\mathcal{O}(m)$ is the line bundle on $\mathbb{P}_1$ corresponding to the divisor $m\infty$.*

More particularly we shall need the following Corollary which says that any rank n holomorphic vector bundle over $\mathbb{P}^1$ has n meromorphic sections which form a basis of the fiber above every point in $\mathbb{C}$.

Corollary 1.4.4 *We cover $\mathbb{P}^1$ with $\mathbb{C}$ and an open disc U around ∞. For any holomorphic vector bundle E over $\mathbb{P}^1$ there exist integers $m_1, \ldots, m_n$ and a local trivialisation $f : \mathbb{C}\times\mathbb{C}^n \to E$ with the property that for any local trivialisation $h_U : U \times \mathbb{C}^n \to E$ the map $h_U^{-1} \circ f$ of $U^* \times \mathbb{C}^n$ to itself has the form $Z \cdot \mathrm{diag}(z^{m_1}, \ldots, z^{m_n})$, where $Z : U \to GL(n, \mathbb{C})$ is holomorphic.*

Proof of Plemelj's theorem. Without loss of generality we can assume that $a = \infty$. Choose z_0 in such a way that it does not lie on any line connecting two points of S. For each i we now draw the line connecting z_0 and s_i and denote the part going from s_i to ∞ by l_i. So the lines l_i forms a set of rays to ∞. We number the indices such that $l_1, l_2, l_3, \ldots$ have a counterclockwise ordering. Denote the open set $\mathbb{C} \setminus (\cup_{i=1}^k l_i)$ by U_0. Now choose $\epsilon > 0$. To each i we associate the open set $D_i \subset \mathbb{C}$ given by all points $z \in \mathbb{C}$ whose distance to l_i is less than ϵ. By choosing ϵ sufficiently small we can see to it that the D_i are disjoint. Choose R such that $|s_i| < R$ for all i and denote by U_∞ the set $z : \ |z| > R$ together with the point at infinity. So the open sets $U_0, U_\infty, D_1, \ldots, D_k$ form an open cover of $\mathbb{P}^1$. Using this cover we construct a vector bundle of rank r.
For each i we choose a matrix G_i such that $e^{2\pi\sqrt{-1}G_i} = M_i$. Note there is some ambiguity in the choice of G_i. We define the holomorphic map $f_i : D_i \cap U_0 \to GL(n, \mathbb{C})$ by some choice of $(z - s_i)^{G_i}$. Note that $U_0 \cap U_\infty$ consists of m open sectors around the point ∞. We call these sectors $T_1, \ldots, T_m$ and order them so that T_i lies between l_i and l_{i+1}

(indices considered modulo $m+1$). We define the map $h : U_0 \cap U_\infty \to GL(n,\mathbb{C})$ by $h(z) = M_1 \cdots M_i$ if $z \in T_i$. We define the holomorphic map $g_i : D_i \cap U_\infty \to GL(n,\mathbb{C})$ by the function which coincides with $M_1 \cdots M_i f_i$ on T_i. Note that by analytic continuation this implies that g_i coincides with $M_1 \cdots M_{i-1} f_i$ on T_{i-1}. As a consequence we have for each i that $g_i = hf_i$ on $U_0 \cap U_\infty \cap D_i$.
Construct a vector bundle as follows. Glue $U_0 \times \mathbb{C}^n$ to $D_i \times \mathbb{C}^n$ via the equivalence relation

$$(z,v) \sim (z_i,v_i) \iff z = z_i \in U_0 \cap D_i \text{ and } v_i = f_i^{-1}v.$$

Glue $U_\infty \times \mathbb{C}^n$ to $D_i \times \mathbb{C}^n$ via the equivalence relation

$$(z,v) \sim (z_i,v_i) \iff z = z_i \in U_\infty \cap D_i \text{ and } v_i = g_i^{-1}v.$$

Finally glue $U_0 \times \mathbb{C}^n$ to $U_\infty \times \mathbb{C}^n$ via the relation

$$(z_0,v_0) \sim (z_\infty,v_\infty) \iff z = z_i \in U_0 \cap U_\infty \text{ and } v_\infty = hv_0.$$

Because of the relation $g_i = hf_i$ for all i this can be done in a compatible way. We thus obtain a holomorphic vector bundle E over $\mathbb{P}^1$. According to the Corollary of the Birkhoff-Grothendieck theorem there is a meromorphic local trivialisation $f : \mathbb{C} \times \mathbb{C}^n \to E$. This implies that there exist holomorphic functions $t_0 : U_0 \to GL(n,\mathbb{C})$ and $t_i : D_i \to GL(n,\mathbb{C})$ and $t_\infty : U_\infty \to GL(n,\mathbb{C})$ with the property that $t_0^{-1}t_i = f_i^{-1}$ for $i = 1,\ldots,m$ and $t_0^{-1}t_\infty z^{G_\infty} = h$, where $G_\infty = \mathrm{diag}(m_1,\ldots,m_n)$.
Now observe that t_0 is a fundamental solution matrix of the system of equations $\frac{d}{dz}\mathbf{y} = A\mathbf{y}$ where $A = \frac{dt_0}{dz}t_0^{-1}$. Note that A has holomorphic entries on U_0. The continuation of t_0 to D_i is given by $t_i f_i$. Hence the continuation of A to D_i has the form

$$(\frac{d}{dz}(t_i(z-s_i)^{G_i})(z-s_i)^{-G_i}t_i^{-1} = \frac{dt_i}{dz}t_i^{-1} + \frac{t_iG_it_i^{-1}}{z-s_i}.$$

Similarly the continuation of A to U_∞ has the form

$$(\frac{d}{dz}(t_\infty z^{G_\infty}h^{-1})hz^{-G_\infty}t_\infty^{-1} = (\frac{dt_\infty}{dz})t_\infty^{-1} + \frac{t_\infty G_\infty t_\infty^{-1}}{z}.$$

Note that $\lim_{z\to\infty} A = 0$. Hence our system is Fuchsian. Since $t_0 = t_i(z-s_i)^{G_i}$ in every D_i, the functions t_0 have the correct local monodromy behaviour at every point s_i. We have thus found our desired Fuchsian system of equations. $\square$

1.5 Fuchsian equations of order two

It is an interesting exercise to write down all Fuchsian differential equations with a given number of singular points. Let us start with first order Fuchsian equations

Exercise 1.5.1 *Show that any Fuchsian equation of order one can be written in the form*

$$\frac{dy}{dz}+\left(\frac{A_1}{z-a_1}+\ldots+\frac{A_k}{z-a_k}\right)y=0$$

for suitable $a_i, A_i \in \mathbb{C}$. Solve this equation.

Let us now turn to higher order Fuchsian equations

Exercise 1.5.2 *Show that any Fuchsian equation having only ∞ as singular point is of the form $\frac{d^ny}{dz^n}=0$.*

More generally, Fuchsian equations having only one singularity are not very interesting since, by a fractional linear transformation, the singularity can be moved to ∞.

Exercise 1.5.3 *Show that any Fuchsian equation having only 0 and ∞ as singular points is of the form*

$$z^ny^{(n)}+a_1z^{n-1}y^{(n-1)}+\cdots+a_{n-1}zy'+a_ny=0$$

for suitable $a_1,\ldots,a_n \in \mathbb{C}$. Verify that the indicial equation has the form

$$X(X-1)\cdots(X-n+1)+a_1X\cdots(X-n+2)+\cdots+a_{n-1}X+a_n=0.$$

Equations such as these are known as Euler equations. Suppose that the local exponents at $z=0$ are all distinct. Then write down a basis of solutions.

More generally, any Fuchsian equation with two singularities can be transformed into an Euler equation.
The underlying reason why Fuchsian equations with one or two singularities are not very exciting is that the fundamental groups of $\mathbb{P}^1 \setminus \infty$ and $\mathbb{P}^1 \setminus \{0,\infty\}$ are trivial and $\mathbb{Z}$ respectively, i.e. they are both abelian groups. Interesting equations can be expected when there are three or more singular points.

Exercise 1.5.4 *Suppose we have a second order Fuchsian equation with singularities $0, 1, \infty$ and suppose the local exponents at these points are given by the following scheme,*

0	1	∞
0	0	a
$1-c$	$c-a-b$	b

The second exponent at 1 is chosen to satisfy Fuchs's relation for exponents. Show that the corresponding second order equation is uniquely determined and reads,

$$z(z-1)F''+((a+b+1)z-c)F'+abF=0.$$

This is the hypergeometric equation with parameters a, b, c.

Suppose we have a second order equation with three singularities, say A, B, C. To eqach singularity we have local exponents which we put in the following (Riemann) scheme,

$$\begin{array}{ccc} A & B & C \\ \hline \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array}$$

Via a Möbius transformation we can map A, B, C to any three distinct points of $\mathbb{P}^1$. Let us take the mapping $A, B, C \to 0, 1, \infty$. So we have to deal with the Fuchsian equation having Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array}$$

If we multiply the solutions of the latter equation by z^μ we obtain a set of functions that satisfy the Fuchsian equation with Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \alpha+\mu & \beta & \gamma-\mu \\ \alpha'+\mu & \beta' & \gamma'-\mu \end{array}$$

A fortiori, after multiplication of the solutions with $z^{-\alpha'}(1-z)^{-\beta'}$ we obtain a Fuchsian equation with a scheme of the form

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \alpha'' & \beta'' & \gamma'' \\ 0 & 0 & 1-\alpha''-\beta''-\gamma'' \end{array}$$

Hence any second order Fuchsian equation with three singularities can be transformed into a hypergeometric equation. Any hypergeometric equation is uniquely determined by its local exponents and, a fortiori, any second order Fuchsian equation is uniquely determined by the location of its singularities and their local exponents.
Here are some remarks on the solutions of the hypergeometric equation. When c is not integral a basis of solutions is given by

$$F(a, b, c|z) := \sum \frac{(a)_n (b)_n}{(c)_n n!} z^n. \tag{1.5}$$

and

$$z^{1-c} F(a+1-c, b+1-c, 2-c|z)$$

The *Pochhammer symbol* $(x)_n$ is defined by $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$. The function $F(a, b, c|z)$ is known as Gauss' *hypergeometric function*.

Exercise 1.5.5 *Show directly that the power series* (1.5) *satisfies the differential equation*

$$z(\theta+a)(\theta+b)F = \theta(\theta+c-1)F, \qquad \theta = z\frac{d}{dz}$$

Using the above theory it is very simple to prove some quadratic relations between hypergeometric functions, such as

$$F(a,b,a+b+1/2|4t-4t^2) = F(2a,2b,a+b+1/2|t)$$

and

$$F(a,b,a+b+1/2|t^2/(4t-4)) = (1-t)^a F(2a,a+b,2a+2b|t).$$

they were discovered by E.Kummer. Let us prove for example the quadratic relation

$$F(a,b,a+b+1/2|t^2/(4t-4)) = (1-t)^a F(2a,a+b,2a+2b|t).$$

Substitute $z = t^2/(4t-4)$ in the hypergeometric equation with parameters $a, b, a+b+1/2$. We obtain a new Fuchsian equation. The map $t \to z = t^2/(4t-4)$ ramifies above $0, 1$ in $t = 0, 2$ respectively. Above $z = 1$ we have the point $t = 2$, above $z = 0$ the point $t = 0$ and above $z = \infty$ the two points $t = 1, \infty$. Notice that our equation has local exponents $0, 1/2$ in $z = 1$. Hence the new equation has local exponents $0, 1$ in $t = 2$, with regular solutions, and $t = 2$ turns out to be a regular point. At $t = 0$ we get the local exponents $0, 2(1/2 - a - b)$ and in $t = 1, \infty$, the points above $z = \infty$, we have the local exponents a, b and a, b. Thus our equation in t has again three singular points and Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & a & a \\ 1-2a-2b & b & b \end{array}$$

By the method sketched above, one easily sees that $(1-t)^a F(2a, a+b, 2a+2b|t)$ is a solution of this equation. Moreover, this is the unique (up to a constant factor) solution holomorphic near $t = 0$. At the same time $F(a,b,a+b+1/2|t^2/(4t-4))$ is a solution, and by the uniquess equality follows.

Exercise 1.5.6 *Prove in a similar way the equality*

$$F(a,b,a+b+1/2|4z-4z^2) = F(2a,2b,a+b+1/2|z).$$

Chapter 2

Gauss hypergeometric functions

2.1 Definition, first properties

Let $a, b, c \in \mathbb{R}$ and $c \notin \mathbb{Z}_{\leq 0}$. Define *Gauss' hypergeometric function* by

$$F(a, b, c|z) = \sum \frac{(a)_n (b)_n}{(c)_n n!} z^n. \tag{2.1}$$

The *Pochhammer symbol* $(x)_n$ is defined by $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$. The radius of convergence of (2.1) is 1 unless a or b is a non-positive integer, in which cases we have a polynomial.

Examples.

$$\begin{array}{rcl}
(1-z)^{-a} & = & F(a, 1, 1|z) \\
\log \dfrac{1+z}{1-z} & = & 2zF(1/2, 1, 3/2|z^2) \\
\arcsin z & = & zF(1/2, 1/2, 3/2|z^2) \\
K(z) & = & \dfrac{\pi}{2} F(1/2, 1/2, 1, z^2) \\
P_n(z) & = & 2^n F(-n, n+1, 1|(1+z)/2) \\
T_n(z) & = & (-1)^n F(-n, n, 1/2|(1+z)/2)
\end{array}$$

Here $K(z)$ is the Jacobi's elliptic integral of the first kind given by

$$K(z) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}}.$$

The polynomials P_n, T_n given by $P_n = (1/n!)(d/dz)^n(1-z^2)^n$ and $T_n(\cos z) = \cos(nz)$ are known as the *Legendre* and *Chebyshev* polynomials respectively. They are examples of orthogonal polynomials.

One easily verifies that (2.1) satisfies the linear differential equation

$$z(\theta+a)(\theta+b)F=\theta(\theta+c-1)F, \qquad \theta=z\frac{d}{dz}.$$

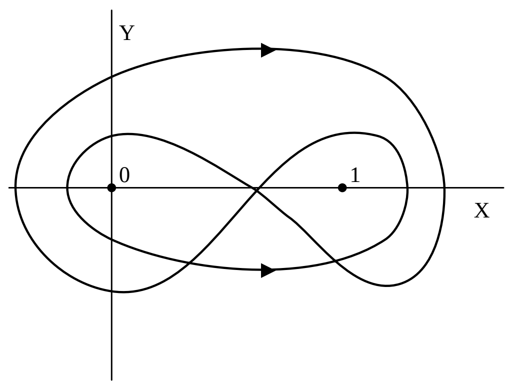
Written more explicitly,

$$z(z-1)F''+((a+b+1)z-c)F'+abF=0. \tag{2.2}$$

There exist various ways to study the analytic continuation of (2.1), via *Euler integrals*, *Kummer's solutions* and *Riemann's* approach. The latter will be discussed in later sections. The Euler integral reads

$$F(a,b,c|z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \quad (c>b>0)$$

and allows choices of z with $|z|>1$. The restriction $c>b>0$ is included to ensure convergence of the integral at 0 and 1. We can drop this condition if we take the Pochhammer contour γ given by



as integration path. Notice that the integrand acquires the same value after analytic continuation along γ.
It is a straightforward exercise to show that for any $b, c-b \notin \mathbb{Z}$ we have

$$F(a,b,c|z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{1}{(1-e^{2\pi ib})(1-e^{2\pi i(c-b)})}\int_\gamma t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

Kummer gave the following 24 solutions to (2.2)

$$\begin{aligned}
& F(a,b,c|z)\\
= \; & (1-z)^{c-a-b}F(c-a,c-b,c|z)\\
= \; & (1-z)^{-a}F(a,c-b,c|z/(z-1))\\
= \; & (1-z)^{-b}F(a-c,b,c|z/(z-1))
\end{aligned}$$

$$\begin{aligned}
& z^{1-c}F(a-c+1,b-c+1,2-c|z)\\
= \; & z^{1-c}(1-z)^{c-a-b}F(1-a,1-b,2-c|z)\\
= \; & z^{1-c}(1-z)^{c-a-1}F(a-c+1,1-b,2-c|z/(z-1))\\
= \; & z^{1-c}(1-z)^{c-b-1}F(1-a,b-c+1,2-c|z/(z-1))
\end{aligned}$$

$$\begin{aligned}
& F(a,b,a+b-c+1|1-z) \\
= \; & x^{1-c}F(a-c+1,b-c+1,a+b-c+1|1-z) \\
= \; & z^{-a}F(a,a-c+1,a+b-c+1|1-1/z) \\
= \; & z^{-b}F(b-c+1,b,a+b-c+1|1-1/z)
\end{aligned}$$

$$\begin{aligned}
& (1-z)^{c-a-b}F(c-a,c-b,c-a-b+1|1-z) \\
= \; & (1-z)^{c-a-b}z^{1-c}F(1-a,1-b,c-a-b+1|1-z) \\
= \; & (1-z)^{c-a-b}z^{a-c}F(1-a,c-a,c-a-b+1|1-1/z) \\
= \; & (1-z)^{c-a-b}z^{b-c}F(c-b,1-b,c-a-b+1|1-1/z)
\end{aligned}$$

$$\begin{aligned}
& z^{-a}F(a,a-c+1,a-b+1|1/z) \\
= \; & z^{-a}(1-1/z)^{c-a-b}F(1-b,c-b,a-b+1|1/z) \\
= \; & z^{-a}(1-1/z)^{c-a-1}F(a-c+1,1-b,2-c|1/(1-z)) \\
= \; & z^{-a}(1-1/z)^{-a}F(a,c-b,a-b+1|1/(1-z))
\end{aligned}$$

$$\begin{aligned}
& z^{-b}F(b,b-c+1,b-a+1|1/z) \\
= \; & z^{-b}(1-1/z)^{c-a-b}F(1-a,c-a,b-a+1|1/z) \\
= \; & z^{-b}(1-1/z)^{c-b-1}F(b-c+1,1-a,2-c|1/(1-z)) \\
= \; & z^{-b}(1-1/z)^{-b}F(b,c-a,b-a+1|1/(1-z))
\end{aligned}$$

Strictly speaking, the above six 4-tuples of functions are only distinct when $c, c-a-b, a-b \notin \mathbb{Z}$. If one of these numbers is an integer we find that there are other solutions containing logarithms. For example, when $c = 1$ we find that z^{1-c} becomes $\log z$ and a second solution near $z = 0$ reads

$$(\log z)F(a,b,1|z) + \sum_{n=1}^{\infty} \frac{(a)_n(b_n)}{(n!)^2} z^n \left[\sum_{k=1}^{n} \left(\frac{1}{a+k-1} + \frac{1}{b+k-1} - \frac{2}{k} \right) \right].$$

Notice that this solution can be obtained by taking the difference of solutions $z^{1-c}F(a-c+1,b-c+1,2-c|z) - F(a,b,c|z)$, divide it by $c-1$ and take the limit as $c \to 1$.
Later it will turn out that Riemann's approach to hypergeometric functions gives a remarkably transparent insight into these formulas as well as the quadratic transformations of Kummer and Goursat.
Examples of such transformations are

$$F(a,b,a+b+1/2|4z-4z^2) = F(2a,2b,a+b+1/2|z)$$

and

$$F(a,b,a+b+1/2|z^2/(4z-4)) = (1-z)^a F(2a,a+b,2a+2b|z).$$

Finally we mention the 6 *contiguous* functions

$$F(a\pm 1,b,c|z),\qquad F(a,b\pm 1,c|z),\qquad F(a,b,c\pm 1|z).$$

Gauss found that $F(a,b,c|z)$ and any two contiguous functions satisfy a linear relation with coefficients which are linear polynomials in z or constants, for example,

$$(c-a)F(a-1,b,c|z)+(2a-c-az+bz)F(a,b,c|z)+a(z-1)F(a+1,b,c|z)=0.$$

Notice also that $F'(a,b,c|z)=(ab/c)F(a+1,b+1,c+1|z)$. These observations are part of the following theorem.

Theorem 2.1.1 *Suppose $a,b\not\equiv 0,c(\mathrm{mod}\ \mathbb{Z})$ and $c\notin\mathbb{Z}$. Then any function $F(a+k,b+l,c+m|z)$ with $k,l,m\in\mathbb{Z}$ equals a linear combination of F,F' with rational functions as coefficients.*

Proof. One easily verifies that

$$\begin{aligned} F(a+1,b,c|z) &= \frac{1}{a}(z\frac{d}{dz}+a)F(a,b,c|z)\\ F(a-1,b,c|z) &= \frac{1}{c-a}(z(1-z)\frac{d}{dz}-bz+c-a)F(a,b,c|z) \end{aligned}$$

and similarly for $F(a,b+1,c|z),F(a,b-1,c|z)$. Furthermore,

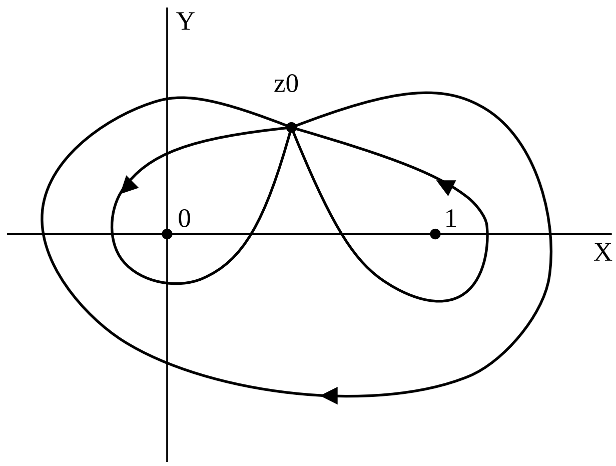
$$\begin{aligned} F(a,b,c+1|z) &= \frac{c}{(c-a)(c-b)}(z(1-z)\frac{d}{dz}+c-a-b)F(a,b,c|z)\\ F(a,b,c-1|z) &= \frac{1}{c-1}(z\frac{d}{dz}+c-1)F(a,b,c|z) \end{aligned}$$

Hence there exists a linear differential operator $\mathcal{L}_{k,l,m}\in\mathbb{C}(z)[\frac{d}{dz}]$ such that $F(a+k,b+l,c+m|z)=\mathcal{L}_{k,l,m}F(a,b,c|z)$. Since F satifies a second order linear differential equation, $\mathcal{L}_{k,l,m}F$ can be written as a $\mathbb{C}(z)$-linear combination of F and F'. $\square$

In general we shall call any function $F(a+k,b+l,c+m|z)$ with $k,l,m\in\mathbb{Z}$ contiguous with $F(a,b,c|z)$. Thus we see that, under the assumptions of Theorem 2.1.1, any three contiguous functions satisfy a $\mathbb{C}(z)$-linear relation.
For many more identities and formulas we refer to [AS] and [E].

2.2 Monodromy of the hypergeometric function

Let us now turn to the monodromy of the hypergeometric equation. Consider the three loops g_0,g_1,g_∞ which satisfy the relation $g_0g_1g_\infty=1$.



We denote the corresponding monodromy matrices by M_0, M_1, M_∞. They also satisfy $M_0 M_1 M_\infty = 1$ and M_0, M_∞ generate the monodromy group. Since the local exponents at $0, 1, \infty$ are $0, 1-c$, $0, c-a-b$ and a, b respectively, the eigenvalues of the matrices M_0, M_1 and M_∞ are $1, \exp(2\pi i(1-c))$, $1, \exp(2\pi i(c-a-b))$ and $\exp(2\pi i a), \exp(2\pi i b)$ respectively. The monodromy group can be considered as being generated by M_0, M_∞ and we know that $M_\infty M_0 = M_1^{-1}$ has eigenvalue 1. This scant information already suffices to draw some important conclusions.

Lemma 2.2.1 *Let $A, B \in GL(2, \mathbb{C})$. Suppose that AB^{-1} has eigenvalue 1. Then there exists a common eigenvector of A, B if and only if A, B have a common eigenvalue.*

Proof. Notice that $\ker(A-B)$ has dimension at least 1. If the dimension were 2 we would have $A = B$ and our lemma would be trivial. So we can assume $\dim(\ker(A-B)) = 1$. In this proof we let $v \in \ker(A-B)$, $v \neq 0$.
Suppose there exists a common eigenvector, w say, of A, B with eigenvalues λ_A, λ_B. If these eigenvalues are equal, we are done. Suppose they are not equal. Then w, v span $\mathbb{C}^2$. Choose α, β such that $Av = \alpha v + \beta w$. Since $Av = Bv$ we also have $Bv = \alpha v + \beta w$. Hence with respect to the basis v, w the matrices of A, B read

$$\begin{pmatrix} \alpha & \beta \\ 0 & \lambda_A \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta \\ 0 & \lambda_B \end{pmatrix}$$

Hence they have the common eigenvalue α.
Suppose A, B have a common eigenvalue λ. If v is an eigenvectore of A we are done, since $Av = Bv$ implies that it is also an eigenvector of B. So suppose v is not an eigenvector of A. Consider the vector $w = (A-\lambda)v$. Since $A - \lambda$ has non-trivial kernel we have $< w >_{\mathbb{C}} = (A-\lambda)\mathbb{C}^2$. In particular, $(A-\lambda)w$ is a scalar multiple of w, i.e. w is an eigenvector of A. We also have $w = (B-\lambda)v$ and a similar argument shows that w is an eigenvector of B. Hence A, B have a common eigenvector. $\square$

Corollary 2.2.2 *The monodromy group of* (2.2) *acts reducibly on the space of solutions if and only if at least one of the numbers $a, b, c-a, c-b$ is integral.*

Proof. This follows by application of the previous lemma to the case $A = M_\infty, B = M_0^{-1}$. Since $M_1^{-1} = M_\infty M_0$ the condition that AB^{-1} has eigenvalue 1 is fullfilled. Knowing the eigenvalues of M_0, M_∞ one easily checks that equality of eigenvalues comes down to the non-empty intersection of the sets $\{0, c\}$ and $\{a, b\}$ considered modulo $\mathbb{Z}$.

Definition 2.2.3 *A hypergeometric equation is called reducible if its monodromy group is reducible. A hypergeometric equation is called abelian if its monodromy group is abelian.*

Typical examples of abelian equations are (2.2) with $a = c = 0$ having solutions $1, (1-z)^{-(b+1)}$ and $a = b = 1, c = 2$ having solutions $1/z, \log(1-z)/z$. Here is a simple necessary condition for abelian equations, which has the pleasant property that it depends only on $a, b, c (\mathrm{mod}\ \mathbb{Z})$.

Lemma 2.2.4 *If* (2.2) *is abelian then at least two of the numbers $a, b, c-a, c-b$ are integral.*

Proof. Abelian monodromy implies reducibility of the monodromy, hence at least one of the four numbers is integral. Let us say $a \in \mathbb{Z}$, the other cases can be dealt with similarly. It suffices to show that in at least one of the points $0, 1, \infty$ the local exponent difference of (2.2) is integral. Then clearly, $1 - c \in \mathbb{Z}$ implies $c - a \in \mathbb{Z}$, $c - a - b \in \mathbb{Z}$ implies $c - b \in \mathbb{Z}$ and $a - b \in \mathbb{Z}$ implies $b \in \mathbb{Z}$.
Suppose that all local exponent differences are non-integral. In particular the eigenvalues of each of the generating monodromy elements M_0, M_1, M_∞ are distinct. Then abelian monodromy implies that the monodromy group acts on the solution space in a completely reducible way as a sum of two one-dimensional representations. In particular the generators of these representations are functions of the form

$$z^\lambda (1-z)^\mu q(z) \qquad z^{\lambda'} (1-z)^{\mu'} p(z)$$

where $p(z), q(z)$ are polynomials with the property that they do not vanish at $z = 0$ or 1. The local exponents can be read off immediately, λ, λ' at 0, μ, μ' at 1 and $-\lambda - \mu - \deg(q), -\lambda' - \mu' - \deg(p)$ at ∞. The sum of the local exponents must be 1, hence $-\deg(p) - \deg(q) = 1$. Clearly this is a contradiction. □

Lemma 2.2.5 *Suppose that $A, B \in GL(2, \mathbb{C})$ have disjoint sets of eigenvalues and suppose that AB^{-1} has eigenvalue 1. Then, letting $X^2 + a_1 X + a_2$ and $X^2 + b_1 X + b_2$ be the characteristic polynomials of A, B, we have up to common conjugation,*

$$A = \begin{pmatrix} 0 & -a_2 \\ 1 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -b_2 \\ 1 & -b_1 \end{pmatrix}.$$

Proof. Choose $v \in \ker(A - B)$ and $w = Av = Bv$. Since A, B have disjoint eigenvalue sets, v is not an eigenvector of A and B. Hence w, v form a basis of $\mathbb{C}^2$. With respect to this basis A, B automatically obtain the form given in our Lemma. □

Corollary 2.2.6 *Suppose that* (2.2) *is irreducible. Then, up to conjugation, the monodromy group depends only on the values of* a, b, c *modulo* $\mathbb{Z}$.

Let us now assume that $a, b, c \in \mathbb{R}$, which is the case most frequently studied. The eigenvalues of M_0, M_1, M_∞ then lie on the unit circle.

Definition 2.2.7 *Let* R, S *be two disjoint finite subsets of the unit circle of equal cardinality. The sets* R, S *are said to interlace if every segment on the unit circle, connecting two points of* R*, contains a point of* S.

Lemma 2.2.8 *Let* A, B *be non-commuting elements of* $GL(2, \mathbb{C})$. *Suppose that the eigenvalues of* A, B *have absolute value* 1 *and that* AB^{-1} *has eigenvalue 1. Let* G *be the group generated by* A, B. *Then there exists a unique (up to a constant factor) non-trivial hermitian form* F *on* $\mathbb{C}^2$ *such that* $F(g(x), g(y)) = F(x, y)$ *for every* $g \in G$ *and every pair* $x, y \in \mathbb{C}^2$. *Moreover,*

$$F \text{ degenerate} \iff A, B \text{ have common eigenvalues}$$

Supposing A, B *have disjoint eigenvalue sets, we have in addition,*

$$F \text{ definite} \iff \text{eigenvalues of } A, B \text{ interlace}$$

$$F \text{ indefinite} \iff \text{eigenvalues of } A, B \text{ do not interlace}$$

We call these three cases the euclidean, spherical and hyperbolic case respectively.

Proof. Let $v \in \ker(A - B)$ and $w = Av$. Suppose first that v, w form a basis of $\mathbb{C}^2$. Of course, with respect to this basis A and B have the form given in the previous lemma. In particular we see that A, B cannot have the same characteristic equation, since this would imply that $A = B$.
We have to find a hermitean form F such that

$$\begin{aligned} F(gv, gv) &= F(v, v) \qquad F(gv, gw) = F(v, w) \\ F(gw, gv) &= F(w, v), \qquad F(gw, gw) = F(w, w) \end{aligned}$$

for every $g \in G$. It suffices to take $g = A, B$. Let $X^2 + a_1X + a_2$ and $X^2 + b_1X + b_2$ be the characteristic polynomials of A, B. Since the roots are on the unit circle we have $a_2\bar{a}_2 = 1, a_2\bar{a}_1 = a_1$ and similarly for b_1, b_2.
Let us first take $g = A$. Then $F(Av, Av) = F(v, v)$ implies

$$F(w, w) = F(v, v).$$

The conditions $F(Av, Aw) = F(v, w)$ and $F(Aw, Av) = F(w, v)$ imply $F(w, A^2v) = F(v, w)$ and $F(A^2v, w) = F(w, v)$. Hence, using $A^2 = -a_1A - a_2$,

$$-\bar{a}_1F(w, w) - \bar{a}_2F(w, v) = F(v, w) \tag{2.3}$$

$$-a_1F(w, w) - a_2F(v, w) = F(w, v) \tag{2.4}$$

Because of the relations $a_2 = \bar{a}_2^{-1}$ and $a_2\bar{a}_1 = a_1$ these equations are actually the same. The condition $F(Aw, Aw) = F(w, w)$ yields $F(A^2v, A^2v) = F(w, w)$ and hence

$$|a_1|^2F(w, w) + a_1\bar{a}_2F(w, v) + \bar{a}_1a_2F(v, w) + |a_2|^2F(v, v) = F(w, w).$$

Using $|a_2|^2 = 1$, $a_2\bar{a}_1 = a_1$ and $F(w, w) = F(v, v)$ this is equivalent to

$$a_1\bar{a}_1F(w, w) + a_1\bar{a}_2F(w, v) + a_1F(v, w) = 0$$

which is precisely (2.3) times a_1. Hence A-invariance of F is equivalent to

$$F(v, v) = F(w, w), \qquad F(w, v) + a_1F(w, w) + a_2F(v, w) = 0.$$

Invariance of F with respect to B yields the additional condition

$$F(w, v) + b_1F(w, w) + b_2F(v, w) = 0.$$

Since A and B do not have the same characteristic equation the solutionspace for F is one-dimensional. When $a_2 = b_2$ a solution is given by

$$F(w, w) = F(v, v) = 0, \qquad F(w, v) = (-a_2)^{1/2}, \qquad F(v, w) = (-a_2)^{-1/2},$$

when $a_2 \neq b_2$ a solution is given by

$$F(w, w) = F(v, v) = 1, \qquad F(w, v) = \epsilon, \qquad F(v, w) = \bar{\epsilon}, \qquad \epsilon = \frac{a_1 - b_1}{b_2 - a_2}.$$

We formally take $\epsilon = \infty$ if $a_2 = b_2$. In both cases cases we see that F is definite, degenerate, indefinite according to the conditions $|\epsilon| < 1$, $|\epsilon| = 1$, $|\epsilon| > 1$. It now a straightforward excercise to see that these inequalities correspond to interlacing, coinciding or non-interlacing of the eigenvalues of A and B.
We are left with the case when v is an eigenvector of A and B. Let α be the eigenvalue. If both A and B have only eigenvalues α they automatically commute, which case is excluded. So either A or B has an eigenvalue different from α. Let us say that A has the distinct eigenvalues α, α'. Let w be an eigenvector corresponding to α'. Then, with respect to v, w the matrix of B must have the form

$$\begin{pmatrix} \alpha & b_{12} \\ 0 & \beta \end{pmatrix}.$$

with $b_{12} \neq 0$. It is now straightforward to verify that $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is the unique invariant hermitean matrix. Moreover it is degenerate, which it should be as A, B have a common eigenvector. □

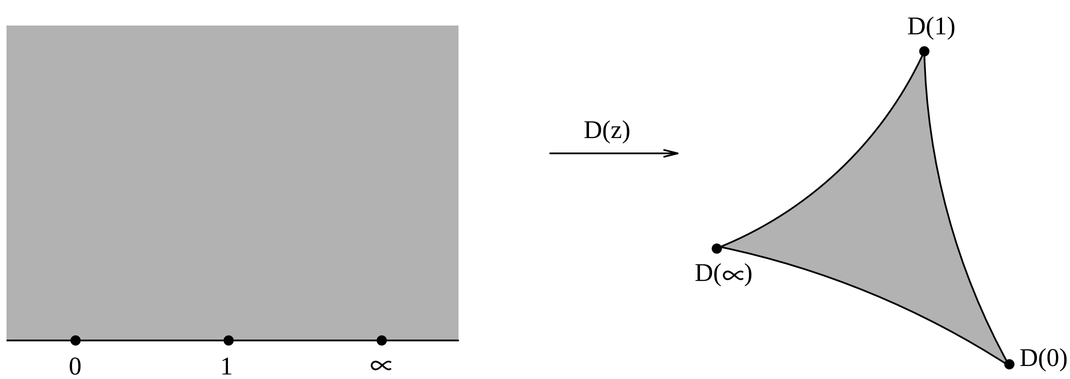
Definition 2.2.9 *With the assumptions as in the previous lemma let G be the group generated by A and B. Then G is called hyperbolic, euclidean, spheric if F is indefinite, degenerate, definite respectively.*

Corollary 2.2.10 *Let $\{x\}$ denote the fractional part of x (x minus largest integer $\le x$). Suppose that* (2.2) *is irreducible. Let F be the invariant hermitean form for the monodromy group. In particular, the sets $\{\{a\},\{b\}\}$ and $\{0,\{c\}\}$ are disjoint. If $\{c\}$ is between $\{a\}$ and $\{b\}$ then F is positive definite (spherical case). If $\{c\}$ is not between $\{a\}$ and $\{b\}$ then F is indefinite (hyperbolic case).*

The most pittoresque way to describe the monodromy group is by using *Schwarz' triangles.* First a little geometry.

Definition 2.2.11 *A curvilinear triangle is a connected open subset of $\mathbb{C}\cup\infty = \mathbb{P}^1$ whose boundary is the union of three open segments of a circle or straight line and three points. The segments are called the edges of the triangles, the points are called the vertices.*

It is an exercise to prove that, given the vertices and the corresponding angles ($< \pi$), a curvilinear triangle exists and is uniquely determined This can be seen best by taking the vertices to be $0, 1, \infty$. Then the edges connected to ∞ are actually straight lines.
More generally, a curvilinear triangle in $\mathbb{C}\cup\infty = \mathbb{P}^1$ is determined by its angles (in clockwise ordering) up to a Möbius transformation.
Let z_0 be a point in the upper half plane $\mathcal{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$ and let f, g be two independent solutions of the hypergeometric equation near z_0. The quotient $D(z) = f/g$, considered as a map from $\mathcal{H}$ to $\mathbb{P}^1$, is called the *Schwarz map* and we have the following picture and theorem.



Theorem 2.2.12 (Schwarz) *Let $\lambda = |1-c|, \mu = |c-a-b|, \nu = |a-b|$ and Suppose $0 \le \lambda, \mu, \nu < 1$. Then the map $D(z) = f/g$ maps $\mathcal{H}\cup\mathbb{R}$ one-to-one onto a curvilinear triangle. The vertices correspond to the points $D(0), D(1), D(\infty)$ and the corresponding angles are $\lambda\pi, \mu\pi, \nu\pi$.*

As to the proof of Schwarz' theorem, the following three ingredients are important.

– The map $D(z)$ is locally bijective in every point of $\mathcal{H}$. Notice that $D'(z) = (f'g - fg')/g^2$. The determinant $f'g - fg'$ is the *Wronskian determinant* of our equation and equals $z^{-c}(1-z)^{c-a-b-1}$. In particular it is non-zero in $\mathcal{H}$. When g has a zero at some point z_1 we simply consider $1/D(z)$ instead. Since f and g cannot vanish at the same time in a regular point, we have $f(z_1) \neq 0$.

– The map $D(z)$ maps the segments $(\infty, 0), (0, 1), (1, \infty)$ to segments of circles or straight lines. For example, since $a, b, c \in \mathbb{R}$ we have two real solutions on $(0, 1)$ (see Kummer's solutions). Call them $\tilde{f}, \tilde{g}$. Clearly, the function $\tilde{D}(z) = \tilde{f}/\tilde{g}$ maps $(0, 1)$ on a segment of $\mathbb{R}$. Since f, g are $\mathbb{C}$-linear combinations of $\tilde{f}, \tilde{g}$ we see that $D(z)$ is a Möbius transform of $\tilde{D}(z)$. Hence $D(z)$ maps $(0, 1)$ to a segment of a circle or a straight line.

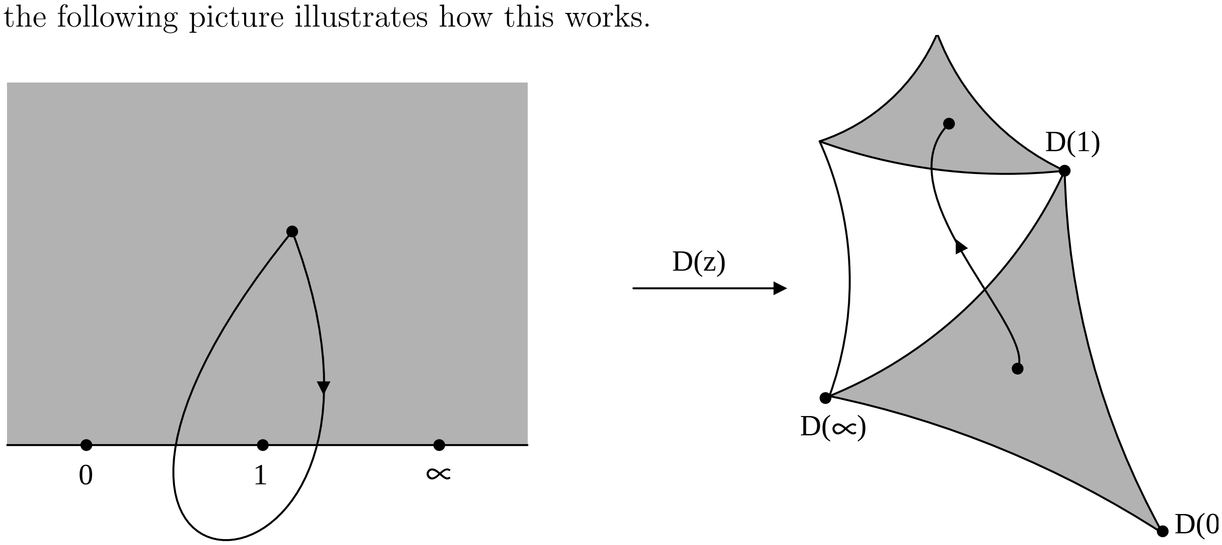
– The map $D(z)$ maps a small neighbourhood of 0 to a sector with angle $|1 - c| = \lambda$ and similarly for $1, \infty$. This follows from the fact that near $z = 0$ the functions f, g are $\mathbb{C}$-linear combinations of $F(a, b, c|z)$ and $z^{1-c}F(a - c + 1, b - c + 1, 2 - c|z)$.

For the exact determination of the image of the Schwarz map we need the following additional result.

Proposition 2.2.13 (Gauss) *Suppose that $a, b, c \in \mathbb{R}$, $c \notin \mathbb{Z}_{\leq 0}$ and $c > a + b$. Prove that*

$$F(a, b, c|1) = \frac{\Gamma(c - a)\Gamma(c - b)}{\Gamma(c)\Gamma(c - a - b)}.$$

This can be proven by evaluation of Euler's integral using the Euler Beta-function.
To study the analytic continuation of $D(z)$ we use *Schwarz' reflection principle*. Hopefully, the following picture illustrates how this works.



The monodromy group modulo scalars arises as follows. Let W be the group generated by the reflections in the edges of the curvilinear triangle. The monodromy group is the subgroup of W consisting of all elements which are product of an even number of reflections. In the following section we shall study precisely such groups.

2.3 Triangle groups

In this section we let S be either the Poincaré disk $\{z \in \mathbb{C}|\ |z| < 1\}$, $\mathbb{C}$ or $\mathbb{P}^1$. equipped with the hyperbolic, euclidean and spherical metric respectively.

Definition 2.3.1 *A (geodesic) triangle is an connected open subset of S, of finite volume, whose boundary in S is a union of three open segments of a geodesic and at most three points. The segments are called the edges of the triangles, the points are called the vertices.*

We first point out that under very mild conditions any curvilinear triangle can be thought of as a geodesic triangle.

Lemma 2.3.2 *Let λ, μ, ν be real numbers in the interval $[0, 1)$. There exists a geodesic triangle with angles $\lambda\pi, \mu\pi, \nu\pi$ if and only if $\lambda + \mu + \nu < 1 + 2\min(\lambda, \mu, \nu)$.*

Proof. Suppose first that $\lambda+\mu+\nu < 1$. Our condition is then trivially satisfied. For any such curvilinear triangle we can take the common orthogonal circle of the three edges, which will become the boundary of a Poincaré disk. The edges are then automatically geodesics.
Suppose that $\lambda + \mu + \nu = 1$. Our condition is equivalent to saying that all angles are positive. In this case geodesic triangles are planar triangles in the euclidean geometry with finite area. The latter property is equivalent to positivity of all angles.
Suppose that $\lambda + \mu + \nu > 1$. From spherical geometry it follows that a spherical triangle exists if and only if our condition is satisfied.

We let $W(\Delta)$ be the group of isometries of S generated by the 3 reflections through the edges of a geodesic triangle Δ. First we look at subgroups generated by reflection in two intersecting geodesics.

Lemma 2.3.3 *Let ρ, σ be two geodesics intersecting in a point P with an angle $\pi\lambda$. Let r, s be the reflections in ρ, σ respectively. Then the group G generated by r, s is a dihedral group consisting of rotations $(rs)^n$ around P with angles $2n\pi\lambda$, $n \in \mathbb{Z}$ and reflections in the lines $(rs)^n(\rho), (rs)^n(\sigma)$. In particular G is finite of order $2m$ if and only if $\lambda = q/m$ for some $q \in \mathbb{Z}$ with $q \neq 0$ and $\gcd(m, q) = 1$. Furthermore, G is discrete if and only if λ is either zero or a rational number.*

Theorem 2.3.4 *For any geodesic triangle Δ we have $S = \cup_{\gamma\in W(\Delta)}\gamma(\overline{\Delta})$, where $\overline{\Delta}$ denotes the closure of Δ in S.*

Proof. First of all we note that there exists a positive d_0 with the following property. For any point P whose distance to Δ is less than d_0 there exists $\gamma \in W(\Delta)$ such that $P \in \gamma(\overline{\Delta})$. For γ we can simply take a suitable element from one of the dihedral reflection groups around the vertices.
A fortiori, any point P with distance less than d_0 from $\cup_{\gamma\in W(\Delta)}\gamma(\overline{\Delta})$ belongs to this set. As a consequence the set $\cup_{\gamma\in W(\Delta)}\gamma(\overline{\Delta})$ is open and closed in S, hence our theorem follows. □

Definition 2.3.5 *An elementary triangle is a geodesic triangle whose vertex angles are all of the form π/n, $n \in \mathbb{Z}_{\geq 2} \cup \infty$.*

Theorem 2.3.6 *Let Δ be an elementary triangle. Then, for any $\gamma \in W(\Delta), \gamma \neq \mathrm{Id}$ we have $\gamma(\Delta) \cap \Delta = \emptyset$.*

Proof. This is a special case of the theorem of Coxeter-Tits on representations of Coxeter groups. See Humphreys book on Reflection groups and Coxeter groups [H].

A group G of isometries acting on S is said to act discretely if there exists a point $P \in S$ and a positive d_0 such that distance$(P, g(P)) > d_0$ whenever $g \neq \mathrm{Id}$. In particular it follows from the previous theorem that triangle groups generated by elementary triangles act discretely. The following theorem characterises all groups $W(\Delta)$ which act discretely on the symmetric space S.

Theorem 2.3.7 *Suppose $W = W(\Delta)$ acts discretely. Then there exists an elementary triangle Δ_{el} such that $W(\Delta) = W(\Delta_{\mathrm{el}})$. Moreover, $\overline{\Delta}$ is a finite union of copies of $\overline{\Delta_{\mathrm{el}}}$ under elements of W.*

Proof. First of all note that the vertex angles must be either 0 or rational multiples of π, otherwise the corresponding dihedral group is not discrete.
We shall show that if Δ is not elementary, then there exists a geodesic triangle Δ' such that $W(\Delta) = W(\Delta')$ and $\mathrm{Vol}(\Delta') \leq \mathrm{Vol}(\Delta)/2$. If Δ' is not elementary we repeat the process and so on. However, there is a limit to these processes since, by discreteness, there is a positive lower bound to $\mathrm{Vol}(\Delta'')$ for any Δ'' satisfying $W(\Delta) = W(\Delta'')$. Hence we must hit upon an elementary triangle Δ_{el} such that $W(\Delta) = W(\Delta_{\mathrm{el}})$.
Let α, β, γ be the edges of Δ and $r_\alpha, r_\beta, r_\gamma$ the corresponding reflections. Suppose that the vertex angle between α and β is of the form $m\pi/n$ with $\gcd(m, n) = 1$, but $m > 1$. Let δ be the geodesic between α and β whose angle with α is π/n. Let r_δ be the reflection in δ. Then the dihedral group generated by r_α and r_β is the same as the one generated by r_α and r_δ. Let Δ' be the triangle with edges α, δ, γ. Then, clearly, $W(\Delta) = W(\Delta')$. If the volume of Δ' is larger than half the volume of Δ we simply perform the above construction with α and β interchanged. $\square$

Below we give a list of non-elementary triangles $\Delta = (\lambda, \mu, \nu)$ with vertex angles $\lambda\pi, \mu\pi, \nu\pi$ which allow a dissection with elementary triangles Δ_{el} such that $W(\Delta) = W(\Delta_{\mathrm{el}})$. In the spherical case discreteness of $W(\Delta)$ implies finiteness. The list of spherical cases was already found by H.A.Schwarz and F.Klein (see [Kl]). In the following table N denotes the number of congruent elementary triangles needed to cover Δ.

λ	μ	ν	N		elementary
$2/n$	$1/m$	$1/m$	2	$\times$	$(1/2, 1/n, 1/m)$ n odd
$1/2$	$2/n$	$1/n$	3	$\times$	$(1/2, 1/3, 1/n)$ n odd
$1/3$	$3/n$	$1/n$	4	$\times$	$(1/2, 1/3, 1/n)$ $n \not\equiv 0 \bmod 3$
$2/n$	$2/n$	$2/n$	6	$\times$	$(1/2, 1/3, 1/n)$ n odd
$4/n$	$1/n$	$1/n$	6	$\times$	$(1/2, 1/3, 1/n)$ n odd
$2/3$	$1/3$	$1/5$	6	$\times$	$(1/2, 1/3, 1/5)$
$1/2$	$2/3$	$1/5$	7	$\times$	$(1/2, 1/3, 1/5)$
$3/5$	$2/5$	$1/3$	10	$\times$	$(1/2, 1/3, 1/5)$
$1/3$	$2/7$	$1/7$	10	$\times$	$(1/2, 1/3, 1/7)$

As an application we construct a hypergeometric function which is algebraic over $\mathbb{C}(z)$. Take the triangle $(4/5, 1/5, 1/5)$, which is spherical. Corresponding values for a, b, c can be taken to be $1/10, -1/10, 1/5$. Hence the quotient of any two solutions f, g of the corresponding hypergeometric is algebraic. Its derivative $(f'g - fg')/g^2$ is algebraic and so is the Wronskian determinant $f'g - fg' = z^{-c}(1-z)^{c-a-b-1}$. Hence g and, a fortiori, f are algebraic. In particular, $F(1/10, -1/10, 1/5|z)$ is an algebraic function.
In many cases it is also possible to find elementary triangles $\Delta_{\rm el}$ which can be dissected into isometric copies of a smaller elementary triangle $\Delta'_{\rm el}$. Hence $W(\Delta_{\rm el}) \subset W(\Delta'_{\rm el})$. The most spectacular example is the dissection of the triangle $(1/7, 1/7, 1/7)$ into 24 copies of $(1/2, 1/3, 1/7)$. As a corollary of this dissection we find the remarkable identity

$$ {}_2F_1\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7} \middle| z\right) = b(z)^{-1/28}\ {}_2F_1\left(\frac{1}{84}, \frac{29}{84}, \frac{6}{7} \middle| 12^3 \frac{z(z-1)(z^3 - 8z^2 + 5z + 1)}{b(z)^3}\right) $$

where $b(z) = 1 - 236z + 1666z^2 - 3360z^3 + 3395z^4 - 1736z^5 + 42z^6 + 228z^7 + z^8$. For a complete list of such dissections and the corresponding identities we refer to [V].

2.4 Some loose ends

In the Schwarz map we have assumed that the parameters a, b, c are such that $\lambda = |1-c|, \mu = |c-a-b|, \nu = |a-b|$ are all less than 1. It turns out that in the irreducible case this is no restriction, since we can shift a, b, c by integers without affecting the monodromy group. In fact,

Lemma 2.4.1 *Assume that none of the numbers $a, b, c-a, c-b$ is integral. There exist $a' \in a(\text{mod } \mathbb{Z}),\ b' \in b(\text{mod } \mathbb{Z}),\ c' \in c(\text{mod } \mathbb{Z})$ such that*

$$ 0 \le \lambda, \mu, \nu < 1 \qquad \lambda + \mu + \nu < 1 + 2\min(\lambda, \mu, \nu) $$

where $\lambda = |1-c'|, \mu = |c'-a'-b'|, \nu = |a'-b'|$. In the case $\lambda + \mu + \nu < 1$ there exists only one choice for a', b', c' and in the case $\lambda + \mu + \nu > 1$ there exist four possible choices.

Proof. First of all let us suppose that $0 \le a, b, c < 1$. Without loss of generality we can assume that $a \le b$. We consider the following cases.
Case i) $0 < a < c < b < 1$. We take $a' = a, b' = b, c' = c$. Then, $\lambda = 1-c, \mu = a+b-c, \nu = b-a$ and the inequalities are satisfied. Moreover, $\lambda + \mu + \nu = 1 + 2b - 2c > 1$.
Case ii) $0 < a \le b < c < 1$. We take $a' = a, b' = b, c' = c$. When $c \ge a + b$ we get $\lambda = 1-c, \mu = c-a-b, \nu = b-a$ and the inequalities hold. Moreover, $\lambda+\mu+\nu = 1-2a < 1$. When $c \le a + b$ we get $\lambda = 1-c, \mu = a+b-c, \nu = b-a$ and the inequalities hold. Moreover, $\lambda + \mu + \nu = 1 + 2b - 2c < 1$.
Case iii) $0 \le c < a \le b < 1$. We take $a' = a, b' = b, c' = c+1$. Then, $\lambda = c, \mu = c+1-a-b, \nu = b-a$ and the inequalities are readily verified. Moreover, $\lambda+\mu+\nu = 1+2c-2a < 1$.
As to uniqueness we note that an integral shift in the a, b, c such that the corresponding values of λ, μ, ν stay below 1 necessarily gives the substitutions of the form $\lambda \to 1-\lambda, \mu \to$

$1-\mu, \nu \to \nu$ and similar ones where two of the parameters are replaced by 1 minus their value. In casethe condition $\lambda+\mu+\nu < 1+2\min(\lambda,\mu,\nu)$ is violated by such a substitution. For example, $\lambda+\mu+\nu \le 1$ implies $1-\lambda+1-\mu+\nu = 2-(\lambda+\mu+\nu)+2\nu \ge 1+2\nu$. In the spherical case the condition is not violated.

When we have obtained a geodesic Schwarz triangle in our construction we automatically have a metric which is invariant under the projective monodromy group. This closely reflects the nature of the natural hermitian form on the monodromy group itself.

Theorem 2.4.2 *Let $a, b, c \in \mathbb{R}$ be such that*

$$0 \le \lambda, \mu, \nu < 1 \qquad \lambda+\mu+\nu < 1+2\min(\lambda,\mu,\nu)$$

where $\lambda = |1-c|, \mu = |c-a-b|, \nu = |a-b|$. Let M be the monodromy group of (2.2). *Then,*

$$M \textit{ is spheric} \iff \lambda+\mu+\nu > 1$$

$$M \textit{ is euclidean} \iff \lambda+\mu+\nu = 1$$

$$M \textit{ is hyperbolic} \iff \lambda+\mu+\nu < 1.$$

Proof. In the case when none of the numbers $a, b, c-a, c-b$ is integral, this statement can already be inferred from the proof of the previous lemma (we get only the hyperbolic and spheric case). It remains to show that if one of the numbers $a, b, c-a, c-b$ is integral, we have $\lambda+\mu+\nu = 1$. Let us suppose for example that $a \in \mathbb{Z}$. Notice that $|a-b| < 1$ and $|a+b| < |c|+1 < 3$. Hence $|a| \le |a-b|/2 + |a+b|/2 < 2$. So, $a = 0, \pm 1$. A case by case analysis using the inequalities for λ, μ, ν yields our statement. □

Chapter 3

Generalised hypergeometric functions ${}_nF_{n-1}$

3.1 Definition, first properties

Let $\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n$ be any complex numbers and consider the generalised hypergeometric equation in one variable,

$$z(\theta+\alpha_1)\cdots(\theta+\alpha_n)F = (\theta+\beta_1-1)\cdots(\theta+\beta_n-1)F, \qquad \theta = z\frac{d}{dz} \tag{3.1}$$

This is a Fuchsian equation of order n with singularities at $0, 1, \infty$. The local exponents read,

$$\begin{array}{ll} 1-\beta_1, \dots, 1-\beta_n & \text{at } z=0 \\ \alpha_1, \dots, \alpha_n & \text{at } z=\infty \\ 0, 1, \dots, n-2, \ -1+\sum_1^n(\beta_i-\alpha_i) & \text{at } z=1 \end{array}$$

When the β_i are distinct modulo 1 a basis of solutions at $z=0$ is given by the functions

$$z^{1-\beta_i}\ {}_nF_{n-1}\left(\left.\begin{array}{c}\alpha_1-\beta_i+1, \dots, \alpha_n-\beta_i+1 \\ \beta_1-\beta_i+1, ..\overset{\vee}{}.., \beta_n-\beta_i+1\end{array}\right| z\right) \qquad (i=1,\dots,n).$$

Here $..^{\vee}..$ denotes suppression of the term $\beta_i-\beta_i+1$ and ${}_nF_{n-1}$ stands for the generalised hypergeometric function in one variable

$${}_nF_{n-1}\left(\left.\begin{array}{c}\alpha_1, \dots, \alpha_n \\ \beta_1, \dots, \beta_{n-1}\end{array}\right| z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_{n-1})_k k!} z^k.$$

At $z=1$ we have the following interesting situation.

Theorem 3.1.1 (Pochhammer) *The equation* (2.2) *has* $n-1$ *independent holomorphic solutions near* $z=1$.

The proof of this result follows from the observation that the coefficient of $\left(\frac{d}{dz}\right)^n$ in (2.2) equals $z^{n+1}-z^n$ and the following theorem.

Theorem 3.1.2 *Consider the linear differential equation*

$$p_n(z)y^{(n)} + p_{n-1}(z)y^{(n-1)} + \cdots + p_1(z)y' + p_0(z)y = 0$$

where the $p_i(z)$ are analytic around a point $z = a$. Suppose that $p_n(z)$ has a zero of order one at $z = a$. The the differential equations has $n-1$ independent holomorphic solutions around $z = a$.

Proof. Without loss of generality we can assume that $a = 0$ and $p_n(z) = z$. Then we determine a power series solution $\sum_{k\ge 0} f_k z^k$ by substituting it into the equation. We obtain a recursion relation of the following form,

$$k\cdots(k-n+2)(k-n+1+p_{n-1}(0))f_k = \sum_{i=1}^{\infty}(k-i)\cdots(k-n+2)c_i(k)f_{k-i} \qquad \text{(R)}$$

where $c_i(k)$ are polynomials in k of degree $< n$, the product $(k-i)\cdots(k-n+2)$ is considered 1 if $i > n-2$ and f_m is considered 0 if $m < 0$. With these conventions we see that (R) holds for $k = 0, 1, \ldots, n-1$ independent of the choice of $f_0, f_1, \ldots, f_{n-2}$. When $p_{n-1}(0) \notin \mathbb{Z}_{\le 0}$ we see that recursion (R) allows us to determine f_k for $k \ge n-1$ in a unique manner. In this way we find $n-1$ independent holomorphic solutions. When $p_{n-1}(0) \in \mathbb{Z}_{\le 0}$ we need to refine our argument slightly in the sense that there is a linear relation between $f_0, \ldots, f_{n-2}$ and $f_{n-1-p_{n-1}(0)}$ can be chosen arbitrarily. $\square$

Finally we mention the Euler integral for ${}_nF_{n-1}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_{n-1}|z)$,

$$\prod_{i=1}^{n-1}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}\int_0^1\cdots\int_0^1 \frac{\prod_{i=1}^{n-1} t_i^{\alpha_i-1}(1-t_i)^{\beta_i-\alpha_i-1}}{(1-zt_1\cdots t_{n-1})^{\alpha_n}}dt_1\cdots dt_{n-1}$$

for all $\Re\beta_i > \Re\alpha_i > 0$ $(i = 1, \ldots, n-1)$.

3.2 Monodromy

Fix a base point $z_0 \in \mathbb{P}^1 - \{0, 1, \infty\}$, say $z_0 = 1/2$. Denote by G the fundamental group $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$. Clearly G is generated by the simple loops g_0, g_1, g_∞ around the corresponding points together with the relation $g_0 g_1 g_\infty = 1$. Let $V(\alpha, \beta) = V(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ be the local solution space of (2.2) around z_0. Denote by

$$M(\alpha, \beta): \ G \to GL(V(\alpha, \beta))$$

the monodromy representation of (2.2). Write

$$h_0 = M(\alpha,\beta)g_0 \qquad h_1 = M(\alpha,\beta)g_1 \qquad h_\infty = M(\alpha,\beta)g_\infty.$$

The eigenvalues of h_0 and h_∞ read $\exp(-2\pi i\beta_j)$ and $\exp(2\pi i\alpha_j)$ respectively. Since there are $n-1$ independent holomorphic solutions near $z = 1$ the element h_1 has $n-1$ eigenvalues

1 together with $n-1$ independent eigenvectors. Equivalently, $\mathrm{rank}(h_1 - Id) \le 1$. An element $h \in GL(V)$ such that $\mathrm{rank}(h - Id) = 1$ will be called a *(pseudo)-reflection*. The determinant of a reflection will be called the *special eigenvalue*. From the relation between the generators of the fundamental group it follows that $h_1^{-1} = h_\infty h_0$ is a (pseudo)reflection.

Theorem 3.2.1 *Let $H \subset GL(n,\mathbb{C})$ be a subgroup generated by two matrices A, B such that AB^{-1} is a reflection. Then H acts irreducibly on $\mathbb{C}^n$ if and only if A and B have disjoint sets of eigenvalues.*

Proof. Suppose that H acts reducibly. Let V_1 be a nontrivial invariant subspace and let $V_2 = \mathbb{C}^n/V_1$. Since $A - B$ has rank 1, A and B coincide on either V_1 or V_2. Hence they have a common eigenvalue.
Suppose conversely that A and B have a common eigenvalue λ. Let $W = \ker(A - B)$. Since $AB^{-1} - Id$ has rank one, the same holds for $A - B$. Hence $\dim(W) = n - 1$. If any eigenvector of A belongs to W, it must also be an eigenvector of B, since A and B coincide on W. Hence there is a one-dimensional invariant subspace. Suppose W does not contain any eigenvector of A or B. We show that the subspace $U = (A-\lambda)\mathbb{C}^n$ is invariant under H. Note that $A - \lambda Id$ has a non-trivial kernel which has trivial intersection with W. Hence U has dimension $n-1$ and $U = (A-\lambda)W$. Since $A - \lambda$ and $B - \lambda$ coincide on W we conclude that also $U = (B-\lambda)W$ and hence, by a similar argument as for A, $U = (B-\lambda)\mathbb{C}^n$. Notice that U is stable under A, as follows trivially from $A(A-\lambda)\mathbb{C}^n = (A-\lambda)A\mathbb{C}^n = (A-\lambda)\mathbb{C}^n$. For a similar reason U is stable under B. Hence H has the invariant subspace U. $\square$

Corollary 3.2.2 *The monodromy group of* (2.2) *acts irreducibly if and only if all differences $\alpha_i - \beta_j$ are non-integral.*

This Corollary follows by application of our Theorem with $A = h_\infty$ and $B = h_0^{-1}$.
From now on we shall be interested in the irreducible case only.

Theorem 3.2.3 (Levelt) *Let $a_1, \ldots, a_n; b_1, \ldots, b_n \in \mathbb{C}^*$ be such that $a_i \neq b_j$ for all i, j. Then there exist $A, B \in GL(n,\mathbb{C})$ with eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively such that AB^{-1} is a reflection. Moreover, the pair A, B is uniquely determined up to conjugation.*

Proof. First we show the existence. Let

$$\prod_i (X - a_i) = X^n + A_1 X^{n-1} + \cdots + A_n$$

$$\prod_i (X - b_i) = X^n + B_1 X^{n-1} + \cdots + B_n$$

and

$$A=\begin{pmatrix}0 & 0 & \dots & 0 & -A_n\\ 1 & 0 & \dots & 0 & -A_{n-1}\\ 0 & 1 & \dots & 0 & -A_{n-2}\\ \vdots & & & & \vdots\\ 0 & 0 & \dots & 1 & -A_1\end{pmatrix}\qquad B=\begin{pmatrix}0 & 0 & \dots & 0 & -B_n\\ 1 & 0 & \dots & 0 & -B_{n-1}\\ 0 & 1 & \dots & 0 & -B_{n-2}\\ \vdots & & & & \vdots\\ 0 & 0 & \dots & 1 & -B_1\end{pmatrix}$$

Then $\mathrm{rank}(A-B)=1$, hence $\mathrm{rank}(AB^{-1}-Id)=1$ and AB^{-1} is a reflection.
To prove uniqueness of A,B we let $W=\ker(A-B)$. Note that $\dim W=n-1$. Let $V=W\cap A^{-1}W\cap\cdots\cap A^{-(n-2)}W$. Then $\dim V\geq 1$. Suppose $\dim V>1$. Choose $v\in V\cap A^{-(n-1)}W$. Then $A^iv\in W$ for $i=0,1,\dots,n-1$. Hence $U=<A^iv>_{i\in\mathbb{Z}}\subset W$ is A-stable. In particular, W contains an eigenvector of A. Since $B=A$ on W this is also an eigenvector of B with the same eigenvalue, contradicting our assumption on A,B. Hence $\dim V=1$. Letting $v\in V$ we take $v,Av,\dots,A^{n-1}v$ as basis of $\mathbb{C}^n$. Since $A=B$ on W we have that $A^iv=B^iv$ for $i=0,1,\dots,n-2$ and with respect to this basis A and B have automatically the form given above. □

Corollary 3.2.4 *With the same hypotheses and A_i,B_j as in the proof of the previous theorem we have that $<A,B>$ can be described by matrices having elements in $\mathbb{Z}[A_i,B_j,1/A_n,1/B_n]$.*

Levelt's theorem is a special case of a general rigidity theorem which has recently been proved by N.M.Katz. In the last section we shall give an elementary proof of Katz's theorem.

3.3 Hypergeometric groups

Definition 3.3.1 *Let $a_1,\dots,a_n;b_1,\dots,b_n\in\mathbb{C}^*$. such that $a_i\neq b_j$ for every i,j. The group generated by A,B such that A and B have eigenvalues $a_1,\dots,a_n$ and $b_1,\dots,b_n$ respectively and such that AB^{-1} is a pseudoreflection, will be called a hypergeometric group with parameters a_i and b_j. Notation: $H(a,b)=a_1,\dots,a_n;b_1,\dots,b_n$.*

In particular, the monodromy group of (2.2) is a hypergeometric group with $a_k=e^{2\pi i\alpha_k}$ and $b_k=e^{2\pi i\beta_k}$.

Theorem 3.3.2 *Let H be a hypergeometric group with parameters $a_1,\dots,a_n$ and $b_1,\dots,b_n$. Suppose that these parameters lie on the unit circle in $\mathbb{C}$. Then there exists a non-degenerate hermitean form $F(x,y)=\sum F_{ij}x_iy_j$ on $\mathbb{C}^n$ such that $F(hx,hy)=F(x,y)$ for all $h\in H$ and all $x,y\in\mathbb{C}^n$.*
Denote by $\prec,\preceq$ the total ordering on the unit circle corresponding to increasing argument. Assume that the $a_1\preceq\dots\preceq a_n$ and $b_1\preceq\dots\preceq b_n$. Let $m_j=\#\{k|b_k\prec a_j\}$ for $j=1,\dots,n$. Then the signature (p,q) of the hermitean form F is given by

$$|p-q|=\left|\sum_{j=1}^n(-1)^{j+m_j}\right|.$$

Definition 3.3.3 *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be sets on the unit circle. We say that these sets interlace on the unit circle if and only if either*

$$a_1 \prec b_1 \prec a_2 \prec b_2 \cdots \prec a_n \prec b_n$$

or

$$b_1 \prec a_1 \prec b_2 \prec a_2 \cdots \prec b_n \prec a_n.$$

Corollary 3.3.4 *Let the hypergeometric group H have all of its parameters on the unit circle. Then H is contained in $U(n, \mathbb{C})$ if and only if the parametersets interlace on the unit circle.*

Theorem 3.3.5 *Suppose the parameters $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are roots of unity, let us say h-th roots of unity for some $h \in \mathbb{Z}_{\geq 2}$. Then the hypergeometric group $H(a, b)$ is finite if and only if for each $k \in \mathbb{Z}$ with $(h, k) = 1$ the sets $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace on the unit circle.*

Proof. The Galois group of $\mathbb{Q}(\exp(2\pi i/h))$ over $\mathbb{Q}$ is given by elements of the form

$$\sigma_k : \exp(2\pi i/h) \to \exp(2\pi i k/h)$$

for any $k, (k, h) = 1$. The group $H(a, b)$ can be represented by matrices with entries in the ring of cyclotomic integers $\mathbb{Z}[\exp(2\pi i/h)]$. The Galois automorphsim σ_k establishes an isomorphism between $H(a, b)$ and the hypergeometric group H_k with parameters $a_1^k, \ldots, a_n^k$, $b_1^k, \ldots, b_n^k$. Each group H_k has an invariant hermitian form F_k for $(k, h) = 1$.
Suppose $H(a, b)$ is finite. Then each F_k is definite, hence every pair of sets $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace on the unit circle.
Suppose conversely that $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace for every $k, (k, h) = 1$. Then each group H_k is subgroup of a unitary group with definite form F_k. In particular the entries of each element are bounded in absolute value by some constant, C say. This implies that any entry of any element of $H(a, b)$ has conjugates which are all bounded by C. Since there exist only finitely many elements of $\mathbb{Z}[\exp(2\pi i/h)]$ having this property, we conclude the finiteness of $H(a, b)$. $\square$

An immediate consequence of this theorem is that, for example, the hypergeometric function

$${}_8F_7\left(\begin{array}{c} 1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30 \\ 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8 \end{array} \middle| z\right)$$

is an algebraic function.

3.4 Rigidity

In this section we formulate and prove Katz's result on rigidity, see [Ka]. Let k be a field and $g_1, g_2, \ldots, g_r \in GL(n, k)$ Let G be the group generated by $g_1, \ldots, g_r$. We say that the

r-tuple is *irreducible* if the group G acts irreducibly on k^n. We call the r-tuple $g_1, \ldots, g_r$ *linearly rigid* if for any conjugates $\tilde{g}_1, \ldots, \tilde{h}_r$ of $g_1, \ldots, g_r$ with $\tilde{g}_1\tilde{g}_2 \cdots \tilde{g}_r = Id$ there exists $u \in GL(n, k)$ such that $\tilde{g}_i = ug_iu^{-1}$ for $i = 1, 2, \ldots, r$.
For example, it follows from Levelt's theorem that the generators $g_1 = A, g_2 = B^{-1}, g_3 = BA^{-1}$ of a hypergeometric group form a linearly rigid system.

Theorem 3.4.1 (Katz) *Let $g_1, g_2, \ldots, g_r \in GL(n, k)$ be an irreducible r-tuple with $g_1g_2 \ldots g_r = Id$. Let, for each i, δ_i be the codimension of the linear space $\{A \in M_n(k) | g_iA = Ag_i\}$ (centralizer of g_i). Then,*

i) $\delta_1 + \cdots + \delta_r \geq 2(n^2 - 1)$

ii) If $\delta_1 + \cdots + \delta_r = 2(n^2 - 1)$, the system is linearly rigid.

iii) If k is algebraically closed, then the converse of part ii) holds

We note that the centraliser of $g \in GL(n, k)$ depends only on the Jordan normal form of g. If g is diagonalisable, the dimension of the centraliser is the sum of the squares of the dimensions of the eigenspaces of g. When g has distinct eigenvalues this dimension is n, when g is a (pseudo)reflection this dimension is $(n - 1)^2 + 1 = n^2 - 2n + 2$. The corresponding codimensions are $n^2 - n$ and $2n - 2$ respectively.
By way of example consider a hypergeometric group generated by $g_1 = A, g_2 = B^{-1}, g_3 = BA^{-1}$. In general A and B each have distinct eigenvalues, so $\delta_1 = \delta_2 = n^2 - n$. Since g_3 is a (pseudo)reflection we have $\delta_3 = 2n - 2$. Notice that $\delta_1 + \delta_2 + \delta_3 = 2n^2 - 2$. Hence the triple A, B^{-1}, BA^{-1} is linearly rigid. As a bonus we get that the eigenspaces of A and B all have dimension one. Hence to each eigenvalue there is precisely one Jordan block in the Jordan normal form.
Another example comes from the Jordan-Pochhammer equation, which is an n-th order Fuchsian equation with $n + 1$ singular points and around each singular point the local monodromy is (up to a scalar) a pseudo-reflection. So for each singularity we have $\delta_i = 2n - 2$. The sum of these delta's is of course $2(n^2 - 1)$. So if the monodromy is irreducible we have again a rigid system. This case has been elaborated by [Ha].
The proof of Katz's theorem is based on the following Theorem from linear algebra. In this Theorem we consider a group G acting on a finite dimensional linear space V. For every $X \subset G$ we denote by $d(X)$ resp. $d^*(X)$ the codimension of the common fixed point space in V resp. V^*, the dual of V, of all elements of X.

Theorem 3.4.2 (L.Scott) *Let $H \in GL(V)$ be the group generated by $h_1, h_2, \ldots, h_r$ with $h_1h_2 \cdots h_r = \mathrm{Id}$. Then*

$$d(h_1) + d(h_2) + \cdots + d(h_r) \geq d(H) + d^*(H).$$

Proof. Let W be the direct sum $\oplus_{i=1}^r (1 - h_i)V$. Define the linear map $\beta : V \to W$ by

$$\beta : v \mapsto ((1 - h_1)v, \ldots, (1 - h_r)v).$$

Define the linear map $\delta : W \to V$ by

$$\delta : (v_1, \ldots, v_r) \mapsto v_1 + h_1 v_2 + h_1 h_2 v_3 + \cdots + h_1 \cdots h_{r-1} v_r$$

Because of the identity

$$1 - h_1 h_2 \cdots h_r = (1 - h_1) + h_1(1 - h_2) + \cdots + h_1 \cdots h_{r-1}(1 - h_r)$$

we see that the image of β is contained in the kernel of δ. Hence $\dim(\Im\beta) \le \dim(\ker\delta)$. Moreover, the kernel of β is precisely $\cap_{i=1}^r \ker(1 - h_i)$. The dimension of the latter space equals $n - d(H)$. Hence $\dim(\Im\beta) = n - (n - d(H)) = d(H)$.
The image of δ is

$$(1 - h_1)V + h_1(1 - h_2)V + \cdots + h_1 \cdots h_{r-1}(1 - h_r)V$$

which is equal to $(1-h_1)V + (1-h_2)V + \cdots + (1-h_r)V$. Note that any $w \in \cap_{i=1}^r \ker(1-h_i^*)$ in the dual space V^* we vanishes on $\Im\delta$. Hence $\dim(\Im\delta) \ge d^*(H)$.
Finally notice that $\dim(W) = \sum_{i=1}^r d(h_i)$. Putting everything together we get

$$\begin{aligned} \sum_{i=1}^r d(h_i) = \dim(W) &= \dim(\ker\delta) + \dim(\Im\delta) \\ &\ge \dim(\Im\beta) + \dim(\Im\delta) \\ &\ge d(H) + d^*(H) \end{aligned}$$

This is precisely the desired inequality. □

Proof of Katz's theorem. We follow the approach of Völklein-Strambach [VS]. For the first part of Katz's theorem we apply Scott's Theorem to the vector space of $n \times n$-matrices and the group generated by the maps $h_i : A \mapsto g_i^{-1} A g_i$. Notice that $d(h_i)$ is now precisely the codimension of the centraliser of g_i, hence $d(h_i) = \delta_i$ for all i. The number $d(H)$ is precisely the codimension of the space $\{A \in M_n(k) | gA = Ag \text{ for all } g \in G\}$. By Schur's Lemma the irreducibility of the action of G implies that the dimension of this space is 1 and the codimension $n^2 - 1$. So $d(H) = n^2 - 1$. To determine $d^*(H)$ we note that the matrix space $V = M_n(k)$ is isomorphic to its dual via the map $V \to V^*$ given by $A \mapsto (X \mapsto \mathrm{Trace}(AX)$. Let us identify V with V^* in this way. Since $\mathrm{Trace}(Ag^{-1}Xg) = \mathrm{Trace}(gAg^{-1}X)$ we see that the action of g on the dual space is given by $A \mapsto gAg^{-1}$. hence $d^*(H) = n^2 - 1$. Application of Scott's Theorem now shows that

$$\delta_1 + \cdots + \delta_2 \ge d(H) + d^*(H) = 2(n^2 - 1)$$

To prove the second part of the theorem we apply Scott's Theorem with $V = M_n(k)$ again, but now with the maps $h_i : A \mapsto g_i^{-1} A \tilde{g}_i$. For each i choose $u_i \in GL(n, k)$ such that $\tilde{g}_i = u_i g_i u_i^{-1}$. Now note that

$$\begin{aligned} d(h_i) &= \mathrm{codim}\{A|g_i^{-1}A\tilde{g}_i = A\} \\ &= \mathrm{codim}\{A|A\tilde{g}_i = g_iA\} \\ &= \mathrm{codim}\{A|Au_ig_iu_i^{-1} = g_iA\} \\ &= \mathrm{codim}\{A|(Au_i)g_i = g_i(Au_i)\} \\ &= \mathrm{codim}\{A|Ag_i = g_iA\} = \delta_i \end{aligned}$$

The sum of the δ_i is given to be $2(n^2-1)$. Together with Scott's Theorem this implies $d(H) + d^*(H) \le 2(n^2-1)$. This means that either $d(H) < n^2$ or $d^*(H) < n^2$ or both. Let us assume $d(H) < n^2$, the other case being similar. Then there is a non-trivial $n \times n$ matrix A such that $A\tilde{g}_i = g_iA$ for all i. From these inequalities we see in particular that the image of A is stable under the group generated by the g_i. Since the r-tuple $g_1, \ldots, g_r$ is irreducible this means that $A(k^n)$ is either trivial or k^n itself. Because A is non-trivial we conclude that $A(k^n) = k^n$ and A is invertible. We thus conclude that $\tilde{g}_i = A^{-1}g_iA$ for all i. In other words, our system $g_1, \ldots, g_r$ is rigid.
The proof of part iii) uses a dimension argument. Let C_i be the conjugacy class of g_i $i = 1, 2, \ldots, r$. Consider the multiplication map $\Pi : C_1 \times C_2 \times \cdots \times C_r \to GL(n,k)$ given by $(c_1, c-2, \ldots, c_r) \mapsto c_1c_2\cdots c_r$. We have

$$\dim(C_1 \times \cdots \times C_r) \le \dim(\Pi^{-1}(\mathrm{Id}) + \dim(\Im\Pi)$$

First of all note that $\dim(C_1 \times \cdots \times C_r) = \sum_{i=1}^r \dim(C_i) = \sum_{i=1}^r \delta_i$. Secondly, by the rigidity and irreducibility assumptions we have $\dim(\Pi^{-1}(\mathrm{Id})) = n^2 - 1$. Finally, $\Im\Pi$ is contained in the hypersuface of all matrices whose determinant is $\det(g_1g_2\cdots g_r) = 1$. Hence $\dim(\Im\Pi) \le n^2 - 1$.
These three facts imply that $\sum_{i=1}^r \delta_i \le 2(n^2-1)$. Together with part i) this implies the desired equality. □

In many practical situations the local monodromies of differential equations have eigenvalues which are complex numbers with absolute value 1. In that case there exists also a monodromy invariant Hermitian form on the solution space. We formulate this as a Lemma.

Lemma 3.4.3 *Let $g_1, g_2, \ldots, g_r \in GL(n,\mathbb{C})$ be a rigid, irreducible system with $g_1g_2\cdots g_r =$ Id. Suppose that for each i the matrices g_i and $\tilde{g}_i = (\overline{g}_i^t)^{-1}$ are conjugate. Then there exists a non-trivial matrix $H \in M_n(\mathbb{C})$ such that $\overline{g}_i^t H g_i = H$ for each i and $\overline{H}^t = H$.*

Proof. Notice that, $\tilde{g}_1 \cdots \tilde{g}_r = \mathrm{Id}$. Moreover, the g_i and $\tilde{g}_i$ are conjugate so by rigidity there exists a matrix $H \in GL(n,\mathbb{C})$ such that $\tilde{g}_i = Hg_iH^{-1}$ for all i. Hence $H = \overline{g}_i^t H g_i$ for all i. Moreover, since the system $g_1, \ldots, g_r$ is irreducible, the matrix H is uniquely determined up to a scalar factor. Since $\overline{H}^t$ is also a solution we see that $\overline{H}^t = \lambda H$ for some $\lambda \in \mathbb{C}$. Moreover $|\lambda| = 1$ and writing $\lambda = \mu/\overline{\mu}$ we see that μH is a Hermitian matrix. Now take $H := \mu H$. □

Chapter 4

Explicit monodromy for hypergeometric equations

4.1 Introduction

Notice that the proof of Levelt's Theorem **??** provides us a very explicit construction of the monodromy matrices of the hypergeometric equation. However, the basis with respect to which these monodromy matrices occur do appear in the proof. So Levelt's Theorem gives us only a determination of the monodromy group up to conjugation. In many applications it is desirable to have the explicit matrices with respect to an explicitly given basis of solutions. This is precisely the purpose of this chapter.
We consider the hypergeometric equation

$$z(\theta+\alpha_1)\cdots(\theta+\alpha_n)F = (\theta+\beta_1-1)\cdots(\theta+\beta_n-1)F, \qquad \theta = z\frac{d}{dz}. \tag{4.1}$$

We consider it in the complex plane with the positive real axis deleted. That is, all complex z with $|\arg(-z)| < \pi$. We fix a basis of local solutions at $z=0$ and at $z=\infty$. We continue the local solutions around 0 analytically in $\mathbb{C}-\mathbb{R}_{\geq 0}$ to ∞ and compare the continued solutions with the local solutions at ∞. The coefficients that occur enable us to compute the desired monodromy matrices.
To simplify matters we assume that the local bases of solutions do not contain logarithms, that is the parameters α_i are distinct modulo 1 and the parameters β_i are distinct modulo 1. A solution basis around $z=0$ can then be given by

$$(-z)^{1-\beta_i}\ {}_nF_{n-1}\left(\begin{matrix}\alpha_1-\beta_i+1,\ldots,\alpha_n-\beta_i+1\\ \beta_1-\beta_i+1,..^{\vee}..,\beta_n-\beta_i+1\end{matrix}\middle|\, z\right) \qquad (i=1,\ldots,n)$$

where the sign $^{\vee}$ denotes suppression of $\beta_i-\beta_i+1$. To make the final formulas more elegant we multiply this solution with the constant

$$\frac{\Gamma(\alpha_1-\beta_i+1)\cdots\Gamma(\alpha_n-\beta_i+1)}{\Gamma(\beta_1-\beta_i+1)\cdots\Gamma(\beta_n-\beta_i+1)}$$

and get the solution

$$F_i := (\epsilon z)^{1-\beta_i} \sum_{k\ge 0} \frac{\Gamma(\alpha_1-\beta_i+k+1)\cdots\Gamma(\alpha_n-\beta_i+k+1)}{\Gamma(\beta_1-\beta_i+k+1)\cdots\Gamma(\beta_n-\beta_i+k+1)}\ z^k$$

where we introduced the extra factor $\epsilon=(-1)^{n-1}$ for reasons that will become clear later. We also agree that for the determination of $(\epsilon z)^{1-\beta_i}$ we choose $-(n+2)\pi < \arg(\epsilon z) < -n\pi$. Similarly, around $z=\infty$ a basis can be given by

$$G_i = (\epsilon z)^{-\alpha_i} \sum_{k\ge 0} \frac{\Gamma(\alpha_i-\beta_1+k+1)\cdots\Gamma(\alpha_i-\beta_n+k+1)}{\Gamma(\alpha_i-\alpha_1+k+1)\cdots\Gamma(\alpha_i-\alpha_n+k+1)}\ (1/z)^k$$

In order to determine the connection between these solution sets we propose to use the technique of Mellin-Barnes integrals.

4.2 Mellin-Barnes integrals

In [GM] Golyshev anf Mellit describe a way to determine explicit monodromy of hypergeometric functions by studying Fourier transforms of products of factors of the form $1/\Gamma(\gamma\pm s)$. In this section we adopt an approach inspired by them and which gives precisely the same formulas, namely Mellin-Branes type integrals of products of factors $\Gamma(\gamma\pm s)$.

Let $\alpha_i,\beta_j\in\mathbb{C}$ for $i,j=1,2,\ldots,n$ and suppose from now on that the α_i,β_j are all distinct modulo 1. This means we can write explicit solution bases for the hypergeometric equations as in the previous section and, by Corollary **??**, the monodromy representation is irreducible. Let Γ be a path in the complex plane from $i\infty$ to $-i\infty$ and which bends in such a way that all points $-\alpha_i-k$ ($i=1,\ldots,n$, $k\in\mathbb{Z}_{\ge 0}$) are on the left of Γ and all points $-\beta_i+k+1$ ($i=1,\ldots,n$, $k\in\mathbb{Z}_{\ge 0}$) are on the right of Γ. Let $i\in\{1,\ldots,n\}$ and consider the integral

$$I=\frac{1}{2\pi i}\int_\Gamma \Gamma(\alpha_1+s)\cdots\Gamma(\alpha_n+s)\Gamma(1-\beta_1-s)\cdots\Gamma(1-\beta_n-s)\ (\epsilon z)^s ds$$

where $\epsilon=(-1)^{n-1}$.

From Stirling's formula (see [AAR, p21]) it follows that when $s=a+bi$ and $a_1<a<a_2$ and $|b|\to\infty$ that

$$|\Gamma(a+bi)|=\sqrt{2\pi}|b|^{a-1/2}e^{-\pi|b|/2}[1+O(1/|b|)].$$

Notice also that $|(-z)^{a+bi}|=|(\epsilon z)^a|e^{-b\arg(\epsilon z)}|$ for all $a,b\in\mathbb{R}$. Putting these estimates together we see that the integral converges absolutely for all $z\in\mathbb{C}$ with $|\arg(\epsilon z)|<n\pi$. The interval $(-n\pi,n\pi)$ can be subdivided into n intervals of the form $((2r-n-2)\pi,(2r-n)\pi)$. So, depending on the choice of determination of $(\epsilon z)^s$ the integral I represents n functions I_r on $\mathbb{C}\setminus\mathbb{R}_{\ge 0}$ indexed by r. We now compute I_r as a power series in z. So we

assume $|z| < 1$ and $(2r - n - 2)\pi < \arg(\epsilon z) < (2r - n)\pi$. For any real a we denote by Γ_a the vertical path from $a + i\infty$ to $a - i\infty$. Let a_0 be larger than all real parts of the $-\alpha_i$. We deform the contour Γ to Γ_{a_0} such that all points $-\alpha_i - k$ with $k \in \mathbb{Z}_{\geq 0}$ stay on the left. Then we continue to shift Γ_{a_0} to the right via the paths Γ_a with $a \to \infty$. In the process the deformed paths may pass through a pole of $\Gamma(1 - \beta_1 - s) \cdots \Gamma(1 - \beta_n - s)$ and no others. That is, the points $1 - \beta_i, 2 - \beta_i, \ldots$ for $i = 1, \ldots, n$. The residue of the integrand of I_r at the pole $s = k + 1 - \beta_i$ equals

$$(\epsilon z)^k (\epsilon z)^{1-\beta_i} \Gamma(\alpha_1 - \beta_i + k + 1) \cdots \Gamma(\alpha_n - \beta_i + k + 1) \Gamma(-\beta_1 + \beta_i - k) \cdots \Gamma(-\beta_n + \beta_i - k)$$

where the factor $\Gamma(-\beta_i + \beta_i - k)$ is to be read as $1/k!$. Once again we apply the identity $\Gamma(x)\Gamma(1 - x) = \sin \pi x$ to obtain

$$(\epsilon z)^k (\epsilon z)^{1-\beta_i} \prod_{l=1}^{n} \frac{\Gamma(\alpha_l - \beta_i + k + 1)}{\Gamma(\beta_l - \beta_i + k + 1)} \prod_{l \neq i} \frac{\pi}{\sin \pi(-\beta_l + \beta_i - k)}$$

where the factor $\sin \pi(-\beta_i + \beta_i - k)$ is omitted. So we get

$$I_r = I_{i,a} + \sum_{i=1}^{n} (\epsilon z)^{1-\beta_i} \prod_{l \neq i} \frac{\pi}{\sin \pi(-\beta_l + \beta_i)} \sum_k \prod_{l=1}^{n} \frac{\Gamma(\alpha_l - \beta_i + k + 1)}{\Gamma(\beta_l - \beta_i + k + 1)} z^k$$

where the summation is over $k = 0, 1, 2, \ldots \lfloor a + \Re(\beta_i) \rfloor$ and $I_{i,a}$ denotes integration over Γ_a. Finally we note that $|I_{i,a}| \to 0$ as $a \to \infty$, simply because $|z| < 1$ and so $|(-z)^{a+bi}|$ decreases exponentially in a as $a \to \infty$. Therefore we conclude that

$$I_r = \pi^{n-1} \sum_{i=1}^{n} e^{-2\pi i r \beta_i} \frac{F_i}{\prod_{l \neq i} \sin \pi(-\beta_l + \beta_i)} \qquad r = 1, 2, \ldots, n.$$

We now compute I_r as a power zeries in $1/z$. So we assume $|z| > 1$ and $(2r - n - 2)\pi < \arg(\epsilon z) < (2r - n)\pi$. Let a_0 be a real number smaller than all real parts of the $-\beta_i$. We now deform the contour Γ to Γ_{a_0} while keeping all points $-\beta_i + k + 1$ with $k \in \mathbb{Z}_{\geq 0}$ on the right. From then on we shift Γ_{a_0} to Γ_a where we let $a \to -\infty$. In the process the deformed paths may pass through the poles of $\Gamma(s + \alpha_1) \cdots \Gamma(s + \alpha_n)$ and no others. That is, the points $-\alpha_j - k$ with $k \in \mathbb{Z}_{\geq 0}$. Use the fact that the residue of $\Gamma(x)$ at $x = k$ with $k \leq 0$ is given by $(-1)^k/k!$. We obtain that the residue of the integrand equals

$$(\epsilon z)^{-k} (\epsilon z)^{-\alpha_j} \Gamma(\alpha_1 - \alpha_j - k) \cdots \Gamma(\alpha_n - \alpha_j - k) \Gamma(-\beta_1 + \alpha_j + k + 1) \cdots \Gamma(-\beta_n + \alpha_j + k + 1)$$

where the factor $\Gamma(\alpha_j - \alpha_j - k)$ is to be read as $1/k!$. We use the identity $\Gamma(x)\Gamma(1 - x) = \pi / \sin \pi x$ once again. We get

$$(\epsilon)^{-k} (\epsilon z)^{-\alpha_j} \prod_{l=1}^{n} \frac{\Gamma(\alpha_j - \beta_1 + k + 1)}{\Gamma(\alpha_j - \alpha_1 + k + 1)} \prod_{l \neq i} \frac{\pi}{\sin(\pi(\alpha_l - \alpha_j - k))}.$$

The integral over Γ_a tends to zero as $a \to -\infty$ because $|z| > 1$ and $|(-z)^{a+bi}|$ tends exponentially to 0 as $a \to -\infty$. Hence we conclude that for $|z| > 1$ we have

$$I_r = \pi^{n-1} \sum_{j=1}^{n} e^{-2\pi i r \alpha_j} \frac{G_j}{\prod_{l \neq j} \sin(\pi(\alpha_l - \alpha_j))}, r = 1, 2, \ldots, n.$$

There is an interesting consequence.

Corollary 4.2.1 *Let notation be as above. With respect to the basis of solutions $I_1, \ldots, I_n$ the monodromy matrix around $z = 0$ reads*

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -B_n & -B_{n-1} & -B_{n-2} & \cdots & -B_1 \end{pmatrix}$$

where $X^n + B_1 X^{n-1} + \cdots + B_{n-1} X + B_n$ is the polynomial with zeros $e^{-2\pi i \beta_k}$, $k = 1, \ldots, n$. Similarly, around the point $z = \infty$ the monodromy matrix with respect to $\mathbf{I}$ reads

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -A_n & -A_{n-1} & -A_{n-2} & \cdots & -A_1 \end{pmatrix}$$

where $X^n + A_1 X^{n-1} + \cdots + A_{n-1} X + A_n$ is the polynomial with zeros $e^{-2\pi i \alpha_k}$, $k = 1, \ldots, n$.

So we see that we have found an explicit basis of solutions of the hypergeometric equation with respect to which the monodromy has the shape given in Levelt's Theorem **??**.

Proof. Let us denote $\mathbf{I} = (I_1, \ldots, I_n)^t$ and denote by $\mathbf{F}$ the vector with coordinates

$$\frac{\pi^{n-1} F_i}{\prod_{l \neq i} \sin \pi(-\beta_l + \beta_i)}.$$

We have seen above that $\mathbf{I} = M_\beta \mathbf{F}$ where M_β is the VanderMonde type matrix

$$\begin{pmatrix} e^{-2\pi i \beta_1} & e^{-2\pi i \beta_2} & \cdots & e^{-2\pi i \beta_n} \\ e^{-4\pi i \beta_1} & e^{-4\pi i \beta_2} & \cdots & e^{-4\pi i \beta_n} \\ \vdots & \vdots & & \vdots \\ e^{-2n\pi i \beta_1} & e^{-2n\pi i \beta_2} & \cdots & e^{-2n\pi i \beta_n} \end{pmatrix}.$$

A closed loop around $z = 0$ in positive direction gives the local monodromy $\mathbf{F} \to D_\beta \mathbf{F}$ where D_β is the diagonal matrix with entries $e^{-2\pi i\beta_j}$. Hence the solutions $\mathbf{I}$ goes over into $M_\beta I_\beta \mathbf{F} = M_\beta I_\beta M_\beta^{-1}\mathbf{I}$. The local monodromy matrix with respect to $\mathbf{I}$ reads

$$M_\beta I_\beta M_\beta^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -B_n & -B_{n-1} & -B_{n-2} & \cdots & -B_1 \end{pmatrix}.$$

The calculation around $z = \infty$ runs similarly.

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