Homework problem VI
Elementary Number theory 2009

1. Do problem 7.5.2 of the course notes.

2. In this problem we analyse which integers n can be written as the *difference* between two squares. In other words, for which integers n is $x^2 - y^2 = n$ solvable in integers x, y. (Hint to use: $x^2 - y^2 = (x + y)(x - y)$).

   (a) Show that every odd integer can be written as the difference between two squares.

   (b) Show that every multiple of 4 is a difference of two squares.

   (c) Show that if $n \equiv 2 (\text{mod } 4)$ then n cannot be written as difference of two squares.

   (d) Let $\tau(n)$ be the number of divisors of n. Show that for any odd positive n, not a square, the number of solutions to $n = x^2 - y^2$ in non-negative integers equals $\tau(n)/2$.

3. We are give the following identity,

$$(a^2 + 3b^2)(c^2 + 3d^2) = (ac + 3bd)^2 + 3(ad - bc)^2.$$

   Prove that every prime p with $p \equiv 1 (\text{mod } 3)$ can be written in the form $p = x^2 + 3y^2$. (Hint: you can use the same ideas that we used in the proof of the part of Lagrange's theorem that every prime is sum of four squares).