Homework problem VII

Elementary Number theory 2009

In the following problems we practice a little with the *abc*-conjecture. By $N(x)$ we denote the product of the distinct primes of x (the so-called radical of x). *Let* $\epsilon > 0$. *Then there exists* $C(\epsilon) > 0$ *such that for any* $a, b, c \in \mathbb{N}$ *with* $a+b = c$ *and* $\gcd(a, b) = 1$ *we have*

$$c < C(\epsilon)N(abc)^{1+\epsilon}.$$

Problems:

1. Show, by way of example, that the *abc*-conjecture cannot be true if we drop the condition $\gcd(a, b) = 1$.
2. Assuming the *abc*-conjecture, show that the equation $x^5 - y^2z^3 = 1$ in $x, y, z \in \mathbb{Z}$ has, beside the solutions with $x = 0, 1$, at most finitely many solutions. (It seems no other solutions besides the ones with $x = 0, 1$ exist, but it is not known how to prove this).
3. Assuming the *abc*-conjecture, show that the equation $m! + 1 = n^2$ has at most finitely many solutions. (The known solutions are $4!+1 = 5^2, 5!+1 = 11^2, 7! + 1 = 71^2$)
4. (challenge problem, not graded) An *abc*-triple is a triple $a, b, c \in \mathbb{N}$ such that $a+b = c$, $\gcd(a, b) = 1$ and $C > N(abc)$. In exercise 9.8.10 it is shown that there are infinitely many *abc*-triples with $a = 1$ (see the elaboration). Given any $a_0 \in \mathbb{N}$ show that there are infinitely many *abc*-triples with $a = a_0$.