Homework problem VIII
Elementary Number theory 2006

1. In this problem we analyse which integers n can be written as the *difference* between two squares. In other words, for which integers n is $x^2 - y^2 = n$ solvable in integers x, y. (Hint to use: $x^2 - y^2 = (x + y)(x - y)$).
   (a) Show that every odd integer can be written as the difference between two squares.
   (b) Show that every multiple of 4 is a difference of two squares.
   (c) Show that a $n \equiv 2 (\text{mod } 4)$ cannot be written as difference of two squares.

2. We are give the following identity,
$$(a^2 + 3b^2)(c^2 + 3d^2) = (ac + 3bd)^2 + 3(ad - bc)^2.$$
Prove that every prime p with $p \equiv 1 (\text{mod } 3)$ can be written in the form $p = x^2 + 3y^2$. (Hint: use exactly the same steps that we used in the proof of the part of Lagrange's theorem that every prime is sum of four squares).

3. (You can use Mathematica). The number $n = 10000000001 (= 10^{10} + 1)$ can obviously be written as sum of two squares. Which way is that? Determine at least one essentially different way to write n as sum of two squares. In order to do this, follow the procedure of page 64 of the course notes by factoring n in $\mathbb{Z}[i]$ and then recombine the factors in different ways.