Lower bounds of heights of points on hypersurfaces

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1 Introduction

Let us first recall Lehmer's conjecture [Le] on lower bounds for the height of an algebraic number which was stated in 1933. Let K be an algebraic number field of degree D over $\mathbf{Q}$. For any valuation v we denote $D_v = [K_v : \mathbf{Q}_v]$, where $K_v, \mathbf{Q}_v$ are the completions of $K, \mathbf{Q}$ with respect to v. For archimedean v we normalise the valuation by $|x_v| = |x|^{D_v/D}$ where $|.|$ is the ordinary complex absolute value. When v is non-archimedean we take $|p|_v = p^{-D_v/D}$ where p is the unique rational prime such that $|p|_v < 1$. The height of an algebraic number $\alpha \in K$ is defined by

$$H(\alpha) = \prod_v \max(1, |x|_v)$$

Because of our normalisation $H(\alpha)$ does not depend on the choice of the field K in which α is contained. We can now state Lehmer's conjecture.

Conjecture 1.1 *There exists a number $c > 1$ such that for any algebraic number α, not a root of unity and of degree D we have*

$$H(\alpha)^D \geq c.$$

Presumably $c = 1.1762808\ldots$, which is the larger real root of $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$.

The best unconditional result so far follows from work of Dobrowolski, Cantor and Louboutin [Lo], stating that there exists $\gamma > 0$ such that

$$H(\alpha)^D \geq 1 + \gamma \left(\frac{\log\log D}{\log D} \right)^3.$$

It came as a great surprise when S.Zhang [Zh] showed in 1992 that there does exist a number $c_1 > 1$ such that

$$H(\alpha)H(1-\alpha) \geq c_1$$

for all $\alpha \in \overline{\mathbf{Q}}$ such that $\alpha \neq 0, 1, \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$. This was proved by using Arakelov intersection theory on $\mathbf{P}^1$. It was almost immediately realised by one of us (see [Za]) that an elementary proof could be given which at the same time yields the best possible c_1, namely $\sqrt{\eta}$ where

$\eta = (1+\sqrt{5})/2$, the golden ratio. The minimum is attained when α is minus a fifth root of unity. In [Za] there is also a generalisation of the following sort. For any K-rational point $P = (P_0 : P_1 : \ldots : P_n)$ in n-dimensional projective space $\mathbf{P}^n$ we define the height by

$$H(P) = \prod_v \max(|P_0|_v, \ldots, |P_n|_v).$$

In particular the height of an algebraic number α is nothing but the projective height of $(1 : \alpha) \in \mathbf{P}^1(K)$. Then it is shown in [Za] that for any $(x_0 : x_1 : x_2) \in \mathbf{P}^2(\overline{\mathbf{Q}})$ such that $x_0 + x_1 + x_2 = 0$, $x_0x_1x_2 \neq 0$ and $(x_0 : x_1 : x_2) \neq (1 : \omega^{\pm 1} : \omega^{\mp 1})$, $(\omega^3 = 1)$, we have

$$H(x_0, x_1, x_2) \geq c_2$$

where c_2 is the larger real root of $x^6 - x^4 - 1$. The minimum is attained when the x_i are the roots of $x^3 + x - 1$.

Inspired by [Za], H.P.Schlickewei and E.Wirsing [SW] showed the following result. Consider the line $L : \ \lambda x + \mu y + \nu z = 0$ in $\mathbf{P}^2$ with $\lambda\mu\nu \neq 0$. Suppose that $\lambda + \mu + \nu = 0$. Then, for any two points $P_1, P_2 \in L(\overline{\mathbf{Q}})$ with non-zero coordinates and such that $(1 : 1 : 1), P_1, P_2$ are distinct, we have

$$H(P_1)H(P_2) > \exp(1/2400) = 1.00041\ldots$$

This result was applied by Schlickewei [Schl] to estimating numbers of solutions of three term S-unit equations in a strikingly successful way. Although very useful, the derivation of the Schlickewei-Wirsing result did not look optimal. It is the goal of this paper to remedy this situation and also give a generalisation which encompasses the previous results. We finish the introduction by giving a description of our general setup and main result.

Consider a hypersurface S of multidegree $d_1, \ldots, d_r$ on $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_r}$ given by a polynomial F with coefficients in $\mathbf{Z}$. Denote the coordinates of $\mathbf{P}^{n_i}$ by $\mathbf{x}_i = (x_{i0}, x_{i1}, \ldots, x_{in_i})$. The degree of F in the variable x_{ij} is denoted by d_{ij}. We define $\tilde{d}_i = -d_i + \sum_j d_{ij}$.

Choose a subset I of $\{i |\ n_i = 1\}$ and let E be the set $\{(i, 0) | \, i \in I\}$, to which we refer as *exceptional index pairs*. For any polynomial with coefficients in $\mathbf{Z}$ we denote by $||P||$ the sum of the absolute values of the coefficients. We define

$$c_{ij} = \left|\left| \frac{\partial F}{\partial x_{ij}} \right|\right|, \qquad c_F = \max_{(i,j) \notin E} c_{ij}$$

The advantage of having the exceptional set E is that the value of c_F may be smaller than the one we would get by taking the maximum over all pairs (i, j). In the first example in [Za] this enables us to get the optimal lower bound for the product $H(\alpha)H(1-\alpha)$. By δ we denote the maximum of the numbers $\max_{i \in I}(\tilde{d}_i + d_{i,1})/2$ and $\max_{i \notin I} \frac{\tilde{d}_i}{n_i + 1}$.

Theorem 1.2 *For each point $(\mathbf{x}_1, \ldots, \mathbf{x}_r) \in \mathbf{P}^{n_1}(\overline{\mathbf{Q}}) \times \cdots \times \mathbf{P}^{n_r}(\overline{\mathbf{Q}})$ such that $F(x_{ij}) = 0$, $x_{ij} \neq 0$ for all i, j and $F(x_{ij}^{-1}) \neq 0$ we have*

$$H(\mathbf{x}_1)^{n_1+1} \cdots H(\mathbf{x}_r)^{n_r+1} \geq \rho,$$

where ρ is the unique real root larger than 1 of $x^{-2} + c_F^{-1} x^{-\delta} = 1$.

During the preparation of this paper W.M.Schmidt informed us that in [Schm] he had already proved a theorem very similar to ours in the case where one works in $(\mathbf{P}^1)^r$. The logarithm of the lower bound given in [Schm] is $1/(2^{4f+2r}H)$, where f is the total degree of F and H the maximum of all coefficients. Although the basic starting point in this paper and [Schm] is the same, we nevertheless found that the principle of our approach and the better values of the constants have some interest.

2 Applications

Before proving the theorem we describe a few consequences. First of all consider r algebraic numbers $\alpha_1, \ldots, \alpha_r$ whose sum is a rational integer N. We like to interpret the r-tuple as a point $(1 : \alpha_1) \times \cdots \times (1 : \alpha_r) \in (\mathbf{P}^1)^r$. For the set I of our theorem we choose $\{1, \ldots, r\}$. Letting F be the homogeneous version of $x_1 + \cdots + x_r - N$ one easily checks that $c_i = 1$ for all i. Note that the coefficient N in F does not appear in the c_i because of our choice of I. So we get $c_F = 1$. Moreover, $n_i = 1$ and $d_i = 1$ for all i. Hence $\delta = 1$. Thus we find,

Corollary 2.1 *Let $\alpha_1 \ldots, \alpha_r \in \overline{\mathbf{Q}}^*$, $N \in \mathbf{Z}$ be such that $\alpha_1 + \cdots + \alpha_r = N$ and $\alpha_1^{-1} + \cdots + \alpha_r^{-1} \neq N$. Then,*

$$H(\alpha_1) \cdots H(\alpha_r) \geq \sqrt{\eta}$$

where η is the golden ratio.

Note that when $r \geq 4$ the lower bound is actually attained for the r-tuple $-\zeta_5, 1 + \zeta_5, 1, \zeta_{r-2}, \ldots, \zeta_{r-2}^{r-3}$ where ζ_k denotes a primitive k-th root of unity. When we take for the α_i the conjugates of an algebraic number α of degree D we get the following consequence.

Corollary 2.2 *Let $\alpha \in \overline{\mathbf{Q}}^*$ be such that* $\mathrm{trace}(\alpha)$ *is integral and* $\mathrm{trace}(\alpha) \neq \mathrm{trace}(\alpha^{-1})$. *Then $H(\alpha)^D \geq \sqrt{\eta}$.*

However, this result is already contained in a result of C.Smyth [Sm] which states that $H(\alpha)^D \geq \theta$ for every non-reciprocal $\alpha \in \overline{\mathbf{Q}}^*$. Here θ is the real root of $x^3 - x - 1$.

We now consider r algebraic numbers α_i whose sum is 1 and give a lower bound for $H(1, \alpha_1, \ldots, \alpha_r)$. The polynomial F can be written $x_1 + \cdots + x_r - x_0$ and we have $c_F = 1,\ d_1 = 1$. Furthermore, $\delta = r/(r+1)$.

Corollary 2.3 *For any $\alpha_1, \ldots, \alpha_r \in \overline{\mathbf{Q}}^*$ such that $\alpha_1 + \cdots + \alpha_r = 1$ and $\alpha_1^{-1} + \cdots + \alpha_r^{-1} \neq 1$ we have*

$$H(1, \alpha_1, \ldots, \alpha_r) \geq \rho.$$

where ρ is the real root larger than 1 of $1 = x^{-2r-2} + x^{-r}$.

As pointed out in the introduction, this result is optimal when $r = 2$. For $r > 2$ this is not true any more. When $r = 3$ for example we find the lower bound $1.14613\ldots$ (which improves the bound $\exp(1/402) = 1.00249\ldots$ from [SW]). However the lowest height we could find was $H = 1.15096\ldots$ when the α_i are the zeros of $x^3 - x^2 + 1$. On the other hand the asymptotic behaviour of ρ as a function of r looks optimal. It is not hard to show that $\rho^{r+1} \rightarrow \eta$ as $r \rightarrow \infty$

while on the other hand the zeros $\alpha_0, \ldots, \alpha_r$ of $x^{r+1} - x - 1$ satisfy $H(\alpha_0, \ldots, \alpha_r)^{r+1} \to 2$ as $r \to \infty$.

We now consider the Schlickewei-Wirsing result. Suppose we have a line $L: \ \lambda x + \mu y + \nu z = 0$ in $\mathbf{P}^2$ with $\lambda\mu\nu \neq 0$. Let $P_1, P_2, P_3 \in L(\overline{\mathbf{Q}})$ be three distinct points with non-zero coordinates. Letting $P_i = (x_i : y_i : z_i)$ $(i = 1, 2, 3)$ we get the relation

$$\Delta := \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

We want a lower bound of $H(P_1)H(P_2)H(P_3)$. Our polynomial F is now the determinant form Δ. First we point out that

$$\tilde{\Delta} := \begin{vmatrix} x_1^{-1} & y_1^{-1} & z_1^{-1} \\ x_2^{-1} & y_2^{-1} & z_2^{-1} \\ x_3^{-1} & y_3^{-1} & z_3^{-1} \end{vmatrix} \neq 0.$$

Suppose $\tilde{\Delta} = 0$. Then there exist α, β, γ, not all zero, such that $\alpha x_i^{-1} + \beta y_i^{-1} + \gamma z_i^{-1} = 0$ for $i = 1, 2, 3$. Hence $\alpha y_i z_i + \beta z_i x_i + \gamma x_i y_i = 0$ $(i = 1, 2, 3)$. The conic $C: \ \alpha yz + \beta zx + \gamma xy = 0$ is reducible if and only if $\alpha\beta\gamma = 0$. So, if $\gamma = 0$ for example, we get $\alpha x_i + \beta y_i = 0$ for $i = 1, 2, 3$. But this contradicts $\nu \neq 0$. So C is an irreducible conic. But then P_1, P_2, P_3 lie both on C and L which is impossible since $|C \cap L| \leq 2$. We must conclude that $\tilde{\Delta} \neq 0$. We can now apply our Theorem with $r = 3, \ n_1 = n_2 = n_3 = 2, \ d_1 = d_2 = d_3 = 1, \ c_F = 2$ and $I = \emptyset$.

Corollary 2.4 *Consider the line $L: \ \lambda x + \mu y + \nu z = 0$ in $\mathbf{P}^2$ with $\lambda\mu\nu \neq 0$. Let $P_1, P_2, P_3 \in L(\overline{\mathbf{Q}})$ be three distinct points with non-zero coordinates. Then,*

$$H(P_1)H(P_2)H(P_3) \geq \rho,$$

where ρ is the real root larger than 1 of $1 = \rho^{-6} + (1/2)\rho^{-2}$.

The numerical value of ρ is $1.09427\ldots$ which compares favourably with the value $1.00041\ldots$ from [SW] or $1.019\ldots$ from [Sch]. Moreover this result was applied successfully to equations of the form $x + y = 1$ with x, y unknows in a finitely generated multiplicative group and to multiplicity estimates for binary recurrences in [BS].

3 Proof of Theorem 1.2

The proof is based on the following observation, which is a direct generalisation of [Za]. Let X be a closed subvariety of $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_r}$ defined over $\mathbf{Q}$. We denote the coordinates by $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ with $\mathbf{x}_i = (x_{i0}, \ldots, x_{in_i})$. Denote by $X(\mathbf{C})_1$ the intersection of $X(\mathbf{C})$ with the polydisc $\{\mathbf{x} \ : \ |x_{ij}| \leq 1 \ \forall i, j\}$. We also give ourselves a collection of multihomogeneous polynomials $G_k(\mathbf{x}) \in \mathbf{Z}[\mathbf{x}]$ of multidegrees $(d_{k1}, \ldots, d_{kr})$

Lemma 3.1 *Let $\nu_k \geq 0$ for all k and set*

$$w_i = \sum_k \nu_k d_{ki}, \qquad \lambda = -\max_{\mathbf{x} \in X(\mathbf{C})_1} \left\{ \sum_k \nu_k \log |G_k(\mathbf{x})| \right\}. \tag{1}$$

Then for any point $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_r) \in X(\overline{\mathbf{Q}})$ *with* $\prod_k G_k(\mathbf{x}) \neq 0$ *we have*

$$\prod_{i=1}^{r} H(x_i)^{w_i} \geq e^{\lambda}. \tag{2}$$

Proof. Suppose that $\mathbf{x} \in X(K)$ with $G_k(\mathbf{x}) \neq 0$ for all k. Here K is an algebraic number field of degree D, say. For any valuation v of K we let $D_v = [K_v : \mathbf{Q}_v]$. Then the inequality

$$\sum_{i=1}^{r} w_i \log(\max_j |x_{ij}|_v) \geq \sum_k \nu_k \log |G_k(\mathbf{x})|_v + \begin{cases} \frac{D_v}{D}\lambda & \text{if } v|\infty \\ 0 & \text{if } v \not|\infty \end{cases}$$

holds for all places v of K, because by the homogeneity condition (1) we may assume that $\max_j |x_{ij}|_v = 1$ for all i and the inequality follows from the definition of λ if v is infinite and is straightforward if v is finite. The lemma follows by summing over all v and using the product formula. □

The following Lemma is saves us a considerable amount of effort in the determination of λ for the sake of the previous Lemma.

Lemma 3.2 *Letting notations be as above, the function* $\Psi := \sum \nu_k \log |G_k(\mathbf{x})|$ *assumes a maximum in* $\mathbf{x} \in X_1(\mathbf{C})$ *and it is attained at a point all of whose coordinates have absolute value* 1 *with at most one exception.*

Proof. Since the ν_k are positive, Ψ is bounded from above in $X_1(\mathbf{C})$. For $\epsilon > 0$ sufficiently small the set $\mathbf{x} \in X_1(\mathbf{C})$ such that $\Psi(\mathbf{x}) \geq \log(\epsilon)$ is compact and not empty. Hence it is clear that Ψ, being continuous, assumes a maximum.

Now suppose that Ψ assumes a maximum at a point P where at least two coordinates have absolute value < 1. Call these coordinates ξ, η and denote the values of these coordinates at P by ξ_0, η_0. Substitute in $F = 0$ the values of all coordinates of P except ξ, η. The equation $F = 0$ reduces to the equation of a curve $f(\xi, \eta) = 0$ containing the point ξ_0, η_0. By choosing a branch of $f = 0$ at the point ξ_0, η_0 we find locally analytic functions $\xi(t), \eta(t)$ such that $\xi(0) = \xi_0, \eta(0) = \eta_0$ and $f(\xi(t), \eta(t)) = 0$ identically in a neighbourhood of $t = 0$. When f was identically zero anyway, we can choose $\xi(t), \eta(t)$ arbitrarily. Choose a disk D in the complex t-plane around 0 such that $|\xi(t)|, |\eta(t)| \leq 1$ for all $t \in D$. Specialise the arguments in Ψ to the values of the point P except for ξ and η where we substitute $\xi(t)$ and $\eta(t)$. In this way we obtain a function $\psi(t)$ in $t \in D$ which assumes a maximum in $t = 0$. Notice that $\psi(t)$ is harmonic in the real and imaginary part of t. A harmonic function assuming a maximum in the interior of its domain is necessarily constant. Hence $\psi(t)$ is constant. But in that case we can continue $\xi(t)$ and $\eta(t)$ analytically until either one of them hits the unit circle. In that new point the value of Ψ is again $\psi(0)$, i.e. maximal. We continue this procedure for other coordinates, if necessary, until we have found an optimal point all of whose coordinates have absolute value one with at most one exception. □

Lemma 3.3 *Let* $\alpha, \beta, \gamma > 0$. *Let* m *be the unique minimum of the function*

$$u \log \frac{\gamma u}{u+v} + v \log \frac{v}{u+v}$$

under the constraints $u, v \geq 0,\ \alpha u + \beta v = 1$. *Then* e^{-m} *is the unique real root larger than* 1 *of* $\gamma^{-1}x^{-\alpha} + x^{-\beta} = 1$.

Proof. Put $x = v/(u+v)$ and $1 - x = u/(u+v)$. Then

$$u = \frac{1-x}{\beta x + \alpha(1-x)} \qquad v = \frac{x}{\beta x + \alpha(1-x)}$$

and $x \in [0,1]$. We must minimize

$$f(x) = \frac{(1-x)\log(\gamma(1-x)) + x\log x}{\beta x + \alpha(1-x)}$$

on $[0,1]$. Differentiate with respect to x,

$$f'(x) = \frac{-\beta\log(\gamma(1-x)) + \alpha\log x}{(\beta x + \alpha(1-x))^2}.$$

This vanishes if $(\gamma(1-x))^\beta = x^\alpha$. Since x is strictly increasing and $1-x$ strictly decreasing there is a unique solution $x_0 \in]0,1[$. Choose $\rho > 0$ such that $x_0 = \rho^{-\beta}$. Then, $\gamma(1-x) = \rho^{-\alpha}$ and thus we see that ρ satisfies $1 - \rho^{-\beta} = \gamma^{-1}\rho^{-\alpha}$. It remains to verify that $f(x_0) = -\log\rho$, which is straightforward. □

Proof of Theorem. We apply Lemma 3.1 to the hypersurface X given by the multihomogeneous polynomial $F(\mathbf{x}) \in \mathbf{Z}[\mathbf{x}]$ with multidegrees $d_1, \ldots, d_r$. For the G_k we take the coordinates x_{ij} and the function

$$\tilde{F}(\mathbf{x}) = F(x_{ij}^{-1})\prod x_{ij}^{d_{ij}}$$

where d_{ij} is the degree of F in x_{ij}. Let $\mu, \nu_{ij} \geq 0$. Let $\Phi(\mathbf{x})$ be the function $\mu\log|\tilde{F}(\mathbf{x})| + \sum_{i,j}\nu_{ij}\log|x_{ij}|$ on $X(\mathbf{C})$. Let $\tilde{d}_i = -d_i + \sum_j d_{ij}$ be the degree of $\tilde{F}$ in $\mathbf{x}_i$ and suppose

$$w_i = \mu\tilde{d}_i + \sum_j \nu_{ij}, \qquad \lambda = -\max_{\mathbf{x}\in X(\mathbf{C})_1}\Phi(\mathbf{x}).$$

Then Lemma 3.1 states that (2) holds for all $\mathbf{x} \in X(\overline{\mathbf{Q}})$ with $x_{ij} \neq 0$ and $F(x_{ij}^{-1}) \neq 0$.

Let us take $w_i = n_i + 1$ for all i. Although there are many other choices for the weights w_i, this choice gives us the particularly simple shape of our main theorem. It remains to choose μ, ν_{ij} in such a way that λ becomes positive and as large as possible. We choose

$$\nu_{ij} = 1 - \frac{\tilde{d}_i}{n_i+1}\mu \text{ if } i \notin I$$

and

$$\nu_{i,0} = 1 - \frac{\tilde{d}_i - d_{i,1}}{2}\mu, \qquad \nu_{i,1} = 1 - \frac{\tilde{d}_i + d_{i,1}}{2}\mu \text{ if } i \in I.$$

Let us determine $\max_{\mathbf{x}\in X(\mathbf{C})_1} \Phi(\mathbf{x})$. By Lemma 3.2 this maximum is attained at a point all of whose coordinates, with possibly one exception, lie on the unit circle. Suppose that $|x_{i_0j_0}| \le 1$ and that $|x_{ij}| = 1$ for all $(i,j) \ne (i_0, j_0)$. Suppose first that $(i_0, j_0) \notin E$. Then,

$$\begin{aligned}
|\tilde{F}(x_{ij})| &= |F(x_{ij}^{-1})| \cdot |\prod_{i,j}(x_{ij})^{d_{ij}}| \\
&= |F(x_{i_0j_0}^{-1}, \overline{x_{ij}})| \cdot |x_{i_0j_0}|^{d_{i_0j_0}} \\
&= |F(x_{i_0j_0}^{-1}, \overline{x_{ij}}) - F(\overline{x_{ij}})| \cdot |x_{i_0j_0}|^{d_{i_0j_0}} \\
&\le c_{i_0j_0}|\frac{1}{x_{i_0j_0}} - \overline{x_{i_0j_0}}| \max(|x_{i_0j_0}|^{-1}, |x_{i_0j_0}|)^{d_{i_0j_0}-1} \cdot |x_{i_0j_0}|^{d_{i_0j_0}} \\
&= c_{i_0j_0}(1 - |x_{i_0j_0}|^2)
\end{aligned}$$

Put $|x_{i_0j_0}|^2 = \xi$. We see that the maximum of Φ is

$$\max_{\xi\in[0,1]} \mu \log(c_{i_0j_0}(1-\xi)) + (\nu_{i_0j_0}/2)\log\xi$$

This maximum is attained at $\xi = \nu_{i_0j_0}/(\nu_{i_0j_0} + 2\mu)$ and its value is

$$\mu \log \frac{2\mu c_{i_0j_0}}{\nu_{i_0j_0} + 2\mu} + \frac{\nu_{i_0j_0}}{2} \log \frac{\nu_{i_0j_0}}{\nu_{i_0j_0} + 2\mu}.$$

Since we have $\nu_{i_0j_0} \ge 1 - \delta\mu$, this maximum is bounded above by

$$\mu \log c_F \frac{2\mu}{(1-\delta\mu) + 2\mu} + \frac{1-\delta\mu}{2} \log \frac{1-\delta\mu}{(1-\delta\mu)+2\mu} \tag{M}$$

We now determine the maximum when $(i_0, j_0) \in E$. In particular, $j_0 = 0$. So suppose we have $|x_{i_0,0}| \le 1$ and $|x_{ij}| = 1$ for all other i, j. Writing down the dependence on $x_{i_0,0}, x_{i_0,1}$ explicitly and putting $z = x_{i_0,0}/x_{i_0,1}$, we find

$$\begin{aligned}
|\tilde{F}(x_{ij})| &= |F(x_{i_0,0}^{-1}, x_{i_0,1}^{-1}, x_{ij}^{-1})| \cdot |x_{i_0,0}|^{d_{i_0,0}} \\
&= |F(1, z, x_{ij}^{-1})| \\
&= |F(1, z, \overline{x_{ij}}) - F(1, 1/\overline{z}, \overline{x_{ij}}| \\
&\le c_{i_0,1}|z - 1/\overline{z} \max(|z|, |z|^{-1})^{d_{i_0,1}-1} \\
&= c_{i_0,1}|1 - |z|^2| \cdot |z|^{-d_{i_0,1}}
\end{aligned}$$

Put $\xi = |z|^2 = |x_{i_0,0}|^2$. We see that the maximum of Φ is

$$\max_{\xi\in[0,1]} [\mu \log(c_{i_0,1}(1-\xi)) - (d_{i_0,1}\mu/2)\log|\xi| + (\nu_{i_0,0}/2)\log|\xi|]$$

which equals

$$\mu \log \frac{2c_{i_0j_0}\mu}{\tilde{\nu} + 2\mu} + \frac{\tilde{\nu}}{2} \log \frac{\tilde{\nu}}{\tilde{\nu} + 2\mu}$$

where $\tilde{\nu} = \nu_{i_0,0} - d_{i_0,1}\mu/2$. Note that by our choice of $\nu_{i_0,0}$, $\tilde{\nu} = 1 - (\tilde{d}_{i_0} + d_{i_0,1})\mu/2 \ge 1 - \delta\mu$. Hence our maximum is again bounded by (M). Now use Lemma 3.3 with $\alpha = \delta, \beta = 2, \gamma = c_F$ to minimize (M) by letting μ vary. The assertion of our theorem follows immediately. □

4 References

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