LAMÉ EQUATIONS WITH ALGEBRAIC SOLUTIONS

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ABSTRACT. In this paper we list all finite groups that occur as the monodromy groups of Lamé equations $L_{n,B}y = 0$ with finite monodromy, together with a list of examples of such equations. We show that the set of such Lamé equations with $n \notin 1/2+\mathbb{Z}$ is countable, up to scaling of the equation. This result follows from the general statement that the set of equivalent second order equations, having algebraic solutions and all of whose integer local exponent differences are 1, is countable.

1. SECOND ORDER EQUATIONS WITH FINITE MONODROMY

Consider the set of second order linear Fuchsian differential equations of the form

$$Ly = 0, \qquad L \in \mathbb{C}(z)[\frac{d}{dz}]$$

having finite monodromy group. Denote this set by $\mathcal{A}$. As is well-known, $\mathcal{A}$ is precisely the set of second order equations over $\mathbb{C}(z)$ whose solution set consists of functions that are algebraic over $\mathbb{C}(z)$. By abuse of language we call the elements of $\mathcal{A}$ *algebraic differential equations.*

Consider an equation $Ly = 0$ from the set $\mathcal{A}$. At every point of $a \in \mathbb{P}^1$ the equation $Ly = 0$ has two rational local exponents $\rho_1(a), \rho_2(a)$. We call $|\rho_1(a) - \rho_2(a)|$ the *local exponent difference* at a. The local exponent difference at every non-singular point is 1.

Suppose conversely that the local exponent difference of $Ly = 0$ at a equals 1. Since $Ly = 0$ belongs to $\mathcal{A}$ there are no local logarithmic solutions. Two local solutions of the equation are $(z-a)^\rho f_1(z)$ and $(z-a)^{\rho+1} f_2(z)$, where f_1, f_2 are locally biholomorphic functions at a. The differential equation $(z-a)^{-\rho}L((z-a)^\rho y) = 0$ then is algebraic and has the solutions $f_1, (z-a)f_2$. In particular, $z = a$ is a non-singular point of the new differential equation.

An equation from $\mathcal{A}$ is called *pure* if the only integral exponent difference that is allowed to occur is 1. In particular, apparent singularities are forbidden with such equations. Denote the subset of $\mathcal{A}$ of pure equations by $\mathcal{A}_0$. Both sets are stable under the substitution

$$L \to A(z)R(z)^{-\rho}L \circ R(z)^\rho$$

for any $A(z), R(z) \in \mathbb{C}(z)$ and $\rho \in \mathbb{Q}$. They are also stable under automorphisms of $\mathbb{P}^1$. That is, given an automorphism $\phi(z)$, there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$$

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such that

$$\phi(z) = \frac{az+b}{cz+d},$$

and vice versa. These two operations give an equivalence relation on $\mathcal{A}$ and $\mathcal{A}_0$ that we denote by $\sim$. We call two equations in $\mathcal{A}$ *equivalent* if they are equivalent with respect to the relation $\sim$. We have the following statement.

Theorem 1.1. *Let the notation be as above. Then $\mathcal{A}_0/\sim$ is a countable set.*

This Theorem is a consequence of Theorem 7.1, which is a more quantitative version of Theorem 1.1. The statement and proof of Thm 7.1 can be found in Section 7 of this paper. As a consequence of Theorem 1.1 one can start an enumeration of the set $\mathcal{A}_0$. We perform this enumeration using an increasing number of singular points of the differential equation. Let us start with Fuchsian equations having two singularities, which we may assume to be $0, \infty$. Such an equation is of the form $z^2\frac{d^2y}{dz^2} + az\frac{dy}{dz} + by = 0$ and is known as Euler's homogeneous equation of order 2. It has a basis of solutions of the form z^{ρ_1}, z^{ρ_2}, where ρ_1, ρ_2 are zeros of $x^2 + (a-1)x + b$. Algebraicity of the solutions is equivalent to $\rho_1, \rho_2 \in \mathbb{Q}$. The latter condition is equivalent to imposing $a, b \in \mathbb{Q}$ with positive rational square $(a-1)^2 - 4b$. In addition, if the equation is supposed to be in $\mathcal{A}_0$, then $(a-1)^2 - 4b$ is not allowed to be an integer square in $\mathbb{Z} \setminus \{1\}$.

The first interesting case of an algebraic Fuchsian equation of order 2 is one with three singularities. By application of an equivalence transformation we can see to it that the singularities are $0, 1, \infty$ and that at both $0, 1$ at least one local exponent is 0. These properties characterise the Gaussian hypergeometric equation, having the famous hypergeometric series

$$F(a, b, c|z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

as solution, where $(x)_n = x(x+1)\cdots(x+n-1)$ is the so-called Pochhammer symbol. The numbers a, b, c are the parameters of the hypergeometric equation. In 1873 H.A.Schwarz [Sch73], using ideas of Riemann, gave a complete list and enumeration of all hypergeometric equations having an algebraic solution set.

The next step would be to study second order equations with four singularities. However, in this case we encounter a difficulty. In the previous cases the equation was determined by the location of the singularities and the local exponents. In other words, by local data. In the case of four singularities there is one parameter which is not determined by local data. This parameter is called the *accessory parameter* of the equation. The dependence of the monodromy group on the accessory parameter is as yet little understood. It is possible, however, to find conditions on the accessory parameter such that the solutions will be algebraic. We do this for the Lamé equation, equation (1). As parameters this equation contains the accessory parameter B and an index n.

Statement and proof of the main results of this paper will appear in the sections to come. Here we give a brief overview. In Theorem 4.4 we deduce all possible finite groups that might occur as finite monodromy for a Lamé equation. We also list the corresponding values of the parameter n, which are usually congruence classes modulo $\mathbb{Z}$. Conversely, in Theorems 5.1 and 6.1 we show that these groups, together with each class $n(\text{mod } \mathbb{Z})$, actually occur. In Theorem 4.6 we show that, given a finite group G and an index n, $n \notin 1/2+\mathbb{Z}$, the number of Lamé equations with given monodromy G and index n is finite, up to equivalence of equations.

Finally we mention that this paper is an elaboration of Alexa van der Waall's PhD thesis [vdW02], in which many of the results of the present paper are contained.

2. The Lamé equation

The Lamé equation with $n \in \mathbb{Q}$ and $g_2, g_3, B \in \mathbb{C}$ as parameters is the differential equation

$$p(z)\frac{d^2y}{dz^2} + \frac{1}{2}p'(z)\frac{dy}{dz} - (n(n+1)z+B)\,y = 0, \tag{1}$$

where $p(z) := 4z^3 - g_2 z - g_3$ has three distinct zeros z_1, z_2, z_3. The Lamé equation will be abbreviated by

$$L_{n,B}y = 0.$$

The local exponents are $0, 1/2$ at the three finite singularities z_1, z_2 and z_3, and $-n/2, (n+1)/2$ at ∞. Since the equation does not change under the transformation $n \mapsto -1-n$ we assume $n \geq -1/2$. The number B is the accessory parameter of the equation.

One can consider the action of the monodromy group $M \subset \mathrm{GL}(2,\mathbb{C})$ of $L_{n,B}$ with respect to a local basis of solutions around a non-singular point. The local monodromy matrices γ_i at the finite singularities z_i, $i = 1,2,3$, have eigenvalues ± 1 and hence satisfy $\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \mathrm{Id}$. They thus are *reflections* by definition. In addition, one has $\gamma_1\gamma_2\gamma_3\gamma_\infty = \mathrm{Id}$ where γ_∞ is the local monodromy matrix at ∞. Moreover, the group M is generated by the complex reflections γ_1, γ_2 and γ_3 and therefore is a *reflection group* by definition.

Lemma 2.1. *The monodromy group M of the Lamé equation is a complex reflection group, whose matrices have determinant ± 1.*

Proof. The lemma follows immediately from the remarks made above. □

In Section 4 we determine the monodromy group M and its projective group $PM \subset \mathrm{PGL}(2,\mathbb{C})$ more specifically. In general there are two particular cases to be mentioned. The first is $n + 1/2 \in \mathbb{Z}$. Then logarithmic solutions of the Lamé equation at ∞ may arise, since $n + 1/2 \in \mathbb{Z}$ is the local exponent difference at ∞. There is always a logarithmic solution for $n = -1/2$.

Theorem 2.2 (Brioschi-Halphen)**.** *Suppose $n + 1/2 \in \mathbb{Z}_{\geq 0}$. Then there exists $p_n \in \mathbb{Z}[g_2/4, g_3/4, B]$ of degree $n+1/2$ in B such that $L_{n,B}y = 0$ has no logarithmic solutions at ∞ if and only if $p_n(g_2, g_3, B) = 0$.*

Proof. Proofs are given in [Poo36, §37] and in [Bal81, Thm 2.6]. □

The polynomial p_n of Theorem 2.2 is known as the *Brioschi-Halphen determinant.* The suggested proofs of the Theorem also show how to calculate p_n in practice.

Theorem 2.3. *Let $n \in 1/2 + \mathbb{Z}_{\geq 0}$ and $g_2, g_3 \in \mathbb{C}$ be given. Then $p_n(g_2, g_3, B) = 0$ if and only if M is finite. In that case PM is Klein's Four group V_4.*

Proof. We use Theorem 2.2. First suppose $p_n(g_2, g_3, B) = 0$. Then there are no logarithmic solutions. The matrix γ_∞ thus acts as a scalar multiplication by $\pm$i. This yields $\gamma_1\gamma_2\gamma_3 = \pm\mathrm{i}\cdot\mathrm{Id}$. It follows from $\gamma_i^2 = \mathrm{Id}$ and $\gamma_i \neq \pm\,\mathrm{Id}$, $i = 1, 2, 3$, that PM is V_4. Lemma 2.1 implies that the scalar matrices of M are contained in $\langle \mathrm{i}\cdot\mathrm{Id}\rangle$. Henceforth, the group M is finite.

Conversely, if M is finite then the Lamé equation only has algebraic solutions. In particular there are no logarithmic solutions and thus $p_n(g_2, g_3, B) = 0$. □

For $n = 3/2$ we have $p_n(g_2, g_3, B) = B^2 - 3g_2/4$. There are overcountably many g_2, g_3, B satifying $B^2 - 3g_2/4 = 0$. Notice also that our equation is not pure for such triples, since the local exponent difference at ∞ is 2. So we see that Theorem 1.1 cannot hold if we drop the purity condition. Table 1 gives a few more polynomials p_n. More examples of p_n are listed in Table A.1 of [vdW02].

n	$p_n(g_2, g_3, B)$
1/2	B
3/2	$B^2 - \frac{3}{4}g_2$
5/2	$B^3 - 7g_2B + 20g_3$
7/2	$B^4 - \frac{63}{2}g_2B^2 + 216g_3B + \frac{945}{16}g_2^2$

TABLE 1.

The remaining case of interest in general is $n \notin 1/2 + \mathbb{Z}$.

3. Lamé equations with integral n

The purpose of this section is to specify the monodromy groups for Lamé equations with integer n. In particular we describe their finite monodromy groups and show that these are neccessarily irreducible.

Theorem 3.1. *Suppose that M acts reducibly. Then $n \in \mathbb{Z}$. Moreover, M does not act completely reducibly and is infinite.*

Proof. Suppose that M acts reducibly. Then, from the local exponents at the finite singularities we infer that the invariant solution is of the form

$$\prod_{i=1}^{3}(z - z_i)^{\epsilon_i}Q(z) \tag{2}$$

with $\epsilon_i \in \{0, 1/2\}$, $Q(z) \in \mathbb{C}[z]$. This implies that there is a local exponent at ∞ which is $-\deg(Q) - \sum_{i=1}^{3}\epsilon_i$. Therefore it is an integer or half integer which is ≤ 0. As a result we have $n \in \mathbb{Z}_{\geq 0}$.

Suppose in addition that M acts completely irreducible. This would imply the existence of a second solution of the form (2) that is linearly independent of the first. If γ_∞ is not a scalar matrix, then this solution should correspond to the second local exponent $(n+1)/2$. This is a contradiction to the exponent being ≤ 0. If γ_∞ is a scalar matrix, then we know that the exponent difference $n + 1/2$ is an integer, which contradicts $n \in \mathbb{Z}$.

Suppose that M is finite and acts reducibly. Then, by Maschke's Theorem, M acts completely reducibly. However, this is not possible by our previous statement. We conclude that if M acts reducibly, it should be infinite. □

In the case that M acts reducibly, the eigenfunctions of the form (2) are called the *Lamé solutions.* The following classical theorem characterises the occurrence of Lamé solutions.

Theorem 3.2. *Suppose $n \in \mathbb{Z}_{\geq 0}$. Then there exists a polynomial $l_n \in \mathbb{Q}[g_2, g_3, B]$ of degree $2n+1$ in B, such that there is a Lamé solution if and only if $l_n(g_2, g_3, B) = 0$.*

Proof. For a discussion of this statement we refer to [WW50, §23.41, 23.42] or to [vdW02, §4.4]. □

A few of the polynomials $l_n(B)$ are given in Table 2. For a more extensive list we refer to Table A.3 of [vdW02].

n	$l_n(B)$
0	B
1	$4B^3 - g_2B - g_3$
2	$(B^2 - 3g_2)(4B^3 - 9g_2B + 27g_3)$
3	$B\left(16B^6 - 504B^4g_2 + 2376B^3g_3 + 4185B^2g_2^2\right.$
	$\left.-36450Bg_2g_3 - 3375g_2^3 + 91125g_3^2\right)$

TABLE 2.

It is now possible to give a fairly complete description of the monodromy group in the case that $n \in \mathbb{Z}_{\geq 0}$.

Theorem 3.3. *Suppose that $n \in \mathbb{Z}_{\geq 0}$ and that M acts irreducibly. Then there is a basis $y_1(z), y_2(z)$ of the vector space of solutions with respect to which M is a subgroup of the infinite dihedral group*

$$D_\infty = \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix} : \lambda \in \mathbb{C}^* \right\rangle \quad \textit{with} \quad \gamma_\infty = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In addition, y_1y_2 is a polynomial $F_n(z)$ of degree n in z.

Proof. Note that $n + 1/2$ is a half integer. The matrix γ_∞ thus is a reflection. We now have

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \gamma_\infty^2 = \mathrm{Id} \quad \text{and} \quad \gamma_1\gamma_2\gamma_3\gamma_\infty = \mathrm{Id}.$$

From these relations it follows easily that the subgroup

$$H := \langle \gamma_1\gamma_2, \gamma_2\gamma_3 \rangle$$

is an abelian subgroup of M of index 2. It is characterised by the fact that H is the subgroup of determinant one elements. Moreover, we have $M = H \cup \gamma_\infty H$. We can now distinguish two cases. Either H has two invariant subspaces of dimension 1 or it has only one.

First suppose H only contains semi-simple (i.e. diagonisable) elements. Since M acts irreducibly, it is non-abelian, and so $|M| \geq 6$. Hence H has order ≥ 3 and consequently contains a non-scalar element. Denote the eigenfunctions of this element by y_1, y_2. With respect to this basis the group H is a subgroup of the infinite cyclic group

$$C_\infty := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}.$$

It follows from $\gamma_\infty H = H\gamma_\infty$, that the element γ_∞ permutes the functions y_1, y_2 up to scalar multiplication. If $\gamma_\infty(y_1) = \lambda_1 y_1,\ \gamma_\infty(y_2) = \lambda_2 y_2$ then M would be an abelian group which acts reducibly, contrary to our assumptions. So we have $\gamma_\infty(y_1) = \lambda_1 y_2,\ \gamma_\infty(y_2) = \lambda_2 y_1$ for certain $\lambda_1, \lambda_2 \in \mathbb{C}$. After replacing $\lambda_1 y_2$ by y_2 and noticing $\gamma_\infty^2 = \mathrm{Id}$ we obtain $\gamma_\infty(y_1) = y_2,\ \gamma_\infty(y_2) = y_1$. This shows that γ_∞ with respect to the basis y_1, y_2 is of the desired form. Moreover, the group generated by this matrix and C_∞ is precisely the group D_∞.

The second case we must consider is that H contains a non semi-simple element h. By a suitable choice of basis we can assume it has the form $h = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The only matrices of determinant 1 that commute with h form the group

$$\left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C} \right\}.$$

Therefore H is a subgroup of this group of matrices. The elements of H have, up to a scalar, an unique common eigenvector y. The monodromy γ_∞ sends y to a multiple of itself, since H is normal in M. Hence the eigenvector y spans a one-dimensional invariant subspace under the action of M on the solution space, contrary to our assumption of irreducibility. This proves the first part of the Theorem.

Finally, let $y_1(z), y_2(z)$ be as in the Theorem. Then the product $y_1 y_2(z)$ is invariant under the monodromy action on the solution space. Hence, it is a rational function in $\mathbb{C}(z)$. Since the local exponents at the finite points are ≥ 0 we decide that $y_1 y_2(z)$ is a polynomial in z. It degree in z is given by -2 times the negative local exponent at $z = \infty$, hence n. □

For every value of n the polynomial F_n of Theorem 3.3 can be computed easily. A list of the first few is given in Table 3. For a more extensive list of the F_n we refer to [vdW02, Table A.2].

n	$F_n(z)$
0	1
1	$z - B$
2	$z^2 - \frac{1}{3}Bz + \frac{1}{9}B^2 - \frac{1}{4}g_2$
3	$z^3 - \frac{1}{5}Bz^2 + (\frac{2}{75}B^2 - \frac{1}{4}g_2)z - \frac{1}{225}B^3 + \frac{1}{15}Bg_2 - \frac{1}{4}g_3$

TABLE 3.

A direct consequence of Theorems 3.1 and 3.3 is the following.

Corollary 3.4. *Let n be an integer and suppose that M is finite. Then M is a dihedral group D_N of order $2N$, for a certain $N \geq 3$.* □

4. Algebraic Lamé equations and their monodromy groups

In this section we suppose that the monodromy group M of the Lamé equation is finite. Due to Theorem 3.1, the monodromy group now necessarily acts irreducibly on the solution space. Besides classifying M we also consider its projective group $PM \subset \mathrm{PGL}(2,\mathbb{C})$, that is the quotient group of M and its subgroup of multiples of the identity matrix. Through the classification of finite subgroups of $\mathrm{PGL}(2,\mathbb{C})$ we know that PM is isomorphic to either one of the following groups:

(1) The cyclic group C_k of order $k \geq 1$,
(2) the dihedral group D_k of order $2k \geq 4$,
(3) the alternating groups A_4, A_5, and
(4) the permutation group S_4.

Moreover, in each of these cases we can find an explicit description of the matrix group in [Kle84]. The following theorem is immediate.

Theorem 4.1 (Baldassarri). *Suppose that M is finite. Then the projective group PM is not C_k nor A_4.*

Proof. If M modulo scalars would be C_k, for a certain integer k, the group M itself would be reducible. The statement $PM \neq A_4$ follows from the fact that γ_1, γ_2 and γ_3 still have order two if we consider them as elements of $\mathrm{PGL}(2,\mathbb{C})$ and that A_4 cannot be generated by elements of order two. □

A more refined description of M rather than PM can be given when we use the classification of Shephard and Todd [ST54] of finite complex reflection groups.

Definition 4.2. *A finite complex reflection group is a finite subgroup of* $\mathrm{GL}(m,\mathbb{C})$, *$m \in \mathbb{Z}_{>0}$ which is generated by complex reflections. A complex reflection is a semi-simple element all of whose eigenvalues except one are* 1.

In the following theorem an element $g \in \mathrm{GL}(m,\mathbb{C})$ acts on $\mathbb{C}[x_1,\ldots,x_m]$ via $(x_1,\ldots,x_m)^t \mapsto g(x_1,\ldots,x_m)^t$. The action of g on a polynomial P is denoted by P^g. We define the *ring of invariant polynomials* of a subgroup $G \subset \mathrm{GL}(m,\mathbb{C})$ by

$$\mathbb{C}[x_1,\ldots,x_m]^G := \{P \in \mathbb{C}[x_1,\ldots,x_m] : P^g = P \text{ for all } \; g \in G\}.$$

Theorem 4.3 (Shephard-Todd). *Let G be a finite subgroup of* $\mathrm{GL}(m,\mathbb{C})$. *Then G is a finite complex reflection group if and only if $\mathbb{C}[x_1,\ldots,x_m]^G$ is a polynomial ring freely generated by m polynomials $I_1,\ldots,I_m$.*

Proof. The theorem is proved for any unitary group in Part II of [ST54]. However, there is no restriction in assuming unitarity, as every finite subgroup of $\mathrm{GL}(m,\mathbb{C})$ is conjugate to a unitary group. □

Let G be a finite complex reflection group and $I_1,\ldots,I_m$ be a set of generating invariants. We may assume them to be homogeneous polynomials. Denote the degree of I_i by d_i and suppose that $d_1 \leq d_2 \leq \cdots \leq d_m$. Then the d_i are uniquely determined and they are called *the degrees of G*. In their paper [ST54] Shephard and Todd give a complete classification of all finite complex reflection groups. We can use their classification to list all possible finite monodromy groups that could occur for the Lamé equation. In the case that $m = 2$ we get, after considering the further restriction that M is generated by order 2 reflections, the following list of possibilities.

$$G(N,N/2,2)\ (N\in 2\mathbb{Z}_{>0}),\quad G(N,N,2)\ (N\geq 3),\quad G_{12},\quad G_{13},\quad G_{22}$$

Here $G(N,N/2,2)$ is the group of order $4N$ generated by

$$\begin{pmatrix} e^{2\pi i/N} & 0 \\ 0 & e^{-2\pi i/N}\end{pmatrix},\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix},\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

The degrees of $G(N,N/2,2)$ are 4 and N corresponding to the invariants

$$(x_1x_2)^2,\qquad x_1^N+x_2^N.$$

The projective group of $G(N,N/2,2)$ is the dihedral group $D_{N/2}$ if N is divisible by 4 and D_N otherwise.

The matrix group

$$G(N,N,2):=\left\langle \begin{pmatrix} e^{2\pi i/N} & 0 \\ 0 & e^{-2\pi i/N}\end{pmatrix},\ \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right\rangle,$$

is a dihedral group of order $2N$. The degrees of $G(N,N,2)$ are 2 and N which correspond to the invariant polynomials

$$x_1x_2,\qquad x_1^N+x_2^N.$$

The projective group of $G(N,N,2)$ is either $D_{N/2}$ or D_N, depending on whether N is even or not.

The group G_{12} is generated by the matrices

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1+i \\ 1-i & 0\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1\end{pmatrix}.$$

and is of order 48. Its center consists of the group generated by $-\mathrm{Id}$. The degrees of G_{12} are 6 and 8, corresponding to the invariants

$$x_1^5x_2-x_1x_2^5,\qquad x_1^8+14x_1^4x_2^4+x_2^8.$$

The group G_{13} of order 96 is the matrix group generated by the elements of G_{12} together with $i\cdot\mathrm{Id}$. The degrees of G_{13} are 8 and 12, and the invariant polynomials are given by

$$x_1^8+14x_1^4x_2^4+x_2^8,\qquad (x_1^5x_2-x_1x_2^5)^2.$$

The projective groups of G_{12} of G_{13} are isomorphic to S_4. We call G_{12} and G_{13} the *octahedral groups*.

Finally, the group G_{22} of order 120 is generated by

$$\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix},\frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5-\zeta_5^4 & \zeta_5^2-\zeta_5^3 \\ \zeta_5^2-\zeta_5^3 & \zeta_5^4-\zeta_5\end{pmatrix},\frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^3-\zeta_5 & 1-\zeta_5 \\ \zeta_5^4-1 & \zeta_5^2-\zeta_5^4\end{pmatrix},$$

in which ζ_5 denotes a primitive 5-th root of unity. It has degrees 12 and 30, belonging to the invariants

$$x_1^{11}x_2-11x_1^6x_2^6-x_1x_2^{11},$$

$$x_1^{20}+228x_1^{15}x_2^5+494x_1^{10}x_2^{10}-228x_1^5x_2^{15}+x_2^{20}.$$

The projective image of G_{22} is A_5. We call G_{22} the *icosahedral group*. We have the following Theorem.

Theorem 4.4. *Suppose the Lamé equation $L_{n,B}y = 0$ has monodromy group M. Then*

(1) $M = G(N, N/2, 2) \Rightarrow n \in \{1/2\} + \mathbb{Z}$ *and* $N = 4$,
(2) $M = G(N, N, 2),\ N \geq 3 \Rightarrow n \in \mathbb{Z}$ *and* $N \neq 4$,
(3) $M = G_{12} \Rightarrow n \in \{\pm 1/4\} + \mathbb{Z}$,
(4) $M = G_{13} \Rightarrow n \in \{\pm 1/6\} + \mathbb{Z}$,
(5) $M = G_{22} \Rightarrow n \in \{\pm 1/10, \pm 3/10, \pm 1/6\} + \mathbb{Z}$.

Moreover, if M is finite and $n \geq -1/2$, then $n > 0$.

Proof. First we prove the cases (1) and (2) of the Theorem.

Case (1), $M = G(N, N/2, 2)$. Since this group has an invariant of degree 4, the fourth symmetric power of the Lamé equation should have a rational solution in z, and since all local exponents at the finite points are ≥ 0 this solution is a polynomial in z. This implies that at least one of the local exponents $-2n, -n + 1/2, 1, n + 3/2, 2n + 2$ of the fourth symmetric power at ∞ is an integer ≤ 0. Hence n is either integral or half integral. The group $G(N, N/2, 2)$ is not contained in a dihedral linear group as otherwise there would be a group invariant of degree 2. So n cannot be integral. We obtain that n is a half integer and that we are in the situation of Theorem 2.2. As the projective group PM is V_4 by this Theorem, we conclude in addition that $N = 4$.

Case (2), $M = G(N, N, 2)$. As we remarked before, this is the dihedral group of order $2N$ as given in the previous section. In this case there is an invariant polynomial of degree 2. Hence the second symmetric power of the Lamé equation has a polynomial solution in z. This implies that at least one of its exponents $-n$, $1/2$ and $2n + 1$ at ∞ is an integer and ≤ 0. It follows that n is integral.

In addition, suppose there exists a Lamé equation with $M \cong D_4$. Let $y_1(z), y_2(z)$ be two solutions with respect to which the group acquires our standard form as given in Theorem 3.3. Then $y_1 y_2(z)$ and $(y_1^4 + y_2^4)(z)$ are invariant under monodromy and hence should be polynomials in z. We relate these solutions to their expansions around $z = \infty$. One has $y_1 = z^{n/2} f_1(1/z) + z^{-(n+1)/2} f_2(1/z)$, where f_1 and f_2 are suitable powerseries satisfying $f_1(0) f_2(0) \neq 0$. The element γ_∞ maps y_1 to y_2, hence $y_2 = (-1)^n (z^{n/2} f_1(1/z) - z^{-(n+1)/2} f_2(1/z))$. A brief computation shows that $y_1^4 + y_2^4 - 2(y_1 y_2)^2$ is equal to $16 z^{-1} f_1(1/z)^2 f_2(1/z)^2$. We conclude that $y_1^4 + y_2^4 - (y_1 y_2)^2$ is a polynomial in z that can be expanded in a series in positive powers of $1/z$. Therefore, it should vanish identically. This, however, is impossible since y_1 and y_2 are supposed to be linearly independent solutions. It follows that $G(4, 4, 2)$ is not the monodromy group of a Lamé equation.

Next we deal with the remaining cases (3), (4) and (5). The groups G_{12}, G_{13} and G_{22} are not subgroups of the infinite dihedral group D_∞. Therefore we can now assume that n is not an integer. We use various arguments to determine the order of the natural projection $\overline{\gamma}_\infty$ of γ_∞ in PM. If this order is N say, then this means that the exponent difference $n + 1/2$ at ∞ should be a rational number with N as denominator. From this the value of n modulo $\mathbb{Z}$ can be determined for each of the remaining groups.

Case (3), $M = G_{12}$. The group PM then is isomorphic to S_4. We identify PM with S_4 to simplify notation. On G_{12} we distinguish two characters. The first is σ, that is the sign of the permutation of M modulo scalars. The second is δ, the determinant of the elements of M. We are interested in the product character $\sigma\delta$. Take an element $g \in G_{12}$ which has order 3 in PM. Clearly this corresponds to a 3-cycle and hence $\sigma(g) = 1$. Moreover, g^3 is a scalar multiple of Id. Since all scalar matrices in G_{12} have determinant 1, we conclude $\delta(g) = 1$. One therefore has $\sigma\delta(g) = 1$ for every $g \in G_{12}$ that has a 3-cycle as projection.

The group generated by the 3-cycles in S_4 is precisely A_4. Furthermore, G_{12} is generated by its scalar matrices $\pm\,\mathrm{Id}$ and all matrices being 2-cycles in PM. Since G_{12} contains matrices of determinant -1, at least one element $h \in M$ with a 2-cycle as projection must have determinant -1. We then have $\sigma\delta(h) = 1$. The group generated by A_4 and the image of h in S_4 is of course S_4. We conclude that $\sigma\delta(g) = 1$ for all $g \in G_{12}$.

Note that the γ_i $(i = 1, 2, 3)$ have order two and determinant -1. From the discussion above they should correspond to 2-cycles in PM. Hence γ_∞, being the product $\gamma_3\gamma_2\gamma_1$, corresponds to an odd permutation. So the order of $\overline{\gamma}_\infty$ is 2 or 4. Therefore the exponent difference $n + 1/2$ at ∞ is contained in either $\{1/2\} + \mathbb{Z}$ or $\{\pm 1/4\} + \mathbb{Z}$. The first case cannot occur since then n would be an integer. So we are left with $n \in \{\pm 1/4\} + \mathbb{Z}$.

Case (4), $M = G_{13}$. This group is simply G_{12} extended by the scalar matrix $\mathrm{i}\cdot\mathrm{Id}$. On G_{13} we can define the characters σ, δ as in the proof of case (3). Notice that then $\sigma\delta(\mathrm{i}\cdot\mathrm{Id}) = -1$ holds. So G_{12} can be considered as the kernel of the character $\sigma\delta$ on G_{13}. Since γ_1, γ_2 and γ_3 generate G_{13}, at least one of these generators must have $\sigma\delta$-value -1. Their determinants are all -1. It follows that at least one of the matrices $\gamma_1, \gamma_2, \gamma_3$ corresponds to an even permutation in S_4. This permutation is necessarily a product of 2-cycles, since each of the γ_i's has eigenvalues -1 and 1. Also, at least one of the projective matrices of the remaining γ_i's is a 2-cycle as the γ_i's generate G_{13}. It is not hard to verify that S_4 cannot be generated by its subgroup V_4 and a 2-cycle. Therefore we conclude that the projections of two of the matrices $\gamma_1, \gamma_2, \gamma_3$ are distinct 2-cycles and that the third is a non-trivial element of V_4. Consequently, the element $\overline{\gamma}_\infty$ is an even permutation and hence is of order 2 or 3. This yields $n + 1/2 \in \{1/2\} + \mathbb{Z}$ or $n + 1/2 \in \pm\{1/3\} + \mathbb{Z}$. The first case implies $n \in \mathbb{Z}$ which we had excluded earlier. The second case is precisely our assertion.

Case (5), $M = G_{22}$. Modulo scalars this group is isomorphic to A_5. A non-trivial element of A_5 has order 2, 3 or 5. This means that $n + 1/2$ is contained in $\{1/2\} + \mathbb{Z}$, $\{\pm 1/3\} + \mathbb{Z}$, $\{\pm 1/5\} + \mathbb{Z}$ or $\{\pm 2/5\} + \mathbb{Z}$. The first case cannot occur since then $n \in \mathbb{Z}$. The remaining cases precisely form the assertion of our Theorem. This finishes the proof of the cases (1) upto (5).

To finish the proof of the Theorem we suppose now that M is finite and assume $-1/2 \leq n \leq 0$. Then both local exponents at ∞ are ≥ 0. This implies that there exists an algebraic solution that is everywhere bounded. Hence it should be constant. This contradicts the irreducibility of our equation. So one has $n > 0$, as asserted. □

Partial results in the direction of Theorem 4.4 were obtained by Baldassarri [Bal81] and Chiarellotto [Chi95]. In [Chi95] there is a method given to count the number of inequivalent Lamé equations with given dihedral monodromy group. To warn the reader and to avoid confusion it should be pointed out, however, that Chiarellotto only considers monodromy modulo scalars. Since, the monodromy groups D_N and $D_{N/2}$ for $N \equiv 2 (\mathrm{mod}\ 4)$ give rise to the same projective monodromy group, they are not distinguished by the counting procedure in [Chi95]. For example, from [Chi95] it follows that there are two distinct Lamé equations with $n = 1$ and projective dihedral monodromy group of order 10. In [vdW02] the author gets only one Lamé equation with exact monodromy D_5, whereas the other intended solution has dihedral monodromy of order 20. Also, it is mentioned in [Chi95] that there is one case with $n = 1$ and projective dihedral monodromy group of order 8. Theorem 4.4 states that D_4 cannot occur as monodromy group. The group alluded to in [Chi95] is in fact dihedral of order 16. Recently Litcanu [Lit02] reconsidered the results of [Chi95] by studying Grothendieck's "dessins d'enfants". He also considered the counting problem for other groups than the dihedral ones.

Finally we point out an error in [Bal81]. There it is stated that the octahedral group does not occur for $n \in \{\pm 1/6\} + \mathbb{Z}$. However, the Lamé equation with $g_2 = 1$, $g_3 = 0$, $B = 0$ and $n = 1/6$ has octahedral monodromy, as it is the rational pull-back of the hypergeometric equation

$$x(x-1)y'' + (5x/4 - 3/4)y' - (7/24^2)y = 0$$

by the substitution $x = z^2$. The latter hypergeometric equation has octahedral monodromy. This error was noticed independently by R.S. Maier in [Mai02].

A question that remains to be answered is whether for any choice of group and n, as allowed by Theorem 4.4, there exists a Lamé equation. We shall not give a complete answer but prove in Theorem 5.1 that G_{12}, G_{13} and G_{22} indeed are monodromy groups of Lamé equations when we take the smallest positive representatives of each of the possible classes $n (\mathrm{mod}\ \mathbb{Z})$. For the dihedral groups we prove a similar result in Theorem 6.1. We expect that with increasing n the number of algebraic Lamé equation also increases.

Besides showing the existence of an algebraic Lamé equation with given M and n one could take this one step further and try to determine the number of these equations. Notice that if we replace z by λz in the Lamé equation, we get a new Lamé equation with parameters $B/\lambda, g_2/\lambda^2, g_3/\lambda^3$ instead of B, g_2, g_3. We call two such Lamé equations *scalar equivalent*. We have the following fact.

Proposition 4.5. *Suppose $n \neq 0$. Let $L_{n,B}y = 0$ and $L_{m,C}y = 0$ be two Lamé equations which are equivalent with respect to the relation $\sim$ defined in Section 1. Then $L_{n.B}$ and $L_{m,C}$ are scalar equivalent. Moreover, one has $n = m$.*

Proof. For a proof we refer to [vdW02, Prop. 5.3.1]. □

The following result is a direct consequence of Theorem 7.1 and the above Proposition.

Theorem 4.6. *The number of scalar equivalence classes of Lamé equations with given finite monodromy group M and parameter $n \notin 1/2 + \mathbb{Z}$ is at most finite.* □

This Theorem also occurs in [Lit02, Thm 4.1] and [vdW02, Thms 5.4.4, 6.7.9]. It seems that B. Dwork has also given a proof, but never published it, see [MR99, Prop. 2.8]. In the latter reference the authors should probably have added: $2n \notin \mathbb{Z}$.

5. Enumeration of algebraic Lamé equations

In this section we describe a computational method to obtain all algebraic Lamé equations for a given monodromy group $M \in \{G_{12}, G_{13}, G_{22}\}$ and fixed n. In particular this algorithm leads to the following Theorem.

Theorem 5.1. *For each of the following pairs of M, n there exists a Lamé equation:*

(1) $M = G_{12}$ *with* $n = 1/4, 3/4$,

(2) $M = G13$ *with* $n = 1/6, 5/6$,
(3) $M = G_{22}$ *with* $n = 1/10, 3/10, 7/10, 9/10, 1/6, 5/6$.

To prove Theorem 5.1 it suffices to exhibit some examples and this is what we shall do. At the same time we describe the methods used to calculate them.

Proof of Theorem 5.1. The desired examples can be recovered from the following table, Table 4.

	Group	n	B	g_2	g_3
I	G_{12}	1/4	0	0	1
II	G_{12}	3/4	3/8	-168	622
III	G_{13}	1/6	0	1	0
IV	G_{13}	5/6	0	1	0
V	G_{22}	1/6	1/6	60	90
VI	G_{22}	5/6	1	α	β
VII	G_{22}	1/10	0	0	1
VIII	G_{22}	3/10	3/100	3	5/4
IX	G_{22}	7/10	0	0	1
X	G_{22}	9/10	1	γ	δ

TABLE 4.

In this table α is any zero of

$$\begin{aligned} &5338003509875218233256058888 - 2607006811617311385916800 0x \\ &+4600758635129502288000000x^2 - 3197796744112375500000x^3 \\ &+3513693527268750000x^4 + 35232241173828125x^5. \end{aligned}$$

The number β then satisfies

$$\begin{aligned} \beta \;=\; & -(1/33989399094889069443707636943360) \times \\ & \times(612667142673340661012556043831296 \\ & +8665098171510473298454731173107 2\alpha \\ & -72836812645800526557682140600 0\alpha^2 \\ & +420474326606171497609110000 0\alpha^3 \\ & -11767969246646316120921875\alpha^4). \end{aligned}$$

Although example VI looks horrifying, it is the simplest we could find with G_{22} and $n = 5/6$. Furthermore, γ is any zero of

$$\begin{aligned} &1874362548828125 - 34160832421875x \\ &+508353114375x^2 + 2235378627x^3 \end{aligned}$$

and

$$\begin{aligned} \delta \;=\; & (-1/11494507500) \times (354707421875 \\ & +34290961875\gamma + 21619224\gamma^2). \end{aligned}$$

Whenever we have an algebraic Lamé equation with group M that is not dihedral, we can verify the data in Table 4 in the following way. Let d be the degree of the lowest invariant of M. Compute the symmetric powers of the Lamé equation up to d-th power and verify that the d-th symmetric power is the first which has a polynomial solution, which we denote by $P(z)$. Nowadays this can be done in several computer algebra packages, for example in Maple 6 and higher versions. In Table 5 the results for the ten cases above are given.

	d	$P(z)$
I	6	1
II	6	$z - 11/6$
III	6	1
IV	6	$z^2 + 1/320$
V	12	$z + 3/2$
VI	12	polynomial of degree 5
VII	12	1
VIII	12	$z + 5/12$
IX	12	$z^3 - 16/189$
X	12	polynomial of degree 4

TABLE 5.

We have to mention how the examples were found. In Chapter 6 of [vdW02] for each choice of M an algorithm is given, that produces for every input n the list of Lamé equations with monodromy group M and parameter n. Here we give only one example of such a construction. All the other cases that are mentioned in Theorem 5.1, are obtained by performing similar calculations.

We want to determine all algebraic Lamé equations with parameter $n = 3/10$. According to Theorem 4.4 the monodromy group must be G_{22}. This group has an invariant of degree 12. Let $y_1(z), y_2(z)$ be two local solutions around infinity that correspond to the local exponents at ∞. The explicit solutions read

$$\begin{aligned} y_1(z) &= z^{3/20}\left(1 + \frac{5B}{4}\frac{1}{z} + \left(\frac{25B^2}{192} - \frac{7g_2}{1280}\right)\frac{1}{z^2} + \cdots\right) \\ y_2(z) &= z^{-13/20}\left(1 + \frac{5B}{36}\frac{1}{z} + \left(\frac{25B^2}{4032} + \frac{299g_2}{8960}\right)\frac{1}{z^2} + \cdots\right). \end{aligned}$$

There exists a binary form $I(x_1, x_2)$ of degree 12 such that $I(y_1, y_2)(z)$ is invariant under monodromy. Hence it is a rational function in z. Moreover, since the local exponents at all finite points are non-negative, we have $I(y_1, y_2)(z) \in \mathbb{C}[z]$. The only degree twelve monomials that occur in $I(y_1, y_2)$ are therefore, $y_1^{11}y_2, y_1^6y_2^6, y_1y_2^{11}$. The others all contain non-integral fractional powers of z.

We must find α and β such that $I = y_1^{11}y_2 + \alpha y_1^6y_2^6 + \beta y_1y_2^{11} \in \mathbb{C}[z]$. Notice that the three relevant monomials of each term are of order $-1, 3, 7$ in $1/z$,respectively.

Up to order $1/z^3$ one has

$$\begin{aligned} I &= z + \frac{125B}{9} + \frac{10000B^2 - 3g_2}{112}\frac{1}{z} + \\ &\quad + \frac{750000B^3 + 650Bg_2 - 63g_3}{2128}\frac{1}{z^2} + O(\frac{1}{z^3}) \end{aligned}$$

The coefficients of $1/z$ and $1/z^2$ must be zero. Recall that through the substition $z \to \lambda z$ in the Lamé equation the parameter B changes into B/λ. Hence, after suitable normalisation we can assume that B has some arbitrarily given value. We take $B = 3/100$. It then follows from the vanishing of our two coefficients that $g_2 = 3$ and $g_3 = 5/4$. □

Notice that the general approach of solving g_2 and g_3 first for a given non-zero B and then again for the case that $B = 0$ would have given all algebraic Lamé equations with given M and n. However, in our example the assumption $B = 0$ leads to $g_2 = g_3 = 0$, which is impossible by the definition of $p(z)$.

6. Existence of Lamé equations with given dihedral group.

It might be possible to extend the counting methods as given in [Chi95] and [Lit02] to establish a count for Lamé equations having exact monodromy group D_N, but we do not pursue this here. Instead we like to make use of explicit solutions to the Lamé equation using elliptic functions. The main goal in this section is to prove the following Theorem.

Theorem 6.1. *For each integer $N \geq 3$, $N \neq 4, 6$ there is a Lamé equation with $n = 1$ having a dihedral monodromy group of order $2N$. There is a Lamé equation with dihedral group of order 12 when $n = 2$.*

The remainder of this section is devoted to a proof of Theorem 6.1.

The elliptic curve

$$v^2 = 4z^3 - g_2 z - g_3$$

can be parametrised by using the standard Weierstrass function $\wp$ so that

$$\begin{aligned} z &= \wp(u), \\ v &= \wp'(u). \end{aligned}$$

We denote the associated period lattice by Λ. To this lattice also the Weierstrass functions $\zeta(u)$ and $\sigma(u)$ are associated, where

$$\sigma(u) = u \prod_{\omega \in \Lambda} \left(1 - \frac{u}{\omega}\right) \exp\left(\frac{u}{\omega} + \frac{u^2}{2\omega^2}\right)$$

and

$$\zeta(u) = \frac{\sigma'(u)}{\sigma(u)}.$$

Furthermore, one has $\wp(u) = -\zeta'(u)$. All this and more can be found in [WW50, Ch. XX]. The reader should bear in mind though that in [WW50] the symbols ω_1, ω_2 denote half-periods, whereas we use them to denote a basis of Λ.

The Lamé equation can be rewritten using the substitution $z = \wp(u)$ as

$$\frac{d^2y}{du^2} - (n(n+1)\wp(u) + B)y = 0. \tag{3}$$

Using elementary properties of elliptic functions one can show the following proposition.

Proposition 6.2. *Let Λ be a rank two lattice in $\mathbb{C}$ and $\wp(u)$ the corresponding Weierstrass function. Let $B \in \mathbb{C}$ and choose $a \in \mathbb{C}$ such that $B = \wp(a)$ Then a solution to*

$$\frac{d^2y}{du^2} - (2\wp(u) + B)y = 0$$

is given by

$$\mathrm{e}^{u\zeta(a)}\frac{\sigma(u-a)}{\sigma(u)}. \tag{4}$$

Moreover, if a is not a half period, then an independent solution can be obtained by changing a into $-a$.

To verify that the proposed solution satisfies our differential equation is an exercise in elliptic functions. In fact, solving the Lamé equation with $n = 2$ in terms of σ-functions was part of the Cambridge Math. Tripos 1912, see [WW50, p.459].

If we denote solution (4) by $f_a(u)$, the transformation properties of the σ-function ensure

$$f_a(u+\omega) = \mathrm{e}^{\zeta(a)\omega - a\eta(\omega)}f_a(u), \tag{5}$$

where $\eta(\omega)$ is the quasi-period defined by $\zeta(u+\omega) - \zeta(u) = \eta(\omega)$.

The monodromy group of Equation (3) should now be the abelian group H discussed in the proof of Theorem 3.3. The eigenvalues of the elements of H consist of the multipliers of the functional equation (5). A set of generators for this multiplier group can be obtained by taking a basis ω_1, ω_2 of Λ and computing

$$e^{\zeta(a)\omega_1 - a\eta_1}, \qquad e^{\zeta(a)\omega_2 - a\eta_2},$$

where $\eta_1 = \eta(\omega_1), \eta_2 = \eta(\omega_2)$. We now choose $a \in \mathbb{C}$ and Λ in such a way that the above to numbers generate the cyclic group of order N. Hence,

$$\zeta(a)\omega_1 - a\eta_1 = \frac{-2\pi\mathrm{i}k_2}{N}, \qquad \zeta(a)\omega_2 - a\eta_2 = \frac{2\pi\mathrm{i}k_1}{N}$$

for some integers k_1 and k_2 with $\gcd(k_1, k_2, N) = 1$. If we consider this as a system of equations in a and $\zeta(a)$, its solution yields

$$\zeta(a) = \frac{\eta_1 k_1 + \eta_2 k_2}{N}, \quad a = \frac{\omega_1 k_1 + \omega_2 k_2}{N}.$$

We conclude that a is a torsion point of precise order N.

In the following we display the dependence of $\zeta(u)$ on Λ more explicitly, by rewriting $\zeta(u)$ as $\zeta(u, \omega_1, \omega_2)$. It now follows that our Lamé equation has a dihedral group of order $2N$ when

$$\zeta\left(\frac{k_1\omega_1 + k_2\omega_2}{N}, \omega_1, \omega_2\right) = \frac{k_1\eta_1 + k_2\eta_2}{N}$$

is satisfied. In Section 6 of [Hec26] E. Hecke studies functions of the form

$$Z\left(\omega_1,\omega_2,\frac{k_1}{N},\frac{k_2}{N}\right)=\zeta\left(\frac{k_1\omega_1+k_2\omega_2}{N},\omega_1,\omega_2\right)-\frac{k_1\eta_1+k_2\eta_2}{N}.$$

If we write $\tau=\omega_1/\omega_2$ and assume $\Im(\tau)>0$, we get that

$$Z\left(\tau,\frac{k_1}{N},\frac{k_2}{N}\right):=\frac{\omega_2 Z(\omega_1,\omega_2,k_1/N,k_2/N)}{2\pi \mathrm{i}}$$

is a modular form of weight 1 with respect to the congruence subgroup $\Gamma(N)$ consisting of all matrices in $\mathrm{SL}(2,\mathbb{Z})$ which are Id modulo N. Moreover, $Z(\tau,k_1/N,k_2/N)$ only depends on k_1,k_2 modulo N. It is also shown in [Hec26] that the different $Z(\tau,k_1/N,k_2/N)$ are permuted under the action of $\mathrm{SL}(2,\mathbb{Z})$. The value of $Z(\tau,k_1/N,k_2/N)$ at $\tau=\infty$ is given by $k_1/N-1/2$ if $0<k_1<N$ and $\cot(k_2\pi/N)/2i$ when $k_1=0$.

To determine the Lamé equations with $n=1$ and monodromy group D_N, we must look for zeros of the functions $Z(\tau,k_1/N,k_2/N)$. To this end we simply take the product of $Z(\tau,k_1/N,k_2/N)$ over all k_1,k_2 with $0\le k_1,k_2<N$ and $\gcd(k_1,k_2,N)=1$. We denote this product by $Z(\tau)$ and let $\Psi(N)$ be the number of factors. Then $Z(\tau)$ is a modular form with respect to $\mathrm{SL}(2,\mathbb{Z})$ of weight $\Psi(N)$. Denote by $\nu_P(Z)$ the zero-multiplicity of $Z(\tau)$ at P in a fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ (including ∞). We have the well-known formula

$$\nu_\infty(Z)+\frac{1}{2}\nu_{\mathrm{i}}(Z)+\frac{1}{3}\nu_\rho(Z)+\sum_{P\neq\infty,\mathrm{i},\rho}\nu_P(Z)=\frac{\Psi(N)}{12},$$

where ρ is a primitive root of unity of order 3. If we can show that $\nu_\infty(Z)<\Psi(N)/12$, then we are done. It is well-known that $\Psi(N)=N^2\prod_p(1-1/p^2)$, where the product is taken over all prime divisors of N.

We finally compute the contributions of each factor to $\nu_\infty(Z)$. The value of $Z(\tau,k_1/N,k_2/N)$ at $\tau=\infty$ is given by $k_1/N-1/2$ if $0<k_1<N$ and $\cot(k_2\pi/N)/2\mathrm{i}$ for $k_1=0$ (see [Hec26]). So the only factors which contribute something to the zero order are the ones with $k_1/N=1/2$. In these cases we deduce from [Hec26] that the local q-expansion of $Z(\tau,k_1/N,k_2/N)$ starts with $q^{1/2}$. So, when N is odd there is no zero of $Z(\tau)$ at ∞. When N is even, the zero-multiplicity of $Z(\tau)$ at ∞ is the number of integers k_2 such that $0\le k_2<N/2$ and $\gcd(k_2,N/2)=1$, i.e $\phi(N/2)$, where ϕ is Euler's ϕ-function. If we define the ϕ-value of a non-integer to be zero, we finally have that $\nu_\infty(Z)=\phi(N/2)$.

Clearly when N is an odd integer ≥ 3, then $\Psi(N)-\phi(N/2)>0$, so Theorem 6.1 is proved. When N is even, by making use of $\phi(N/2)\le\phi(N)$ we obtain

$$\frac{\Psi(N)}{12}-\phi(N/2)\ge\frac{\Psi(N)}{12}-\phi(N)=\frac{\phi(N)}{12}\left(N\prod_p\left(1+\frac{1}{p}\right)-12\right).$$

The latter quantity is clearly positive for $N\ge 12$. However, checking all integers $N<1$ implies $\Psi(N)/12-\phi(N/2)>0$ for $N=8$ and $N=10$ as well. Thus we have proved the first statement of Theorem 6.1.

By being a bit more careful it might be possible to give a counting formula for the number of Lamé equations with $n=1$ and monodromy D_N. It turns out that for N up to 10, the formula $\Psi(N)/12-\phi(N/2)$ gives twice the correct answer. One may realise that we could have taken our product over all pairs k_1,k_2 modulo a common factor ± 1. This corresponds to the fact that the points a and $-a$ belong to the same Lamé equation. But we will not make this any more precise here.

Finally, we must display a Lamé equation with D_6 as its monodromy group. The choice $n = 2, B = 21, g_2 = 327, g_3 = 1727$ provides such an example by using the techniques as explained in the previous section. The second polynomial solution (next to $(y_1 y_2)^3$) of the 6-th symmetric power of the Lamé equation is $z - 11$. This finishes the proof of Theorem 6.1.

7. The number of equivalent equations

In this section we discuss a refinement of Theorem 1.1 and its proof. To any element $Ly = 0$ of $\mathcal{A}_0$ we associate the number $\delta(L)$, which is the sum of all local exponent differences $\neq 1$ of $Ly = 0$. By $\mathcal{A}_0(r)$ we denote the set of equation $Ly = 0$ in $\mathcal{A}_0$ such that $\delta(L) \leq r$. Since $L \sim L'$ implies $\delta(L) = \delta(L')$, we see that $\sim$ is also an equivalence relation on $\mathcal{A}_0(r)$. Notice also that two equivalent equations have the same projective monodromy group. We have

Theorem 7.1. *Let $G \subset \mathrm{PGL}(2, \mathbb{C})$ be a given finite group and $r \in \mathbf{R}_{\geq 0}$. Then the number of elements in $\mathcal{A}_0(r)/\sim$ having projective monodromy group G is finite.*

Proof. Given a linear differential equation from $\mathcal{A}_0$, let $M \subset \mathrm{GL}(2, \mathbb{C})$ be its finite Galois group. The conjugacy class of M in $\mathrm{GL}(2, \mathbb{C})$ depends on the choice of a local basis y_1, y_2 of solutions with respect to which M is determined. According to F. Klein's work, y_1 and y_2 can be chosen in such a way that M modulo scalars is one of a concrete list of possible groups in $\mathrm{PGL}(2, \mathbb{C})$. They are the cyclic group C_N of order N, the dihedral group D_N of order $2N$, the tetrahedral group A_4, the octahedral group S_4 and the icosahedral group A_5. We can assume that G is one of these groups. A rational function $f(z)$ is called *G-invariant* when $f(\frac{az+b}{cz+d}) = f(z)$ holds for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. The G-invariant rational functions form a subfield of $\mathbb{C}(z)$ that we denote by $\mathbb{C}(z)^G$. Klein constructed an explicit rational function $j_G(z) \in \mathbb{C}(z)$ for each G, such that the field $\mathbb{C}(j_G(z))$ is $\mathbb{C}(z)^G$. Moreover, it is shown in [Kle84] that $j_G(z)$ ramifies only in the points above 0,1 and ∞.

Next we consider the composite function

$$R(z) := j_G(y_1/y_2)(z)$$

on $\mathbb{P}^1$. Then $R(z)$ is invariant under monodromy, hence is a meromorphic function on $\mathbb{P}^1$, i.e. $R(z) \in \mathbb{C}(z)$. For a point $z_0 \in \mathbb{P}^1$ the ramification order of $R(z)$ at z_0 is equal to the local exponent difference of $Ly = 0$ at z_0, multiplied by the ramification order of j_G at $y_1(z_0)/y_2(z_0)$. In particular this implies that any point z_0 where the local exponent difference is not an integer, is mapped to a ramification point of j_G by $z_0 \mapsto y_1(z_0)/y_2(z_0)$. Since j_G ramifies only above $0, 1, \infty$, we conclude $R(z_0) \in \{0, 1, \infty\}$.

Let z_0 be any point that satisfies $R(z_0) \neq 0, 1, \infty$. Then z_0 must have integral exponent difference. Since our equation is pure this difference is 1 and therefore $R(z)$ is unramified in z_0. We conclude that $R(z)$ is a so-called Belyi-function, a rational function $R : \mathbb{P}^1 \to \mathbb{P}^1$ such that R ramifies only above $0, 1, \infty$.

From [Kle84] it follows that two branch points $\alpha_1, \alpha_2 \in \mathbb{P}^1$ of j_G, that satisfy $j_G(\alpha_1) = j_G(\alpha_2)$, have the same multiplicities. We denote the ramification indices of j_G above 0, 1 and ∞ by λ_0, λ_1 and λ_∞. From [Kle84] we know that $(\lambda_0, \lambda_1, \lambda_\infty)$

is a permutation of one of the triples $(1, N, N), (2, 2, N), (2, 3, 3), (2, 3, 4), (2, 3, 5)$, depending on G. Let $\lambda_{\max}$ be the maximum of $\lambda_0, \lambda_1, \lambda_\infty$. Is D denotes the degree of $R(z)$, then the Riemann-Hurwitz Theorem yields

$$2D - 2 = \sum_i (e_i - 1),$$

where the e_i are the ramification indices of $R(z)$. As we have seen, all ramification takes place above $0, 1, \infty$. The contribution to $\sum_i (e_i - 1)$ of the singular points of $Ly = 0$ can be bounded above by $\lambda_{\max}\delta(L)$. The contribution of the non-singular points can be bounded by

$$\sum_{i=0,1,\infty} \frac{D}{\lambda_i}(\lambda_i - 1).$$

Hence

$$2D - 2 \le \delta(L)\lambda_{\max} + (3 - \sum_i 1/\lambda_i)D.$$

So we find that

$$(-1 + \sum_i 1/\lambda_i)D \le \delta(L)\lambda_{\max} + 2.$$

Since $-1 + \sum_i 1/\lambda_i > 0$, the latter inequality gives an upper bound on D depending only on the group G and $\delta(L) \le r$.

According to [Sch94, Lemma I.1] the set of Belyi functions of bounded degree is finite when we consider two Belyi-functions $f(z), f(\phi(z))$ as equivalent for every projective linear map ϕ. Therefore the set of ratios $y_1(z)/y_2(z)$ modulo fractional linear transformations in z is finite.

Finally, suppose that two differential equations $\tilde{L}y = 0$ and $Ly = 0$ give rise to the same quotient

$$y_1/y_2 = \tilde{y}_1/\tilde{y}_2. \tag{6}$$

We have to prove that these equations are equivalent. If we differentiate both sides of Equation (6), then we obtain $W/y_2^2 = \tilde{W}/\tilde{y}_2^2$, where W and $\tilde{W}$ are the Wronskian determinants of the differential equations. For example, one has $W(z) = y_1'y_2 - y_1y_2'$. It is well-known that $W(z) = S(z)^a$ for some $S(z) \in \mathbb{C}(z)$ and $a \in \mathbb{Q}$. Similarly we have $\tilde{W}(z) = \tilde{S}(z)^{\tilde{a}}$. This implies $\tilde{y}_2 = \tilde{S}^{\tilde{a}/2}S(z)^{-a/2}y_2$ and a similar result for y_1. We conclude that $Ly = 0$ and $\tilde{L}y = 0$ are equivalent equations. Hence, up to equivalence the set of equations in $\mathcal{A}_0(r)$ is finite, as asserted. □

The set $\mathcal{A}_0$ is a countable union of sets $\mathcal{A}_0(r)$ with prescribed projective monodromy. Theorem 7.1 therefore immediately implies Theorem 1.1.

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