Periods

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These are the slides of a lecture held at the Luminy meeting on Diophantine Approximation on May 6-11, 2002. We discuss only that part of the above mentioned paper which concerns algebraic independence of periods. On this matter the authors make an interesting conjecture which we shall explain and provide some additional examples for.

Of course reading the other part of the paper (section 3) is also highly recommended. It is a beautiful and rich paper.

Definition 1 A *period* is a complex number whose real and imaginary part are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial (in)equalities with rational coefficients.

Examples

$$\sqrt{2} = \int_{2x^2 \le 1} dx$$

$$\pi = \int\int_{x^2+y^2 \le 1} dx dy$$

$$\log(2) = \int_1^2 \frac{dx}{x}$$

$$\zeta(3) = \int\int\int_{0<x<y<z<1} \frac{dx dy dz}{(1-x)yz}$$

Products of periods are again periods.

More examples

Periods and quasi-periods on an elliptic curve $E : y^2 = x^3 + ax^2 + bx + c$:

$$\int_C \frac{dx}{y}, \qquad \int_C \frac{x\,dx}{y}$$

Mahler measure of a Laurent polynomial P:

$$\mu(P) = \int \cdots \int_{|x_i|=1} \log|P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

Euler Beta-integral: Let $a = r/N, b = s/N$ and define

$$B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \int_0^1 \frac{dx}{y}$$

where $y^N = x^{N-r}(1-x)^{N-s}$.

In particular, $\Gamma(p/q)^q$ is a period for any $p, q \in \mathbb{N}$, e.g. $\Gamma(1/3)^3 = B(1/3, 1/3)B(2/3, 1/3)$.

Notation: $\mathcal{P}$ denotes the set of $\overline{\mathbb{Q}}$-linear combinations of all periods and $\hat{\mathcal{P}}$ denotes the set

$$\cup_{r=1}^{\infty} \frac{1}{(2\pi i)^r} \mathcal{P}$$

Fact: $\mathcal{P}$ is a countable set.

$$\overline{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C}$$

Problem Construct a number not belonging to $\mathcal{P}$

Question $e \in \mathcal{P}$? What about $1/\pi$ and Euler's constant?

Bonus example (Beilinson-Deninger-Scholl). Let f be a modular form of weight $k \geq 2$ defined over $\overline{\mathbb{Q}}$. Then $L(f, m) \in \hat{\mathcal{P}}$ for all integers m.

Period relation 1 $\log(4) = 2\log(2)$

$$\begin{aligned} \log(4) &= \int_1^4 \frac{dx}{x} \\ &= \int_1^2 \frac{dx}{x} + \int_2^4 \frac{dx}{x} \\ &= \int_1^2 \frac{dx}{x} + \int_1^2 \frac{dx}{x} = 2\log(2) \end{aligned}$$

Period relation 2: $6\zeta(2) = \pi^2$ (Calabi)

$$I = \int_0^1 \int_0^1 \frac{1}{1 - xy} \frac{dx dy}{\sqrt{xy}}$$

and note $I = 3\zeta(2)$. On the other hand, substitute

$$x = \xi^2 \frac{1 + \eta^2}{1 + \xi^2}, \quad y = \eta^2 \frac{1 + \xi^2}{1 + \eta^2}$$

and deduce

$$I = 2 \int_0^\infty \frac{d\xi}{1 + \xi^2} \int_0^\infty \frac{d\eta}{1 + \eta^2} = \frac{\pi^2}{2}.$$

Rules for passing from one period to the other

1) Additivity

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2) Change of variables, if $y = f(x)$ is invertible change of variables

$$\int_{f(a)}^{f(b)} F(y)dy = \int_a^b F(f(x))f'(x)dx.$$

3) Newton-Leibniz (Stokes in general):

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Conjecture (version 1) If a period has two integral representations, then one can pass from one formula to the other using only rules 1),2),3) in which all functions and domains of integration are algebraic with coefficients in $\overline{\mathbb{Q}}$.

Period relation 3 (Zagier)

$$\mu(x+y+16+1/x+1/y) = \frac{11}{6}\mu(x+y+5+1/x+1/y).$$

Period relation 4 (Legendre)

$E : y^2 = x^3 + ax^2 + bx + c$ and $\omega_1, \omega_2, \eta_1, \eta_2$ periods and quasi-periods of E. Then

$$\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i.$$

Sketch of proof (non-standard): Integrate

$$x_1 \frac{dx_1}{y_1} \wedge \frac{dx_2}{y_2}$$

on $E \times E$ over a 2-cycle homologous to the diagonal of $E \times E$.

Period relation 5 $E_k : y^2 = x(x-1)(x-k^2)$. Periods are given by $K(k)$ and $iK(k')$ where $k' = \sqrt{1-k^2}$ and

$$K(k) = \int_1^\infty \frac{dx}{2y}.$$

$E_{1/\sqrt{2}}$ has CM by $\mathbb{Z}[i]$. Choose k_n be such that E_{k_n} has CM with $End(E) = \mathbb{Z}[ni]$. For example,

$$k_1 = 1/\sqrt{2},\ k_2 = 3-2\sqrt{2},\ k_3 = \frac{1}{2}(\sqrt{3}-1)(\sqrt{2}-3^{1/4}).$$

Then,

$$K(k_n')/K(k_n) = n, \qquad K(k_n)/K(k_1) \in \overline{\mathbb{Q}}.$$

Moreover for all k,

$$F\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, (1-2k^2)^2\right) = \frac{K(k)+K(k')}{2K(1/\sqrt{2})}$$

In particular, if we take $k = k_2 = 3 - 2\sqrt{2}$,

$$F\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 9(11-8\sqrt{2})^2\right) = \frac{3}{8}(2+\sqrt{2}).$$

Definition 2 Let X be a smooth quasiprojective variety and $n \leq \dim(X)$. Let $D \subset X$ be an algebraic subvariety of dimension $n-1$ with normal crossings, and ω a closed algebraic n-form on X, all defined over $\overline{\mathbb{Q}}$. Let γ be the homology class of a singular n-chain on $X(\mathbb{C})$ with boundary contained in $D(\mathbb{C})$. The integral $\int_\gamma \omega$ is called the *numerical period* of the quadruple (X, D, ω, γ).

The space **P** of *effective periods* is defined as the vector space over $\overline{\mathbb{Q}}$ generated by the symbols $[(X, D, \omega, \gamma)]$ representing equivalence classes modulo the following relations:

1. (linearity) $[(X, D, \omega, \gamma)]$ is linear in both ω and γ.

2. (change of variables) If $f : (X_1, D_1) \to (X_2, D_2)$ is a morphism of pairs defined over $\overline{\mathbb{Q}}$, γ_1

relative n-cykel in $X_1 \setminus D_1$ and ω_2 an algebraic closed n-form on X_2 then

$$[(X_1, D_1, f^*\omega_2, \gamma_1)] = [(X_2, D_2, \omega_2, f_*(\gamma_1))].$$

3. (Stokes formula) Denote by $\tilde{D}$ the normalisation of D, the variety $\tilde{D}$ containing a divisor with normal crossings $\tilde{D}_1$ coming from the double points in D. If β is an $n-1$-form, closed on $\tilde{D}$, then

$$[(X, D, d\beta, \gamma)] = [(\tilde{D}, \tilde{D}_1, \beta_{|\tilde{D}}, \partial\gamma)].$$

We call the image of the evaluation homomorphism $[(X, D, \omega, \gamma)] \mapsto \int_\gamma \omega$ from **P** to $\mathbb{C}$ the set $\mathcal{P}$ of *numerical periods*.

Conjecture (version 2) *The evaluation homomorphism* $\mathbf{P} \to \mathcal{P}$ *is an isomorphism.*

I.e. Any polynomial relation between periods can be obtained through manipulation of the defining integrals.

Remark: In the Period paper, Definition 2 and the Conjecture are only stated with ω of top degree $n = \dim(X)$.

Wüstholz' theorem. Let G be a commutative algebraic group of dimension N defined over $\overline{\mathbb{Q}}$ and

$$\phi : \mathbb{C}^N \to \mathbb{C}^N/\Lambda \cong G$$

a parametrization such that the local inverse at the origin is given by

$$P \mapsto (\int_0^P \omega_1, \int_0^P \omega_2, \ldots, \int_0^P \omega_N)$$

where ω_i form a basis of differential 1-forms on G defined over $\overline{\mathbb{Q}}$. Let $\mathbf{u} = (u_1, \ldots, u_N) \in \mathbb{C}^N$, $\mathbf{u} \neq \mathbf{0}$ be such that $\phi(\mathbf{u}) \in G(\overline{\mathbb{Q}})$. Let L_u be the set of $\overline{\mathbb{Q}}$-linear relations between $u_1, \ldots, u_N$ and suppose it is non-trivial. Consider the linear subspace W_u of $\mathbb{C}^N$ defined by

$$W_u = \{(z_1, \ldots, z_N) \mid l(z_1, \ldots, z_N) = 0 \text{ for all } l \in L_u\}$$

Then $H_u := \phi(W_u)$ is a proper algebraic subgroup of G, defined over $\overline{\mathbb{Q}}$.

Corollary Suppose $P \in G(\overline{\mathbb{Q}})$ and

$$a_1 \int_0^P \omega_1 + a_2 \int_0^P \omega_2 + \cdots + a_N \int_0^P \omega_N = 0$$

where $a_1, \ldots, a_N \in \overline{\mathbb{Q}}$ and the integration paths Γ from 0 to P are the same in each integral. Then Wüstholz' theorem implies that there exists a proper algebraic subgroup $H \subset G$ with $\Gamma \subset H$ and such that $a_1\omega_1 + \cdots + a_N\omega_N$ restricted to H is a trivial 1-form. Of course $\int_0^P \text{trivial} = 0$.

Chudnovsky's theorem Let E be a CM elliptic curve defined over $\overline{\mathbb{Q}}$ and let ω be one of its periods. Then the transcendence degree of the field $\overline{\mathbb{Q}}(\omega, \pi)$ over $\overline{\mathbb{Q}}$ is 2.

Gamma relations

Standard relations: (a is assumed rational)

$$\Gamma(a+1) = a\Gamma(a)$$

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+1/2} \Gamma(na).$$

Proof:

$$\frac{\Gamma(a)\Gamma(1)}{\Gamma(a+1)} = \int_0^1 x^{a-1}(1-x)^{1-1} dx = \frac{1}{a}.$$

$$\begin{aligned}
\Gamma(a)\Gamma(1-a) &= \int_0^1 x^{a-1}(1-x)^{-a} dx \\
&= \int_0^1 \left(\frac{x}{1-x}\right)^a \frac{dx}{x} \\
&= \int_0^\infty u^a \frac{du}{1+u}
\end{aligned}$$

Conjecture (Rohrlich): Any multiplicative relation of the form

$$\prod_{a\in\mathbb{Q}} \Gamma(a)^{m_a} \in \pi^{\mathbb{Z}/2}\cdot\overline{\mathbb{Q}},\ \ m_a\in\mathbb{Z}$$

or its square, is generated by the above three relations.

Example

$$\frac{\Gamma(1/3)\Gamma(2/15)}{\Gamma(4/15)\Gamma(1/5)}\in\overline{\mathbb{Q}} \quad \text{P.Das}$$

Formal derivation: Denote product in conjecture formally by $\sum_a m_a[a]$ where $a\in\mathbb{Q}/\mathbb{Z}$. Divide by relations

$$[na]\sim[a]+[a+1/n]+\cdots+[a+(n-1)/n]$$

and

$$[a]+[-a]\sim 0$$

Then, $2[1/3]+2[2/15]-2[4/15]-2[1/5]\sim 0$ (exercise). Another exercise: $[1/3]+[2/15]-[4/15]-[1/5]\not\sim 0$

Alternatively

$$\frac{\Gamma(1/3)\Gamma(2/15)}{\Gamma(4/15)\Gamma(1/5)} \in \overline{\mathbb{Q}} \Rightarrow \frac{B(1/3,2/15)}{B(4/15,1/5)} \in \overline{\mathbb{Q}}$$

$B(1/3,2/15)$ and $B(4/15,1/5)$ are periods of 1-forms on Jacobian J_1 of $y^{15} = x^{10}(1-x)^{13}$ and J_2 of $y^{15} = x^{11}(1-x)^{12}$.

Wüstholz' theorem $\Rightarrow$ $J_1 \times J_2$ contains non-trivial algebraic subgroup, hence J_1 and J_2 contain isogenous factor.

Dilogarithm relations

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

Note:

$$Li_2(1) = \pi^2/6, \quad Li_2(-1) = -\pi^2/12$$

$$Li_2(1/2) = \pi^2/12 - (1/2)\log^2(1/2).$$

Introduce Roger's function

$$L_2(z) = Li_2(z) + \frac{1}{2}\log z \log(1-z)$$

Then,

$$L_2(-z) + L_2(-1/z) = -\pi^2/6$$

$$L_2(z) + L_2(1-z) = \pi^2/6$$

$$\begin{aligned}\pi^2/2 &= L_2(x) + L_2(y) + L_2(1-xy) \\ &\quad + L_2\left(\frac{1-x}{1-xy}\right) + L_2\left(\frac{1-y}{1-xy}\right)\end{aligned}$$

Numerical dilogarithm relations

$$L_2((\sqrt{5}-1)/2) = \pi^2/10, \quad L_2((3-\sqrt{5})/2) = \pi^2/15$$

$$2L_2(\gamma) + L_2(\gamma^2) = 4\pi^2/21,\ \gamma = 2\cos(3\pi/7)$$

$$6L_2(1/3) - L_2(1/9) = \pi^2/3$$

$$3L_2(-1/6) - L_2(1/8) + L_2(1/9) + L_2(1/28) = -\pi^2/12$$

General pattern: F number field and $\mathcal{F}$ the free $\mathbb{Z}$-module generated by the elements of $F^\times$. Denote its elements by $\sum_i n_i[x_i]$, a finite sum with $n_i \in \mathbb{Z}$ and $x_i \in F^\times$. Then $F^\times$ can be considered as $\mathcal{F}$ modulo the relations $[1] = 0$ and $[ab] = [a] + [b]$.

Theorem (Zagier): Suppose that

$$\sum_i n_i[x_i] \wedge [1 - x_i] = 0 \quad \text{in } (\wedge^2 F^\times) \otimes_{\mathbb{Z}} \mathbb{Q}$$

then $\sum_i n_i L_2(x_i) \in \mathbb{Q} \cdot \pi^2$.

Sample derivation:
$6L_2(1/3) - L_2(1/9) \in \mathbb{Q} \cdot \pi^2$

$$\begin{aligned} &6[1/3] \wedge [1-1/3] - [1/9] \wedge [1-1/9] \\ &= 6[1/3] \wedge [2/3] - [1/9] \wedge [8/9] \\ &= -6[3] \wedge ([2]-[3]) + [9] \wedge ([8]-[9]) \\ &= -6[3] \wedge [2] + 6[3] \wedge [2] = 0 \end{aligned}$$

Conjecture (Zagier): Converse also holds.

In particular, all $\mathbb{Q}$-linear relations between L_2-values come from the functional equations.

Let X be an algebraic variety over $\overline{\mathbb{Q}}$ and let Z be an algebraic cycle of dimension r on $X_{\mathbb{C}}$. Let ω be a meromorphic $2r$-form on $X_{\mathbb{C}}$ of the second kind, defined over $\overline{\mathbb{Q}}$. Then

$$\int_Z \omega \in (2\pi i)^r \overline{\mathbb{Q}}.$$

"Grothendieck's conjecture": All period relations arise from algebraic cycles (or more generally: Hodge cycles).

See Yves André's *G-functions*.

Exercise: Prove the above relation using the basic manipulations of Kontsevich-Zagier.