

The equation $x + y = 1$ in finitely generated groups

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1 Introduction

Let H be a finitely generated subgroup of rank r in $(\mathbf{C}^*)^2$. Denote by G the \mathbf{Q} -closure of H , i.e. the subgroup of $(\mathbf{C}^*)^2$ consisting of all pairs $\mathbf{a} = (a_1, a_2) \in (\mathbf{C}^*)^2$ such that $\mathbf{a}^N = (a_1^N, a_2^N) \in H$ for some $N \in \mathbf{N}$. We are interested in an upper bound for the number of solutions $(x, y) \in G$ of the equation

$$x + y = 1 \tag{1}$$

A special case of (1) is obtained if we restrict x and y to the group of so-called S -units in an algebraic number field K . Here S is assumed to be a finite set of places of K including all infinite ones. Supposing that $d = [K : \mathbf{Q}]$, $s = \#S$ and letting $a, b \in K^*$ be fixed, J.H.Evertse [3, Theorem 1] showed that

$$ax + by = 1 \tag{2}$$

has not more than $3 \times 7^{d+2s}$ solutions. Since $s \geq d/2$ this implies that (2) has at most 3×7^{4s} solutions. We can apply this result to equation (1). However, the estimate will depend on the degree of the field containing H , and on s , the number of places for which the elements of H have non-trivial valuation. Note that for fixed r the number s may have arbitrarily large values.

We shall be interested in bounds which depend only on r . The first such uniform result for a general subgroup G of $(\mathbf{C}^*)^2$ was given in [5]. There the bound

$$2^{2^{26}+36r^2}$$

was derived for the number of solutions of equation (1). This was improved in [6] to

$$2^{13r+63r^2}.$$

In this paper we obtain

Theorem 1.1 *Let G be the \mathbf{Q} -closure of a finitely generated subgroup of $(\mathbf{C}^*)^2$ of rank r . Then the equation*

$$x + y = 1, \quad (x, y) \in G$$

has not more than 2^{8r+8} solutions.

Note that this bound, apart from the numerical constants, has the same shape as Evertse's upper bound.

It is well known that a particular application of Theorem 1.1 deals with the multiplicity of binary recurrences. Let $\{u_m\}_{m \in \mathbf{Z}}$ be a sequence of complex numbers satisfying the recurrence relation

$$u_{m+2} = \nu_1 u_{m+1} + \nu_0 u_m$$

with $\nu_0, \nu_1 \in \mathbf{C}$, $\nu_0 \neq 0$. Suppose that we have initial values $(u_0, u_1) \neq (0, 0)$. Write $f(z) = z^2 - \nu_1 z - \nu_0$. Let α, β be its zeros. Note that $\nu_0 \neq 0$ implies $\alpha, \beta \neq 0$. Let us assume that $\alpha \neq \beta$. Then there exist $a, b \in \mathbf{C}$ such that

$$u_m = a\alpha^m + b\beta^m$$

Given $c \in \mathbf{C}$ we are interested in the number of solutions $m \in \mathbf{Z}$ of $u_m = c$. Note that the cases a, b or c equal to zero are uninteresting since they have either at most one solution or infinitely many trivial ones. So we assume they are non-zero. Divide on both sides by c , and from now on we shall be interested in the equation

$$\lambda\alpha^x + \mu\beta^x = 1 \text{ in } x \in \mathbf{Z}, \quad (3)$$

where $\lambda\mu\alpha\beta \neq 0$. We shall also assume that α, β are not both roots of unity.

As a fine point we add that if α, β are roots of unity, then the set

$$\{(\alpha^x, \beta^x) \mid x \text{ solution of (3)}\}$$

consists of at most two elements. This is a consequence of the fact that there exist precisely two triangles in the complex plane two of whose sides have lengths $|\lambda|, |\mu|$, whose third side is the segment $[0, 1]$ and such that the side of length $|\lambda|$ ends in 0.

Straightforward application of Theorem 1.1 with the group H generated by (λ, μ) and (α, β) shows that (3) has not more than 2^{24} solutions. However, one can do much better,

Theorem 1.2 *Under the assumptions just mentioned the equation*

$$\lambda\alpha^x + \mu\beta^x = 1 \quad \text{in } x \in \mathbf{Z}$$

has at most 61 solutions.

As a curiosity we mention that the equation with the largest number of solutions known is

$$\frac{\theta_2 - \theta_3}{\theta_2 - \theta_1} \left(\frac{\theta_1}{\theta_3}\right)^x + \frac{\theta_1 - \theta_3}{\theta_1 - \theta_2} \left(\frac{\theta_2}{\theta_3}\right)^x = 1$$

where the θ_i are the zeros of $X^3 - 2X^2 + 4X - 4$. The solutions are $x = 0, 1, 4, 6, 13, 52$. It would be interesting to have examples with more than 6 solutions, if they exist.

The first result in the situation of Theorem 1.2 with a universal bound was derived in [4] with the bound $2^{2^{23}}$.

The improvements we give in the current paper in comparison with [4],[5] and [6] depend upon two ingredients. First we use an explicit version of Thue's method via hypergeometric polynomials as given in [1], whereas the previous papers are based on a quantitative version of Roth's Theorem. To get bounds that do not depend upon degrees of number fields involved, previously a result from [7] was used on lower bounds for heights of solutions of equations. Here we apply the strongly improved bound given in Corollary 2.4 of [2].

2 Lemmas on algebraic numbers

First we fix our notations concerning heights. Let K be an algebraic number field of degree d over \mathbf{Q} . For any valuation v we write $d_v = [K_v : \mathbf{Q}_v]$, where K_v, \mathbf{Q}_v are the completions of K, \mathbf{Q} with respect to v . For archimedean v we normalise the valuation by $|x|_v = |x|^{d_v/d}$ where $|\cdot|$ is the ordinary complex absolute value. When v is non-archimedean we take $|p|_v = p^{-d_v/d}$ where p is the unique rational prime such that $|p|_v < 1$. The height of an algebraic number $\alpha \in K^*$ is defined by

$$H(\alpha) = \prod_v \max(1, |x|_v)$$

Because of our normalisation $H(\alpha)$ does not depend on the choice of the field K in which α is contained. More generally, for any $n+1$ -tuple $(x_0, x_1, \dots, x_n) \in K^n$, not all x_i zero we define

$$H(x_0, x_1, \dots, x_n) = \prod_v \max(|x_0|_v, \dots, |x_n|_v).$$

Note that by the product formula we have $H(\lambda x_0, \dots, \lambda x_n) = H(x_0, \dots, x_n)$ for any $\lambda \in K^*$, so we can view this height as a height on the K -rational points of the projective space \mathbf{P}^n . In particular we have $H(\alpha) = H(1, \alpha)$.

We start with an easy Lemma.

Lemma 2.1 *Let $a, a', b, b', A, B \in \overline{\mathbf{Q}}^*$ and $c, c' \in \overline{\mathbf{Q}}$ be such that $ab' \neq a'b$ and*

$$aA + bB = c, \quad a'A + b'B = c'$$

Then, $H(A, B, 1) \leq 2H(a, b, c)H(a', b', c')$.

Proof. Fix a number field K in which all numbers involved are contained. For each infinite valuation v let $r_v = 2^{d_v/d}$ and let $r_v = 1$ if v is finite. Notice that $\prod_v r_v = 2$.

One easily finds that

$$A = \frac{bc' - b'c}{\Delta}, \quad B = \frac{a'c - ac'}{\Delta}$$

where $\Delta = a'b - ab'$. Hence

$$\begin{aligned} H(A, B, 1) &= H(bc' - b'c, a'c - ac', ba' - ab') \\ &= \prod_v \max(|bc' - b'c|_v, |a'c - ac'|_v, |ba' - ab'|_v) \\ &\leq \prod_v r_v \max(|a|_v, |b|_v, |c|_v) \max(|a'|_v, |b'|_v, |c'|_v) \\ &= 2H(a, b, c)H(a', b', c') \end{aligned}$$

□

As a corollary we get

Corollary 2.2 *Let $a, b, A, B \in \overline{\mathbf{Q}}^*$ be such that $a \neq b$ and*

$$A + B = 1, \quad aA + bB = 1$$

Then, $H(A, B, 1) \leq 2H(a, b, 1)$.

The next lemma follows from an explicit version of Thue's method using hypergeometric polynomials.

Lemma 2.3 *Let $a, b, A, B \in \overline{\mathbf{Q}}^*$ and $\rho \in \mathbf{N}$ be such that*

$$A + B = 1, \quad aA^{2\rho} + bB^{2\rho} = 1$$

Then, $H(A, B, 1) \leq 2^{1/\rho} c H(a, b, 1)^{1/\rho}$, where $c = 6\sqrt{3}$.

Proof. We infer from Lemma 6 of [1] that there exist three polynomials P_ρ, Q_ρ, R_ρ of degree $\leq \rho$ such that

$$\begin{aligned} z^{2\rho} P_\rho(z) + (1-z)^{2\rho} Q_\rho(z) &= R_\rho(z), & \forall z \in \mathbf{C} \\ bP_\rho(A) &\neq aQ_\rho(A) \end{aligned}$$

and

$$H(P_\rho(A), Q_\rho(A), R_\rho(A)) \leq (6\sqrt{3})^\rho H(A)^\rho.$$

Substitute $z = A$ in the polynomial identity. Application of the previous lemma with $A^{2\rho}, B^{2\rho}$ instead of A, B and $c = 1$, $a' = P_\rho(A)$, $b' = Q_\rho(A)$, $c' = R_\rho(A)$ yields,

$$\begin{aligned} H(A, B, 1)^{2\rho} &\leq 2H(a, b, 1)H(P_\rho(A), Q_\rho(A), R_\rho(A)) \\ &\leq 2c^\rho H(a, b, 1)H(A)^\rho \leq 2c^\rho H(a, b, 1)H(A, B, 1)^\rho \end{aligned}$$

Divide on both sides by $H(A, B, 1)^\rho$ and take ρ -th roots to obtain our Lemma. \square

The following lemma is due to an improvement of [7] by Corollary 2.4 in [2].

Lemma 2.4 *Let $\lambda, \mu \in \overline{\mathbf{Q}}^*$ and suppose that $\lambda + \mu = 1$. Let (p_i, q_i) , $i = 1, 2$ be two solutions in $\overline{\mathbf{Q}}$ of $\lambda p + \mu q = 1$ such that the pairs (p_1, q_1) , (p_2, q_2) and $(1, 1)$ are all distinct. Then,*

$$H(p_1, q_1, 1)H(p_2, q_2, 1) \geq 1.0942711 \dots$$

By application of this Lemma with $\lambda = x_0, \mu = y_0$ and $p_i = x_i/x_0, q_i = y_i/y_0$ we obtain,

Corollary 2.5 *Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ be three distinct solutions of $x + y = 1$ in $x, y \in \overline{\mathbf{Q}}^*$. Then,*

$$\max_{i=1,2} (\max(H(x_i/x_0), H(y_i/y_0))) \geq 1.022777 \dots$$

3 Normed vector spaces

Let $m \in \mathbf{N}$. For any subgroup $H \subset (\overline{\mathbf{Q}}^*)^m$ we let the \mathbf{Q} -closure of H be the set of all $\mathbf{a} \in (\overline{\mathbf{Q}}^*)^m$ such that $\mathbf{a}^N \in H$ for some $N \in \mathbf{N}$. Let G be the \mathbf{Q} -closure of a finitely generated subgroup of $(\overline{\mathbf{Q}}^*)^m$ of rank r . Let T be the torsion subgroup of G . Then $G/T = G \otimes_{\mathbf{Z}} \mathbf{Q}$ has the natural structure of a \mathbf{Q} -vector space of dimension r . Consider the logarithmic height function $h(x) = \log H(x)$. The function

$$\|(x_1, \dots, x_m)\| = \max_{i=1, \dots, m} h(x_i)$$

provides a natural norm on $G \otimes_{\mathbf{Z}} \mathbf{Q}$ as \mathbf{Q} -vector space. By continuity we can extend this norm to the real vector space $V_G = G \otimes_{\mathbf{Z}} \mathbf{R}$.

Lemma 3.1 *The (semi)-norm $\|\cdot\|$ is positive definite on V_G .*

Proof. Let us write down the semi-norm $\|\cdot\|$ in an explicit way. Suppose the \mathbf{Q} -generators of G are given by

$$\mathbf{a}_i = (a_{i1}, \dots, a_{im}), \quad i = 1, \dots, r.$$

Any element of G can be written, modulo roots of unity, as $\mathbf{x} = (x_1, \dots, x_m) = \prod_{i=1}^r (a_{i1}, \dots, a_{im})^{e_i}$ for some $e_i \in \mathbf{Q}$. Hence, using $h(a) = (1/2) \sum_v |\log(|a|_v)|$,

$$\begin{aligned} \|\mathbf{x}\| &= \max_{j=1, \dots, m} h\left(\prod_{i=1}^r a_{ij}^{e_i}\right) \\ &= \max_{j=1, \dots, m} (1/2) \sum_v \left| \sum_{i=1}^r e_i \log(|a_{ij}|_v) \right|. \end{aligned}$$

Extending $\|\cdot\|$ to the reals is now straightforward, simply extend e_i to \mathbf{R} . We also remark that if we take the e_i integral, the components of \mathbf{x} all lie in the same number field, hence the non-trivial elements of the group generated (over \mathbf{Z}) by the \mathbf{a}_i have a norm uniformly bounded below by a positive constant, γ , say.

We now prove positive definiteness of $\|\cdot\|$. Suppose there exists $\mathbf{y} \in V_G$, non-zero, such that $\|\mathbf{y}\| = 0$. This implies that there exist $e_i \in \mathbf{R}$, not all zero, such that $|\sum_{i=1}^r e_i \log(|a_{ij}|_v)| = 0$ for all valuations v and all j . Using Dirichlet's box principle we can then show that to any $\epsilon > 0$ there exist integers m_i , not all zero, such that $|\sum_{i=1}^r m_i \log(|a_{ij}|_v)| < \epsilon$ for all v and j . This contradicts the existence of the uniform lower bound γ . Hence $\|\mathbf{y}\| = 0$ implies that $e_i = 0$ for all i , as desired. \square

From now on we suppose that $G \subset (\overline{\mathbf{Q}}^*)^2$. We want to bound the number of solutions of the equation

$$x + y = 1, \quad (x, y) \in G \tag{M}$$

Consider the natural projection $p : G \rightarrow V_G$

Lemma 3.2 *Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ be three distinct solutions of (M). Then their images under p cannot be all equal.*

Proof. If all three images would be the same then x_i/x_0 and y_i/y_0 would be roots of unity for $i = 1, 2$. But this is impossible in view of Corollary 2.5. \square

Let \mathcal{M} be the image under p of the solution set of (M). Then the number of solutions to (M) is bounded by $2(\#\mathcal{M})$.

We now restate the lemmas of the previous section in terms of the set $\mathcal{M} \subset V_G$. In the derivations we use the fact that $\max(H(a), H(b)) \leq H(a, b, 1) \leq \max(H(a), H(b))^2$.

Corollary 2.2 becomes,

Lemma 3.3 *Let $\mathbf{w}_1, \mathbf{w}_2$ be distinct points of \mathcal{M} . Then,*

$$\|\mathbf{w}_1\| \leq \log 2 + 2\|\mathbf{w}_2 - \mathbf{w}_1\|$$

Lemma 2.3 becomes,

Lemma 3.4 Let $\mathbf{w}_1, \mathbf{w}_2$ be distinct points of \mathcal{M} and $\rho \in \mathbf{N}$. Then,

$$\|\mathbf{w}_1\| \leq \log c + \frac{1}{\rho}(\log 2 + 2\|\mathbf{w}_2 - 2\rho\mathbf{w}_1\|)$$

Corollary 2.5 becomes,

Lemma 3.5 Let $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ be distinct points of \mathcal{M} . Then,

$$\max(\|\mathbf{w}_1 - \mathbf{w}_0\|, \|\mathbf{w}_2 - \mathbf{w}_0\|) \geq 0.022522\dots$$

It will turn out that the cardinality of any set satisfying the inequalities in the above three lemmas can be bounded in terms of the dimension of V_G .

We need some additional lemmas on coverings of convex bodies. The first is straightforward,

Lemma 3.6 Let V be an m -dimensional normed real vector space with norm $\|\cdot\|$. Let $R > \delta > 0$. Consider the ball B of radius R around the origin and suppose it contains a set U such that $\|\mathbf{u}_1 - \mathbf{u}_2\| \geq \delta$ for any two distinct $\mathbf{u}_1, \mathbf{u}_2 \in U$. Then $\#U \leq (1 + 2R/\delta)^m$.

Proof. Let V_0 be the volume of the unit ball $\{\mathbf{x} \mid \|\mathbf{x}\| < 1\}$. Around any point $\mathbf{u} \in U$ we consider the open ball $B_{\mathbf{u}} = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{u}\| \leq \delta/2\}$. Since these balls are disjoint their union fills up a region of volume $(\#U)(\delta/2)^m V_0$ in the ball of radius $R + \delta/2$. The latter ball has volume $(R + \delta/2)^m V_0$. Hence $(\#U)(\delta/2)^m \leq (R + \delta/2)^m$ and our Lemma follows. \square

Lemma 3.7 Let Ψ be a convex symmetric body in \mathbf{R}^r . By $\lambda\Psi$ we denote the convex body obtained by multiplying the points of Ψ by λ . Then, for any $\lambda > 1$, the set $\lambda\Psi$ can be covered by $(4 + 2\lambda)^r$ translated copies of Ψ .

The proof of this Lemma can be found in [6, Lemma 7.2]. However, we really need the following corollary.

Corollary 3.8 Let V be an r -dimensional normed real vector space with norm $\|\cdot\|$. Let $\epsilon > 0$. Then there is a finite set $E \subset V$ of unit vectors such that every $\mathbf{v} \in V$ can be written as $\mathbf{v} = \|\mathbf{v}\|\mathbf{e} + \mathbf{v}'$ with $\mathbf{e} \in E$ and $\|\mathbf{v}'\| \leq \epsilon\|\mathbf{v}\|$. Moreover, E can be chosen such that $\#E < (4 + 4/\epsilon)^r$.

Proof. Let B be the unit ball with respect to $\|\cdot\|$. According to Lemma 3.7 the ball B can be covered by $(4 + 4/\epsilon)^r$ translates of $(\epsilon/2)B$. Consider such a covering and let Δ be the subset of $(\epsilon/2)$ -balls which have non-trivial intersection with the boundary of B . Clearly the balls in Δ give a covering of the boundary of B . For the set E we take the unit vectors $\mathbf{c}/\|\mathbf{c}\|$ where \mathbf{c} runs over the centers of the $(\epsilon/2)$ -balls in Δ .

Now let $\mathbf{v} \in \mathbf{R}^r$ be arbitrary. Let \mathbf{c} be the center of the $(\epsilon/2)$ -ball in Δ which contains $\mathbf{v}/\|\mathbf{v}\|$ and let $\mathbf{e} = \mathbf{c}/\|\mathbf{c}\|$. Notice that $\|\mathbf{c} - \mathbf{e}\| = |1 - \|\mathbf{c}\|| \leq \epsilon/2$. Hence,

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} - \mathbf{e} \right\| \leq \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} - \mathbf{c} \right\| + \|\mathbf{c} - \mathbf{e}\| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus we find $\|\mathbf{v} - \|\mathbf{v}\|\mathbf{e}\| \leq \epsilon\|\mathbf{v}\|$, $\mathbf{e} \in E$ and our corollary follows. \square

4 Proof of Theorem 1.1

Let Σ be a subset of a normed vector space V satisfying

1. $\|\mathbf{w}_1\| \leq \log 2 + 2\|\mathbf{w}_2 - \mathbf{w}_1\|$ for any two distinct $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma$
2. There exists c_1 such that $\|\mathbf{w}_1\| \leq c_1 + \frac{1}{\rho}(\log 2 + 2\|\mathbf{w}_2 - 2\rho\mathbf{w}_1\|)$ for any two distinct $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma$ and any $\rho \in \mathbf{N}$.
3. There exists $c_0 > 0$ such that $\max(\|\mathbf{w}_1 - \mathbf{w}_0\|, \|\mathbf{w}_2 - \mathbf{w}_0\|) \geq c_0$ for any three distinct $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2 \in \Sigma$.

Proposition 4.1 *Let $c_2 = \max(2 \log 2, c_1 + \log 2/20)$. Then,*

$$\#\Sigma \leq \frac{1}{2} \left(44 + 2 \frac{c_2}{c_0} \right)^{r+1}$$

where r is the dimension of V .

Proof. Let ϵ be a real number such that $0 < \epsilon < 0.1$. Let \mathbf{e} be a unit vector in V and consider the cone

$$C_\epsilon = \{\mathbf{v} \in V \mid \mathbf{v} = \|\mathbf{v}\|\mathbf{e} + \mathbf{v}', \|\mathbf{v}'\| \leq \epsilon\|\mathbf{v}\|\}$$

Let

$$c_3(\epsilon) = \frac{c_2}{1 - 10\epsilon}.$$

We will show that for any two $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_\epsilon$ with $c_3(\epsilon) < \|\mathbf{w}_1\| \leq \|\mathbf{w}_2\|$ we have

$$(5/4)\|\mathbf{w}_1\| \leq \|\mathbf{w}_2\| \leq (1 + 4/\epsilon)\|\mathbf{w}_1\| \quad (4)$$

Suppose first $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_\epsilon$ and $\|\mathbf{w}_1\| \leq \|\mathbf{w}_2\| < (5/4)\|\mathbf{w}_1\|$. Write $\mathbf{w}_i = \|\mathbf{w}_i\|\mathbf{e} + \mathbf{w}'_i$. Then, from the first inequality on Σ we infer,

$$\begin{aligned} \|\mathbf{w}_1\| &\leq \log 2 + 2(\|\mathbf{w}_2\| - \|\mathbf{w}_1\|)\mathbf{e} + \mathbf{w}'_2 - \mathbf{w}'_1 \\ &\leq \log 2 + 2(\|\mathbf{w}_2\| - \|\mathbf{w}_1\|) + 2\epsilon(\|\mathbf{w}_2\| + \|\mathbf{w}_1\|) \\ &\leq \log 2 + 2(1/4)\|\mathbf{w}_1\| + 2\epsilon(9/4)\|\mathbf{w}_1\| \end{aligned}$$

We obtain,

$$\|\mathbf{w}_1\| \leq \frac{2 \log 2}{1 - 9\epsilon} \leq c_3(\epsilon).$$

Suppose next that $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_\epsilon$ and $\|\mathbf{w}_2\| > (1 + 4/\epsilon)\|\mathbf{w}_1\|$. Choose $\rho \in \mathbf{N}$ such that $\|\mathbf{w}_2\| = (2\rho + \delta)\|\mathbf{w}_1\|$ with $|\delta| \leq 1$. Notice that $\rho \geq 2/\epsilon$. From the second inequality on Σ it follows that

$$\begin{aligned} \|\mathbf{w}_1\| &\leq c_1 + \frac{1}{\rho}(\log 2 + 2\|\delta\|\|\mathbf{w}_1\|\mathbf{e} + \mathbf{w}'_2 - 2\rho\mathbf{w}'_1) \\ &\leq c_1 + (\log 2/20) + \frac{2}{\rho}(\|\mathbf{w}_1\| + \epsilon(\|\mathbf{w}_2\| + 2\rho\|\mathbf{w}_1\|)) \\ &\leq c_2 + \frac{2}{\rho}\|\mathbf{w}_1\| + \epsilon(8 + 4/\rho)\|\mathbf{w}_1\| \\ &\leq c_2 + \epsilon\|\mathbf{w}_1\| + 9\epsilon\|\mathbf{w}_1\| \end{aligned}$$

We get

$$\|\mathbf{w}_1\| \leq \frac{c_2}{1-10\epsilon} \leq c_3(\epsilon)$$

We now put the above considerations together. Let N be the smallest integer such that $(5/4)^{N-1} > 1 + 4/\epsilon$. Suppose C_e contains N points $\mathbf{w}_1, \dots, \mathbf{w}_N$ larger than $c_3(\epsilon)$. Suppose they are ordered by size. Then, for each i , $\|\mathbf{w}_{i+1}\|/\|\mathbf{w}_i\| \geq 5/4$. This implies $\|\mathbf{w}_N\|/\|\mathbf{w}_1\| > (5/4)^{N-1} > 1 + 4/\epsilon$ which is impossible by inequality (4). Hence any cone C_e contains at most $N - 1$ elements from Σ of norm $\geq c_3(\epsilon)$. According to Lemma 3.8 the space V can be covered by $(4 + 4/\epsilon)^r$ such cones and so the total number of points of Σ larger than $c_3(\epsilon)$ can be estimated by $(N - 1)(4 + 4/\epsilon)^r$. Since $\epsilon < 0.1$ it is not hard to see that $N - 1 < 2/\epsilon$. Hence the number of large points is bounded by $(2/\epsilon)(4 + 4/\epsilon)^r$.

It remains to count the elements of Σ with norm at most $c_3(\epsilon)$. By the third inequality on Σ a ball of radius c_0 around a point of Σ contains at most one other element from Σ . Consider a subset Σ' of Σ such that a ball of radius c_0 around any point of Σ' contains no other point of Σ' . We can do this in such a way that $|\Sigma| \leq 2|\Sigma'|$. According to Lemma 3.6 the number of points in Σ' can be bounded from above by $(1 + 2c_3(\epsilon)/c_0)^r$. Thus we conclude,

$$|\Sigma| \leq \frac{2}{\epsilon} \left(4 + \frac{4}{\epsilon}\right)^r + 2 \left(\frac{2c_3(\epsilon)}{c_0} + 1\right)^r.$$

Now we choose ϵ such that $4/\epsilon = 2c_3(\epsilon)/c_0$, i.e $\epsilon = (10 + 0.5c_2/c_0)^{-1}$. Our proposition then follows immediately. \square

Proof of Theorem 1.1. By a specialisation argument as in [5] we may assume that $G \subset (\overline{\mathbf{Q}}^*)^2$. We now complete the line of argument started in the Section (3). There we had the set \mathcal{M} . This set satisfies the conditions of Proposition 4.1 for the values $c_0 = 0.022522 \dots$ $c_1 = \log(6\sqrt{3}) = 2.3410 \dots$. Hence the cardinality of \mathcal{M} is bounded by $\frac{1}{2} \times 256^{r+1}$. Since the number of solutions of (M) is bounded by $2\#\mathcal{M}$ our theorem follows. \square

5 Proof of Theorem 1.2

We first need a lemma

Lemma 5.1 *Consider the equation $\lambda\alpha^x + \mu\beta^x = 1$ in $x \in \mathbf{Z}$ where $\lambda, \mu, \alpha, \beta$ are as in the Introduction and assumed to be algebraic numbers. Suppose we have the solutions $x = 0, r, s, t$. Suppose that $t \geq 14s$. Then,*

$$s - 8.4r \leq \frac{9.1}{\log H(\alpha, \beta, 1)}.$$

Proof. Application of Corollary 2.2 with $A = \lambda$, $B = \mu$ yields

$$H(\lambda, \mu, 1) \leq 2H(\alpha, \beta, 1)^r$$

Apply Lemma 2.3 with $A = \lambda\alpha^s$, $B = \mu\beta^s$ and ρ such that $t = 2s\rho + \delta$, with $0 \leq \delta < 2s$. Note that $\rho \geq 7$. We obtain,

$$\begin{aligned} H(\lambda\alpha^s, \mu\beta^s, 1) &\leq 2^{1/\rho} cH(\alpha^\delta \lambda^{1-2\rho}, \beta^\delta \mu^{1-2\rho})^{1/\rho} \\ &\leq 2^{1/\rho} cH(\alpha, \beta, 1)^{\delta/\rho} H(\lambda^{-1}, \mu^{-1}, 1)^{2-1/\rho} \end{aligned}$$

Notice that

$$\begin{aligned} H(\alpha, \beta, 1)^s &\leq H(\lambda^{-1}, \mu^{-1}, 1)H(\lambda\alpha^s, \mu\beta^s, 1) \\ &\leq 2^{1/\rho}cH(\alpha, \beta, 1)^{\delta/\rho}H(\lambda^{-1}, \mu^{-1}, 1)^{3-1/\rho} \end{aligned}$$

and use $H(\lambda^{-1}, \mu^{-1}, 1) \leq H(\lambda, \mu, 1)^2 \leq 4H(\alpha, \beta, 1)^{2r}$ to obtain

$$\begin{aligned} H(\alpha, \beta, 1)^{s-\delta/\rho} &\leq 2^{1/\rho}c2^{6-2/\rho}H(\alpha, \beta, 1)^{6r} \\ &< 64cH(\alpha, \beta, 1)^{6r} \end{aligned}$$

Taking log's and using $\log(64c) \leq 6.5$ yields

$$s - \delta/\rho - 6r \leq 6.5/\log(H(\alpha, \beta, 1))$$

from which our Lemma is immediate via $\delta/\rho \leq 2s/7$. \square

Proof of Theorem 1.2. By Theorem 2 of [1] we may assume that $\alpha, \beta, \lambda, \nu \in \overline{\mathbf{Q}}$. Without loss of generality we can also assume that

$$H(\alpha, \beta, 1) \leq H(\alpha^{-1}, \beta^{-1}, 1).$$

Let q be the length of the shortest closed interval containing three solutions. Let $n, n+p, n+q$ be three such solutions. Application of Lemma 2.4 to the equation $\lambda\alpha^{n+p}X + \mu\beta^{n+p}Y = 1$ yields

$$H(\alpha, \beta, 1)^{q-p}H(\alpha^{-1}, \beta^{-1}, 1)^p \geq c_4,$$

where $c_4 = 1.0942711\dots$. Hence $H(\alpha^{-1}, \beta^{-1}, 1)^q \geq c_4$.

Define $\gamma = \log 8 / \log c_4$ and note that $\gamma < 23.1$.

Now let $k < l < m < n$ be any four solutions. First of all application of Corollary 2.2 with $A = \lambda\alpha^k$, $B = \mu\beta^k$ yields

$$H(\lambda\alpha^k, \mu\beta^k, 1) \leq 2H(\alpha, \beta, 1)^{l-k} \quad (5)$$

In a similar way application of Corollary 2.2 with $A = \lambda\alpha^n$, $B = \mu\beta^n$ yields

$$H(\lambda\alpha^n, \mu\beta^n, 1) \leq 2H(\alpha^{-1}, \beta^{-1}, 1)^{n-m} \quad (6)$$

Application of Lemma 2.1 with $A = \alpha^{k-n}$, $B = \beta^{k-n}$ yields

$$\begin{aligned} H(\alpha^{k-n}, \beta^{k-n}, 1) &\leq 2H(\lambda\alpha^n, \mu\beta^n, 1)H(\lambda\alpha^{n-k+l}, \mu\beta^{n-k+l}, 1) \\ &\leq 2H(\lambda\alpha^n, \mu\beta^n, 1)^2H(\alpha^{l-k}, \beta^{l-k}, 1) \end{aligned}$$

With (6) and $H(\alpha, \beta, 1) \leq H(\alpha^{-1}, \beta^{-1}, 1)$ we get

$$H(\alpha^{-1}, \beta^{-1}, 1)^{n-k} \leq 8H(\alpha^{-1}, \beta^{-1}, 1)^{2(n-m)+l-k}$$

Using our lower bound $H(\alpha^{-1}, \beta^{-1}, 1) \geq c_4^{1/q}$ we find that

$$n - 2m + l \geq -\gamma q \text{ hence } n - l - \gamma q \geq 2(m - l - \gamma q)$$

Denote the smallest solution by n_0 and the second smallest by n_1 . Application of the inequality with $k = n_0, l = n_1$ yields

$$n - n_1 - \gamma q \geq 2(m - n_1 - \gamma q) \quad (7)$$

for any two solutions m, n with $n_1 < m < n$. We divide our solutions into three intervals,

- $I_1 = [n_0, n_1 + (0.9 + \gamma)q[$
- $I_2 = [n_1 + (0.9 + \gamma)q, n_1 + (230 + \gamma)q[$
- $I_3 = [n_1 + (230 + \gamma)q, \infty[$

Since any interval of length $< q$ contains at most two solutions, the interval I_1 contains at most $1 + 2([\gamma + 0.9] + 1) \leq 49$ solutions. Because of (7) the interval I_2 contains at most 8 solutions.

We finally show that I_3 contains at most 4 solutions. Suppose I_3 contains 5 solutions, the largest being denoted by N , the smallest by M . Furthermore we let k be a solution such that there exists another solution l such that $k < l < k + q$. Because of (7) we find $k < n_1 + (1 + \gamma)q$. Since there exists at least one closed interval of length q containing three solutions such a k exists and we may moreover assume that $k \geq n_1$. From (7) it follows that $(N - n_1 - \gamma q) \geq 16(M - n_1 - \gamma q)$. Since $k \geq n_1$ this implies $(N - k - \gamma q) \geq 16(M - k - \gamma q)$ and since $N - k > M - k > 229q$ we get $N - k \geq (16 - 15\gamma/229)(M - k) > 14(M - k)$. Application of Lemma 5.1 to the equation $\lambda\alpha^k\alpha^x + \mu\beta^k\beta^x = 1$ with $r = l - k, s = M - k, t = N - k$ yields

$$M - k - 8.4(l - k) \leq \frac{9.1}{\log H(\alpha, \beta, 1)}.$$

Using the lower bound $H(\alpha, \beta, 1) \geq c_4^{1/2q}$ and $l - k < q$ we get $M - k < 211q$, contradicting $M - k > 229q$.

So we conclude that I_3 contains at most 4 solutions, which leaves us with a total of at most $49 + 8 + 4 = 61$ solutions. \square

6 References

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