Hypergeometric functions in one variable

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1.1 Definition, first properties

Let $\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n$ be any complex numbers and consider the generalised hypergeometric equation in one variable,

$$z(D+\alpha_1)\cdots(D+\alpha_n)F=(D+\beta_1-1)\cdots(D+\beta_n-1)F,\qquad D=z\frac{d}{dz} \tag{1}$$

This is a Fuchsian equation of order n with singularities at $0,1,\infty$. The local exponents read,

$$\begin{array}{ll} 1-\beta_1,\ldots,1-\beta_n & \text{at } z=0\\ \alpha_1,\ldots,\alpha_n & \text{at } z=\infty\\ 0,1,\ldots,n-2,\ -1+\sum_1^n(\beta_i-\alpha_i) & \text{at } z=1 \end{array}$$

When the β_i are distinct modulo 1 a basis of solutions at $z=0$ is given by the functions

$$z^{1-\beta_i}\ {}_nF_{n-1}\left(\left.\begin{array}{c}\alpha_1-\beta_i+1,\ldots,\alpha_n-\beta_i+1\\ \beta_1-\beta_i+1,..^\vee..,\beta_n-\beta_i+1\end{array}\right| z\right)\qquad (i=1,\ldots,n).$$

Here $..^\vee..$ denotes suppression of the term $\beta_i-\beta_i+1$ and ${}_nF_{n-1}$ stands for the generalised hypergeometric function in one variable

$${}_nF_{n-1}\left(\left.\begin{array}{c}\alpha_1,\ldots,\alpha_n\\ \beta_1,\ldots,\beta_{n-1}\end{array}\right| z\right)=\sum_{k=0}^\infty\frac{(\alpha_1)_k\cdots(\alpha_n)_k}{(\beta_1)_k\cdots(\beta_{n-1})_k k!}z^k.$$

To show that these functions are not as complicated as their definition suggests, consider for example the hypergeometric function

$${}_4F_3\left(\left.\begin{array}{c}1/4,1/2,3/4\\ 1/3,2/3\end{array}\right| z\right)=\sum_{k=0}^\infty\frac{(1/4)_k(1/2)_k(3/4)_k}{(1/3)_k(2/3)_k k!}z^k.$$

A straightforward computation shows that

$${}_4F_3\left(\left.\begin{array}{c}1/4,1/2,3/4\\ 1/3,2/3\end{array}\right| \frac{256z}{27}\right)=\sum_{k=0}^\infty\binom{4k}{k}z^k.$$

Later we shall show that this is an algebraic function (over $\mathbb{C}(z)$).
The most studied case of one variable hypergeometric functions is that of the *Gaussian hypergeometric function*, which is the case $n=2$. It was already noted by Euler that many classical functions could be recognized as hypergeometric functions for special choices of the parameters α_i,β_j.
At $z=1$ we have the following interesting situation.

Theorem 1.1.1 (Pochhammer) *The equation* (1) *has* $n-1$ *independent holomorphic solutions near* $z=1$.

The proof of this result follows from the observation that the coefficient of $\left(\frac{d}{dz}\right)^n$ in (1) equals $z^{n+1}-z^n$ and the following theorem.

Theorem 1.1.2 *Consider the linear differential equation*

$$p_n(z)y^{(n)}+p_{n-1}(z)y^{(n-1)}+\cdots+p_1(z)y'+p_0(z)y=0$$

where the $p_i(z)$ *are analytic around a point* $z=a$. *Suppose that* $p_n(z)$ *has a zero of order one at* $z=a$. *Then the differential equation has* $n-1$ *independent holomorphic solutions around* $z=a$.

Proof. Without loss of generality we can assume that $a=0$. Write $p_i(z)=\sum_{j\ge 0}p_{ij}z^j$ for every j. Then, in particular, $p_{n0}=0$ and $p_{n1}\ne 0$. we determine a power series solution $\sum_{k\ge 0}f_kz^k$ by substituting it into the equation. We obtain the recursion relations,

$$\sum_{i=0}^{n}\sum_{j=0}^{m}i!p_{ij}f_{m+i-j}\binom{m+i-j}{i}=0,\qquad m=0,1,2,\ldots$$

Since $p_{n0}=0$ we see that there is no term with f_{m+n} in the above recurrence relation. However, the recurrence does express f_{m+n-1} as a linear combination of the f_{m+n-r} with $r\ge 2$. The coefficient of f_{m+n-1} reads

$$(m+n-1)\cdots mp_{n,1}+(m+n-1)\cdots(m+1)p_{n-1,0}.$$

Suppose first that $mp_{n,1}+p_{n-1,0}\ne 0$ for all $m\in\mathbb{Z}_{\ge 0}$. Then we can choose $f_0,f_1,\ldots,f_{n-2}$ arbitrarily and use the recurrence to find f_{m+n-1} for $m\ge 0$. If $mp_{n,1}+p_{n-1,0}$ vanishes for $m=m_0$ say, we must impose a linear relation between the $f_0,\ldots,f_{n-2}$. However, we can now choose f_{m_0} freely and we have an $n-1$-dimensional solution space again. Note that $p_{n,1}\ne 0$ is important to get power series solutions that converge in a disc around $z=0$. **qed**

Finally we mention the Euler integral for ${}_nF_{n-1}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_{n-1}|z)$,

$$\prod_{i=1}^{n-1}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}\int_0^1\cdots\int_0^1\frac{\prod_{i=1}^{n-1}t_i^{\alpha_i-1}(1-t_i)^{\beta_i-\alpha_i-1}}{(1-zt_1\cdots t_{n-1})^{\alpha_n}}dt_1\cdots dt_{n-1}$$

for all $\Re\beta_i>\Re\alpha_i>0$ $(i=1,\ldots,n-1)$. The latter condition is there to assure convergence of the integral. There is an integral representation for more general parameters if we replace the integration $\int_0^1$ by an integration $\int_\gamma$ where γ is a so-called Pochhammer contour. It avoids the points $0,1$ and looks like this, Notice that the integrand acquires the same value after letting any of the t_i run along γ.

1.2 Monodromy

Fix a base point $z_0 \in \mathbb{P}^1 - \{0, 1, \infty\}$, say $z_0 = 1/2$. Denote by G the fundamental group $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, z_0)$. Clearly G is generated by the simple loops g_0, g_1, g_∞ around the corresponding points together with the relation $g_0 g_1 g_\infty = 1$.
Let $V(\alpha, \beta) = V(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ be the local solution space of (1) around z_0. Denote by

$$M(\alpha, \beta): \; G \to GL(V(\alpha, \beta))$$

the monodromy representation of (1). Write

$$h_0 = M(\alpha, \beta)(g_0) \qquad h_1 = M(\alpha, \beta)(g_1) \qquad h_\infty = M(\alpha, \beta)(g_\infty).$$

The eigenvalues of h_0 and h_∞ read $\exp(-2\pi i \beta_j)$ and $\exp(2\pi i \alpha_j)$ respectively. Since there are $n - 1$ independent holomorphic solutions near $z = 1$ the element h_1 has $n - 1$ eigenvalues 1 together with $n - 1$ independent eigenvectors. Equivalently, $\mathrm{rank}(h_1 - \mathrm{Id}) \leq 1$. An element $h \in GL(V)$ such that $\mathrm{rank}(h - \mathrm{Id}) = 1$ will be called a *(pseudo)-reflection.* The determinant of a reflection will be called the *special eigenvalue.* From the relation between the generators of the fundamental group it follows that $h_1^{-1} = h_\infty h_0$ is a (pseudo)reflection.

Theorem 1.2.1 *Let $H \subset GL(n, \mathbb{C})$ be a subgroup generated by two matrices A, B such that AB^{-1} is a reflection. Then H acts irreducibly on $\mathbb{C}^n$ if and only if A and B have disjoint sets of eigenvalues.*

Proof. Suppose that H acts reducibly. Let V_1 be a nontrivial invariant subspace and let $V_2 = \mathbb{C}^n / V_1$. Since $A - B$ has rank 1, A and B coincide on either V_1 or V_2. Hence they have a common eigenvalue.
Suppose conversely that A and B have a common eigenvalue λ. Let $W = \ker(A - B)$. Since $AB^{-1} - \mathrm{Id}$ has rank one, the same holds for $A - B$. Hence $\dim(W) = n - 1$. If any eigenvector of A belongs to W, it must also be an eigenvector of B, since A and B coincide on W. Hence there is a one-dimensional invariant subspace. Suppose W does not contain any eigenvector of A or B. We show that the subspace $U = (A - \lambda)\mathbb{C}^n$ is invariant under H. Note that $A - \lambda \mathrm{Id}$ has a non-trivial kernel which has trivial intersection with W. Hence U has dimension $n - 1$ and $U = (A - \lambda)W$. Since $A - \lambda$ and $B - \lambda$ coincide on W we conclude that also

$U = (B-\lambda)W$ and hence, by a similar argument as for A, $U = (B-\lambda)\mathbb{C}^n$. Notice that U is stable under A, as follows trivially from $A(A-\lambda)\mathbb{C}^n = (A-\lambda)A\mathbb{C}^n = (A-\lambda)\mathbb{C}^n$. For a similar reason U is stable under B. Hence H has the invariant subspace U. **qed**

Corollary 1.2.2 *The monodromy group of* (1) *acts irreducibly if and only if all differences* $\alpha_i - \beta_j$ *are non-integral.*

This Corollary follows by application of our Theorem with $A = h_\infty$ and $B = h_0^{-1}$.
From now on we shall be interested in the irreducible case only.

Theorem 1.2.3 (Levelt) *Let* $a_1, \ldots, a_n; b_1, \ldots, b_n \in \mathbb{C}^*$ *be such that* $a_i \neq b_j$ *for all* i, j. *Then there exist* $A, B \in GL(n, \mathbb{C})$ *with eigenvalues* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *respectively such that* AB^{-1} *is a reflection. Moreover, the pair* A, B *is uniquely determined up to conjugation.*

Proof. First we show the existence. Let

$$\prod_i (X - a_i) = X^n + A_1 X^{n-1} + \cdots + A_n$$

$$\prod_i (X - b_i) = X^n + B_1 X^{n-1} + \cdots + B_n$$

and

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & -A_n \\ 1 & 0 & \ldots & 0 & -A_{n-1} \\ 0 & 1 & \ldots & 0 & -A_{n-2} \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & -A_1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & \ldots & 0 & -B_n \\ 1 & 0 & \ldots & 0 & -B_{n-1} \\ 0 & 1 & \ldots & 0 & -B_{n-2} \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & -B_1 \end{pmatrix}$$

Then $\mathrm{rank}(A - B) = 1$, hence $\mathrm{rank}(AB^{-1} - \mathrm{Id}) = 1$ and AB^{-1} is a reflection.
To prove uniqueness of A, B we let $W = \ker(A - B)$. Note that $\dim W = n - 1$. Let $V = W \cap A^{-1}W \cap \cdots \cap A^{-(n-2)}W$. Then $\dim V \geq 1$. Suppose $\dim V > 1$. Choose $v \in V \cap A^{-(n-1)}W$. Then $A^i v \in W$ for $i = 0, 1, \ldots, n-1$. Hence $U = < A^i v >_{i \in \mathbb{Z}} \subset W$ is A-stable. In particular, W contains an eigenvector of A. Since $B = A$ on W this is also an eigenvector of B with the same eigenvalue, contradicting our assumption on A, B. Hence $\dim V = 1$. Letting $v \in V$ we take $v, Av, \ldots, A^{n-1}v$ as basis of $\mathbb{C}^n$. Since $A = B$ on W we have that $A^i v = B^i v$ for $i = 0, 1, \ldots, n-2$ and with respect to this basis A and B have automatically the form given above. **qed**

Corollary 1.2.4 *With the same hypotheses and* A_i, B_j *as in the proof of the previous theorem we have that* $< A, B >$ *can be described by matrices having elements in* $\mathbb{Z}[A_i, B_j, 1/A_n, 1/B_n]$.

Levelt's theorem is a special case of a general rigidity theorem which has recently been proved by N.M.Katz. In section 1.4 we shall give an elementary proof of Katz's theorem.

1.3 Hypergeometric groups

Definition 1.3.1 *Let $a_1, \ldots, a_n; b_1, \ldots, b_n \in \mathbb{C}^*$. such that $a_i \neq b_j$ for every i, j. The group generated by A, B such that A and B have eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively and such that AB^{-1} is a pseudoreflection, will be called a hypergeometric group with parameters a_i and b_j. Notation: $H(a, b) = H(a_1, \ldots, a_n; b_1, \ldots, b_n)$.*

In particular, the monodromy group of (1) is a hypergeometric group with $a_k = e^{2\pi i \alpha_k}$ and $b_k = e^{2\pi i \beta_k}$.

Theorem 1.3.2 *Let H be a hypergeometric group with parameters $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. Suppose that these parameters lie on the unit circle in $\mathbb{C}$. Then there exists a non-degenerate hermitean form $F(x, y) = \sum F_{ij} x_i y_j$ on $\mathbb{C}^n$ such that $F(hx, hy) = F(x, y)$ for all $h \in H$ and all $x, y \in \mathbb{C}^n$.*
Denote by $\prec, \preceq$ the total ordering on the unit circle corresponding to increasing argument. Assume that the $a_1 \preceq \ldots \preceq a_n$ and $b_1 \preceq \ldots \preceq b_n$. Let $m_j = \#\{k | b_k \prec a_j\}$ for $j = 1, \ldots, n$. Then the signature (p, q) of the hermitean form F is given by

$$|p - q| = \left| \sum_{j=1}^{n} (-1)^{j+m_j} \right|.$$

The proof of the first part of this Theorem follows from the existence of Hermitian forms on rigid irreducible systems which is proved in section 1.4. The second part, on the signature, is more technical and we shall not show it in this text. We refer to the Beukers-Heckman paper instead.

Definition 1.3.3 *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be sets on the unit circle. We say that these sets interlace on the unit circle if and only if either*

$$a_1 \prec b_1 \prec a_2 \prec b_2 \cdots \prec a_n \prec b_n$$

or

$$b_1 \prec a_1 \prec b_2 \prec a_2 \cdots \prec b_n \prec a_n.$$

Corollary 1.3.4 *Let the hypergeometric group H have all of its parameters on the unit circle. Then H is contained in $U(n, \mathbb{C})$ if and only if the parametersets interlace on the unit circle.*

Proof. To see this Corollary we use Theorem 1.3.2. There we see that the signature of the group equals n if and only if m_j and j have the same parity of all j. If one thinks about it this can only happen if at least one b_k is located among every two consecutive a_j. Hence the eigenvalue sets interlace. **qed**

Theorem 1.3.5 *Suppose the parameters $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are roots of unity, let us say h-th roots of unity for some $h \in \mathbb{Z}_{\geq 2}$. Then the hypergeometric group $H(a, b)$ is finite if and only if for each $k \in \mathbb{Z}$ with $(h, k) = 1$ the sets $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace on the unit circle.*

Proof. The Galois group of $\mathbb{Q}(\exp(2\pi i/h))$ over $\mathbb{Q}$ is given by elements of the form

$$\sigma_k : \exp(2\pi i/h) \to \exp(2\pi ik/h)$$

for any $k, (k,h) = 1$. The group $H(a,b)$ can be represented by matrices with entries in the ring of cyclotomic integers $\mathbb{Z}[\exp(2\pi i/h)]$. The Galois automorphsim σ_k maps the group $H(a,b)$ isomorphically to the hypergeometric group H_k with parameters $a_1^k, \ldots, a_n^k$, $b_1^k, \ldots, b_n^k$. Each group H_k has an invariant hermitian form F_k for $(k,h) = 1$.
Suppose $H(a,b)$ is finite. Then each F_k is definite, hence every pair of sets $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace on the unit circle.
Suppose conversely that $\{a_1^k, \ldots, a_n^k\}$ and $\{b_1^k, \ldots, b_n^k\}$ interlace for every $k, (k,h) = 1$. Then each group H_k is subgroup of a unitary group with definite form F_k. In particular the entries of each element are bounded in absolute value by some constant, C say. This implies that any entry of any element of $H(a,b)$ has conjugates which are all bounded by C. Since there exist only finitely many elements of $\mathbb{Z}[\exp(2\pi i/h)]$ having this property, we conclude the finiteness of $H(a,b)$. **qed**

An immediate consequence of this theorem is that, for example, the hypergeometric function

$${}_8F_7\left(\begin{matrix} 1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30 \\ 1/5, 1/3, 2/5, 1/2, 3/5, 2/3, 4/5 \end{matrix} \middle| z\right)$$

is an algebraic function. The Galois group belonging to this function is the Weyl group $W(E_8)$ which has 696729600 elements. It was noticed by F. Rodriguez-Villegas that if we replace z in this function by $2^14 * 3^9 * 5^5 * z$ we get the powerseries

$$\sum_{n=0}^{\infty} \frac{(30n)!n!}{(15n)!(10n)!(6n)!} z^n$$

which is precisely the series studied by Chebyshev during his work on the distribution of prime numbers.

1.4 Rigidity

In this section we formulate and prove Katz's result on rigidity, see [Katz, Theorem 1.1.2]. Let k be a field and $g_1, g_2, \ldots, g_r \in GL(n,k)$. We suppose that $g_1 g_2 \cdots g_r = \mathrm{Id}$. Let G be the group generated by $g_1, \ldots, g_r$. We say that the r-tuple is *irreducible* if the group G acts irreducibly on k^n. We call the r-tuple $g_1, \ldots, g_r$ *linearly rigid* if for any conjugates $\tilde{g}_1, \ldots, \tilde{g}_r$ of $g_1, \ldots, g_r$ with $\tilde{g}_1 \tilde{g}_2 \cdots \tilde{g}_r = \mathrm{Id}$ there exists $u \in GL(n,k)$ such that $\tilde{g}_i = u g_i u^{-1}$ for $i = 1, 2, \ldots, r$.
For example, it follows from Levelt's theorem that the generators $g_1 = A, g_2 = B^{-1}, g_3 = BA^{-1}$ of a hypergeometric group form a linearly rigid system.

Theorem 1.4.1 (Katz) *Let $g_1, g_2, \ldots, g_r \in GL(n,k)$ be an irreducible r-tuple with $g_1 g_2 \ldots g_r =$* Id. *Let, for each i, δ_i be the codimension of the linear space $\{A \in M_n(k) | g_i A = A g_i\}$ (= codimension of the centralizer of g_i). Then,*

i) $\delta_1 + \cdots + \delta_r \geq 2(n^2 - 1)$

ii) If $\delta_1 + \cdots + \delta_r = 2(n^2 - 1)$, the system is linearly rigid.

iii) If k is algebraically closed, then the converse of part ii) holds

We note that the centraliser of $g \in GL(n,k)$ depends only on the Jordan normal form of g. If g is diagonalisable, the dimension of the centraliser is the sum of the squares of the dimensions of the eigenspaces of g. When g has distinct eigenvalues this dimension is n, when g is a (pseudo)reflection this dimension is $(n-1)^2 + 1 = n^2 - 2n + 2$. The corresponding codimensions are $n^2 - n$ and $2n - 2$ respectively.
More generally, suppose that g has eigenspaces of dimensions $n_1, \ldots, n_s$. Then the codimension of the centraliser is equal to $n^2 - n - \sum_{i=1}^{s}(n_i^2 - n_i)$.
By way of example consider a hypergeometric group generated by $g_1 = A, g_2 = B^{-1}, g_3 = BA^{-1}$. For A and B we clearly have $\delta_1, \delta_2 \leq n^2 - n$. Since g_3 is a (pseudo)reflection we have $\delta_3 = 2n - 2$. Notice that $\delta_1 + \delta_2 + \delta_3 \leq 2n^2 - 2$. Together with part i) of Katz's Theorem this implies $\delta_1 + \delta_2 + \delta_3 = 2n^2 - 2$. Hence the triple A, B^{-1}, BA^{-1} is linearly rigid. As a bonus we get that the eigenspaces of A and B all have dimension one. Hence to each eigenvalue there is precisely one Jordan block in the Jordan normal form.
Another example comes from the Jordan-Pochhammer equation, which is an n-th order Fuchsian equation with $n+1$ singular points and around each singular point the local monodromy is (up to a scalar) a pseudo-reflection. So for each singularity we have $\delta_i = 2n - 2$. The sum of these delta's is of course $2(n^2 - 1)$. So if the monodromy is irreducible we have again a rigid system. This case has been elaborated by [Haraoka].
The proof of Katz's theorem is based on the following Theorem from linear algebra. In this Theorem we consider a group G acting on a finite dimensional linear space V. For every $X \subset G$ we denote by $d(X)$ resp. $d^*(X)$ the codimension of the common fixed point space in V resp. V^*, the dual of V, of all elements of X.

Theorem 1.4.2 (L.L.Scott) *Let $H \in GL(V)$ be the group generated by $h_1, h_2, \ldots, h_r$ with $h_1 h_2 \cdots h_r = \mathrm{Id}$. Then*

$$d(h_1) + d(h_2) + \cdots + d(h_r) \geq d(G) + d^*(G).$$

Proof. Let W be the direct sum $\oplus_{i=1}^{r}(1 - h_i)V$. Define the linear map $\beta : V \to W$ by

$$\beta : v \mapsto ((1 - h_1)v, \ldots, (1 - h_r)v).$$

Define the linear map $\delta : W \to V$ by

$$\delta : (v_1, \ldots, v_r) \mapsto v_1 + h_1 v_2 + h_1 h_2 v_3 + \cdots + h_1 \cdots h_{r-1} v_r$$

Because of the identity

$$1 - h_1 h_2 \cdots h_r = (1 - h_1) + h_1(1 - h_2) + \cdots + h_1 \cdots h_{r-1}(1 - h_r)$$

we see that the image of β is contained in the kernel of δ. Hence $\dim(\mathrm{Im}\beta) \leq \dim(\ker\delta)$. Moreover, the kernel of β is precisely $\cap_{i=1}^{r}\ker(1 - h_i)$. The dimension of the latter space equals $n - d(H)$. Hence $\dim(\mathrm{Im}\beta) = n - (n - d(H)) = d(H)$.

The image of δ is

$$(1-h_1)V + h_1(1-h_2)V + \cdots + h_1\cdots h_{r-1}(1-h_r)V$$

which is equal to $(1-h_1)V+(1-h_2)V+\cdots+(1-h_r)V$. Note that any $w \in \cap_{i=1}^r \mathrm{ker}(1-h_i^*)$ in the dual space V^* we vanishes on $\mathrm{Im}\delta$. Hence $\dim(\mathrm{Im}\delta) \ge d^*(H)$.
Finally notice that $\dim(W) = \sum_{i=1}^r d(h_i)$. Putting everything together we get

$$\begin{aligned}\sum_{i=1}^r d(h_i) = \dim(W) &= \dim(\mathrm{ker}\delta) + \dim(\mathrm{Im}\delta)\\ &\ge \dim(\mathrm{Im}\beta) + \dim(\mathrm{Im}\delta)\\ &\ge d(H) + d^*(H)\end{aligned}$$

This is precisely the desired inequality. **qed**

Proof of Katz's theorem following Völklein-Strambach. For the first part of Katz's theorem we apply Scott's Theorem to the vector space of $n \times n$-matrices and the group generated by the maps $h_i : A \mapsto g_i^{-1}Ag_i$. Notice that $d(h_i)$ is now precisely the codimension of the centraliser of g_i, hence $d(h_i) = \delta_i$ for all i. The number $d(H)$ is precisely the codimension of the space $\{A \in M_n(k) | gA = Ag \text{ for all } g \in G\}$. By Schur's Lemma the irreducibility of the action of G implies that the dimension of this space is 1 and the codimension n^2-1. So $d(H) = n^2-1$. To determine $d^*(H)$ we note that the matrix space $V = M_n(k)$ is isomorphic to its dual via the map $V \to V^*$ given by $A \mapsto (X \mapsto \mathrm{Trace}(AX)$. Let us identify V with V^* in this way. Since $\mathrm{Trace}(Ag^{-1}Xg) = \mathrm{Trace}(gAg^{-1}X)$ we see that the action of g on the dual space is given by $A \mapsto gAg^{-1}$. hence $d^*(H) = n^2-1$. Application of Scott's Theorem now shows that

$$\delta_1 + \cdots + \delta_2 \ge d(H) + d^*(H) = 2(n^2-1)$$

To prove the second part of the theorem we apply Scott's Theorem with $V = M_n(k)$ again, but now with the maps $h_i : A \mapsto g_i^{-1}A\tilde{g}_i$. For each i choose $u_i \in GL(n,k)$ such that $\tilde{g}_i = u_i g_i u_i^{-1}$. Now note that

$$\begin{aligned}d(h_i) &= \mathrm{codim}\{A | g_i^{-1}A\tilde{g}_i = A\}\\ &= \mathrm{codim}\{A | A\tilde{g}_i = g_iA\}\\ &= \mathrm{codim}\{A | Au_ig_iu_i^{-1} = g_iA\}\\ &= \mathrm{codim}\{A | (Au_i)g_i = g_i(Au_i)\}\\ &= \mathrm{codim}\{A | Ag_i = g_iA\} = \delta_i\end{aligned}$$

The sum of the δ_i is given to be $2(n^2-1)$. Together with Scott's Theorem this implies $d(H)+d^*(H) \le 2(n^2-1)$. This means that either $d(H) < n^2$ or $d^*(H) < n^2$ or both. Let us assume $d(H) < n^2$, the other case being similar. Then there is a non-trivial $n \times n$ matrix A such that $A\tilde{g}_i = g_iA$ for all i. From these inequalities we see in particular that the image of A is stable under the group generated by the g_i. Since the r-tuple $g_1, \ldots, g_r$ is irreducible this means that $A(k^n)$ is either trivial or k^n itself. Because A is non-trivial we conclude that $A(k^n) = k^n$ and A is invertible. We thus conclude that $\tilde{g}_i = A^{-1}g_iA$ for all i. In other words, our system $g_1, \ldots, g_r$ is rigid.

The proof of part iii) uses a dimension argument. Let C_i be the conjugacy class of g_i $i = 1, 2, \ldots, r$. Consider the multiplication map $\Pi : C_1 \times C_2 \times \cdots \times C_r \to GL(n, k)$ given by $(c_1, c-2, \ldots, c_r) \mapsto c_1 c_2 \cdots c_r$. We have

$$\dim(C_1 \times \cdots \times C_r) \leq \dim(\Pi^{-1}(\mathrm{Id}) + \dim(\mathrm{Im}\Pi)$$

First of all note that $\dim(C_1 \times \cdots \times C_r) = \sum_{i=1}^{r} \dim(C_i) = \sum_{i=1}^{r} \delta_i$. Secondly, by the rigidity and irreducibility assumptions we have $\dim(\Pi^{-1}(\mathrm{Id})) = n^2 - 1$. Finally, $\mathrm{Im}\Pi$ is contained in the hypersuface of all matrices whose determinant is $\det(g_1 g_2 \cdots g_r) = 1$. Hence $\dim(\mathrm{Im}\Pi) \leq n^2 - 1$.
These three facts imply that $\sum_{i=1}^{r} \delta_i \leq 2(n^2 - 1)$. Together with part i) this implies the desired equality. **qed**

In many practical situations the local monodromies of differential equations have eigenvalues which are complex numbers with absolute value 1. In that case there exists also a monodromy invariant Hermitian form on the solution space. We formulate this as a Lemma.

Lemma 1.4.3 *Let $g_1, g_2, \ldots, g_r \in GL(n, \mathbb{C})$ be a rigid, irreducible system with $g_1 g_2 \cdots g_r = \mathrm{Id}$. Suppose that for each i the matrices g_i and $\tilde{g}_i = (\overline{g}_i^t)^{-1}$ are conjugate. Then there exists a non-trivial matrix $H \in M_n(\mathbb{C})$ such that $\overline{g}_i^t H g_i = H$ for each i and $\overline{H}^t = H$.*

Proof. Notice that, $\tilde{g}_1 \cdots \tilde{g}_r = \mathrm{Id}$. Moreover, the g_i and $\tilde{g}_i$ are conjugate so by rigidity there exists a matrix $H \in GL(n, \mathbb{C})$ such that $\tilde{g}_i = H g_i H^{-1}$ for all i. Hence $H = \overline{g}_i^t H g_i$ for all i. Moreover, since the system $g_1, \ldots, g_r$ is irreducible, the matrix H is uniquely determined up to a scalar factor. Since $\overline{H}^t$ is also a solution we see that $\overline{H}^t = \lambda H$ for some $\lambda \in \mathbb{C}$. Moreover $|\lambda| = 1$ and writing $\lambda = \mu/\overline{\mu}$ we see that μH is a Hermitian matrix. Now take $H := \mu H$. **qed**

1.5 References

- Haraoka,Y. Finite monodromy of Pochhammer equation, Ann. Inst. Fourier, Grenoble 44 (1994), 767-810

- Katz, N.M. *Rigid Local Systems*, Annals of Math.Studies 139, Princeton 1996.

- Völklein,Strambach, On linearly rigid tuples, J.reine angew. Mathematik 510 (1999), 57-62.

- Erdélyi,A (Editor), *Higher Transcendental Functions I* Mcgraw Hill, New York, 1953.

- Beukers,F. Heckman,G. Monodromy for the hypergeometric function ${}_nF_{n-1}$, Inv. Math. 95 (1989), 325-354.

- Thomae,J: Über die höheren hypergeometrische Reihen, Math. Ann. 2 (1870), 427-444.

- Levelt, A.H.M. Hypergeometric Functions, Thesis, University of Amsterdam 1961.

- F.Rodriguez-Villegas, Integral ratios of factorials and algebraic hypergeometric functions, Summary talk at Oberwolfach 2004, see; http://www.ma.utexas.edu/users/villegas/publications.