Home work problems for WISB324

The material up to chapter 14 are relevant for a number of these problems and the statement from Ch 15 that the number of non-equivalent irreducible representations equals the number of conjugacy classes of the group.

1. Let, as usual, $CG$ be the group algebra of a finite group $G$.
   
   (a) Show that for every $CG$-homomorphism $\phi : CG \rightarrow CG$ there exists $w \in CG$ such that $\phi(r) = rw$ (hint: take $w = \phi(e)$).
   
   (b) Let $W \subset CG$ be an irreducible $CG$-submodule of $CG$. Let $w \in W$ be a non-zero element. Show that $W = \{rw | r \in CG\}$.

2. Define the group $G = \langle a, b | a^5 = b^4 = e, b^{-1}ab = a^{-1} \rangle$.
   
   (a) Show that $b^2$ commutes with all elements of $G$.
   
   (b) Determine all conjugacy classes of $G$.
   
   (c) Determine all one-dimensional representations of $G$.
   
   (d) Determine the dimensions of all irreducible representations
   
   (e) Determine all higher dimensional (i.e. dim > 1) representations of $G$. Give the matrix images (up to conjugation) of $a, b$ for these representations.

3. Define the vector space
   
   $V = \left\{ \sum_{1 \leq i < j \leq 4} a_{ij}x_ix_j \bigg| a_{ij} \in \mathbb{C} \right\} \subset \mathbb{C}[x_1, \ldots, x_4]$. 
   
   Define the representation $\rho$ of $S_4$ on $V$ by $\sigma : x_ix_j \mapsto x_{\sigma(i)}x_{\sigma(j)}$ for all $i, j$.
   
   (a) Determine the characters of $\rho$.
   
   (b) Determine the irreducible representations that compose $\rho$ (hint: use the character table of $S_4$, to be completed on Monday May 27, or consult p351 of the book).
   
   (c) Determine a basis for each of the irreducible subrepresentations of $\rho$.

4. Consider the representation $\rho$ of $S_5$ on $\mathbb{C}^5$ given by
   
   $\sigma e_i \mapsto e_{\sigma(i)}$
   
   for all $i$, where $e_1, \ldots, e_5$ is the standard basis of $\mathbb{C}^5$. Show that $\rho$ is a direct sum of the trivial representation and an irreducible one.