Exam: Representations of finite groups (WISB324)

Wednesday June 28 2017, 9.00-12.00 h.

- You are allowed to bring one piece of A4-paper, which may contain formulas, theorems or whatever you want (written/printed on both sides of the paper).
- All exercise parts having a number (·) are worth 1 point, except when otherwise stated. With 20 points you have a 10 as grade for this exam. There is one bonus exercise of 2 points.
- Do not only give answers, but also prove statements, for instance by referring to a theorem in the book.

Good luck.

1. Consider the group $D_{2n}$ for $n$ odd and $n > 2$ with generators $a$ and $b$ and relations $a^n = 1$, $b^2 = 1$ and $bab = a^{n-1}$. We define a representation $\rho$ on the vector space of complex polynomials in $n$ variables $\mathbb{C}[x_1, x_2, \cdots x_n]$ by defining that $\rho(a)(x_j) = x_{j+1(\mod n)}$ and $\rho(b)(x_j) = x_{n-j+1}$. We extend this to monomials as follows:

   \[ \rho(g)(x_{i_1}x_{i_2}\cdots x_{i_k}) = \rho(g)(x_{i_1})\rho(g)(x_{i_2})\cdots \rho(g)(x_{i_k}). \]

   (a) Show that this indeed defines a representation of $D_{2n}$.
   (b) Show that $V_m = \{ p \in \mathbb{C}[x_1, x_2, \cdots x_n] \mid p \text{ homogeneous of degree } m \}$ is a $\mathbb{C}D_{2n}$-module.
   (c) Show that $V_m$ is not irreducible.
   (d) (Bonus exercise, 2 points) Decompose $V_1$ into a direct sum of irreducible $\mathbb{C}D_{2n}$-submodules.

2. Let $G$ be a group $\psi$ a non-trivial linear character and $\chi$ the only irreducible character of degree $n > 1$.
   (a) Prove that $\psi\chi$ is also an irreducible character and that $\psi\chi = \chi$.
   (b) Prove that $\chi(g) = 0$ if $\psi(g) \neq 1$.

3. Let $G$ be a group with generators $a$ and $b$ and relations $a^7 = 1$, $b^6 = 1$ and $b^{-1}ab = a^3$. The subgroup generated by $a$ is denoted by $H$.
   (a) Show that $H$ is a normal subgroup of $G$ and that $G/H$ is abelian.
   (b) List all conjugacy classes of $G$ by giving one element in each conjugacy class.
   (c) Determine the degrees of the irreducible characters of $G$.
   (d) (2 points) Determine the complete character table of $G$.
   (e) Determine all normal subgroups of $G$.
   (f) Let $\chi$ be a non-trivial character of the subgroup $H$. Compute the induced character $\chi \uparrow G$ and show that this is an irreducible character.
4. Let $G$ be a finite group with character $\chi$. We call $\chi$ real if $\chi(g) \in \mathbb{R}$ for all $g \in G$.

(a) Prove that all characters of $G$ are real if and only if all irreducible characters of $G$ are real.

Let $p > 2$ be a prime number and assume that $C_{p}$ is a normal subgroup of $G$ such that $|G| = mp$ and $\gcd(m, p - 1) = 1$.

(b) Prove $|\text{Aut } C_{p}| = p - 1$.

Let $a \in G$ and define the automorphism $\rho_{a} : C_{p} \to C_{p}$ by $\rho_{a}(x) = axa^{-1}$ for $x \in C_{p}$.

(c) Show that $\rho_{a}\rho_{b} = \rho_{ab}$ and prove that $\rho_{a}^{m} = 1$.

(d) Prove that $\rho_{a} = 1$.

(e) Let $\phi$ be a character of $C_{p}$. Prove that the induced character $\phi \uparrow G$ satisfies

$$
\phi \uparrow G(x) = \begin{cases} 
  m\phi(x) & \text{if } x \in C_{p}, \\
  0 & \text{if } x \notin C_{p}.
\end{cases}
$$

(f) Prove that not all characters of $G$ are real.

5. (2 points) Let $G$ be a group and $H$ a subgroup. Let $\chi$ be a character of $G$ and $\psi$ a character of $H$. Prove Frobenius Reciprocity Theorem by elementary calculations, using the definitions of or formulas for the induced and restricted characters.

Frobenius Reciprocity Theorem states that

$$
\langle \psi, \chi \downarrow H \rangle_{H} = \langle \psi \uparrow G, \chi \rangle_{G}.
$$