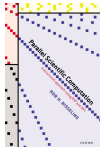
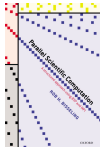
Sequential Fast Fourier Transform
(PSC §3.1–3.2)



Applications of Fourier analysis

- Fourier analysis studies the decomposition of functions into their frequency components.
- Piano Concerto no. 9 by Mozart: enhance high frequencies.
- Chest picture by Computerised Tomography: reconstruct your interior without slicing you up.
- Star picture by pre-repair Hubble Space Telescope: remove blur.



Fourier series

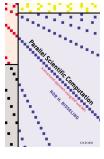
- Let $f : \mathbf{R} \to \mathbf{C}$ be a T-periodic function: $f(t+T) = f(t)$ for all $t \in \mathbf{R}$.
- Fourier series associated with f:

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{2\pi i k t/T}.$$

- Fourier coefficients c_k are given by

$$c_k = \frac{1}{T} \int_0^T f(t) \mathrm{e}^{-2\pi i k t/T} \, dt.$$

- i is the complex number with $i^2 = -1$.
- Series converges if f is piecewise smooth (continuously differentiable).



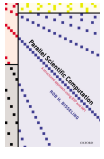
Fourier series for real-valued function

- Complex Fourier coefficients c_k and corresponding real coefficients a_k, b_k for T-periodic $f : \mathbf{R} \to \mathbf{R}$ are given by

$$c_k = a_k - ib_k = \frac{1}{T}\int_0^T f(t)\left(\cos\frac{2\pi kt}{T} - i\sin\frac{2\pi kt}{T}\right)\,dt.$$

- Since $c_{-k} = \overline{c_k}$, $a_k = (c_k + \overline{c_k})/2$, and $b_k = (c_k - \overline{c_k})i/2$:

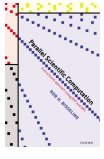
$$\begin{aligned}
\tilde{f}(t) &= \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{2\pi ikt/T} = \sum_{k=1}^{\infty} \overline{c_k}\mathrm{e}^{-2\pi ikt/T} + c_0 + \sum_{k=1}^{\infty} c_k \mathrm{e}^{2\pi ikt/T} \\
&= c_0 + \sum_{k=1}^{\infty}(\overline{c_k} + c_k)\cos\frac{2\pi kt}{T} + \sum_{k=1}^{\infty}(-\overline{c_k} + c_k)i\sin\frac{2\pi kt}{T} \\
&= a_0 + 2\sum_{k=1}^{\infty} a_k \cos\frac{2\pi kt}{T} + 2\sum_{k=1}^{\infty} b_k \sin\frac{2\pi kt}{T}.
\end{aligned}$$



It's a discrete world

One second of audio on a compact disc contains 44,100 function values $f(t_j)$ in regularly spaced sample points

$$t_j = \frac{jT}{n}, \quad 0 \leq j < n.$$



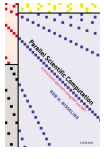
Approximation of Fourier coefficients

- Trapezoidal rule on interval $[t_j, t_{j+1}] = \left[\frac{jT}{n}, \frac{(j+1)T}{n}\right]$:

$$\int_{t_j}^{t_{j+1}} f(t)\,dt \approx \frac{f(t_j) + f(t_{j+1})}{2} \cdot \frac{T}{n}.$$

- On the whole interval $[0, T]$:

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T f(t)e^{-2\pi ikt/T}dt \\ &\approx \frac{1}{T}\cdot\frac{T}{n}\left(\frac{f(0)}{2} + \sum_{j=1}^{n-1} f(t_j)e^{-2\pi ikt_j/T} + \frac{f(T)}{2}\right) \\ &= \frac{1}{n}\sum_{j=0}^{n-1} f(t_j)e^{-2\pi ijk/n} \quad (\text{because } f(0) = f(T) = f(t_0)). \end{aligned}$$



Discrete Fourier transform

- The discrete Fourier transform (DFT) of a vector $\mathbf{x} = (x_0, \ldots, x_{n-1})^T$ is the vector $\mathbf{y} = (y_0, \ldots, y_{n-1})^T$ with

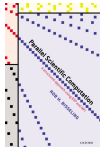
$$y_k = \sum_{j=0}^{n-1} x_j e^{-2\pi ijk/n} = \sum_{j=0}^{n-1} x_j {\omega_n}^{jk}, \text{ for } 0 \leq k < n.$$

  Here, $\omega_n = e^{-2\pi i/n}$.

- Compare:

$$c_k \approx \frac{1}{n} \sum_{j=0}^{n-1} f(t_j) e^{-2\pi ijk/n}$$

  Thus $\mathbf{c} \approx DFT(\mathbf{x})$, where $x_j = f(t_j)/n$.

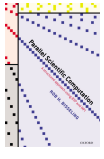


Inverse DFT

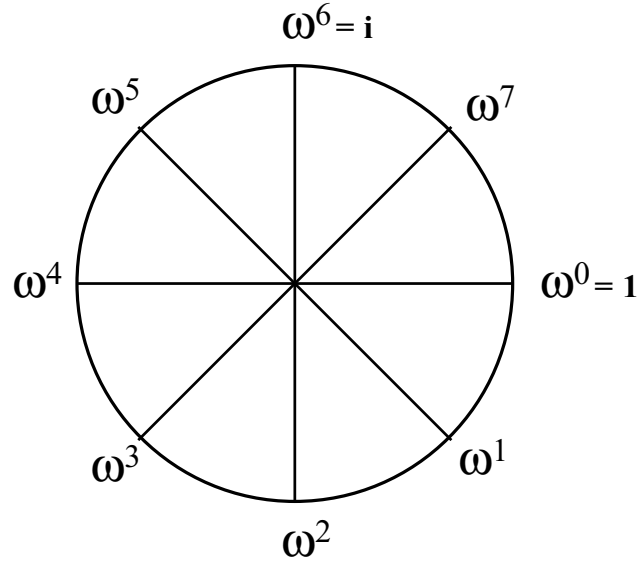
- Easy to prove: the inverse DFT (IDFT) of a vector $\mathbf{x} = (x_0, \ldots, x_{n-1})^T$ is the vector $\mathbf{y} = (y_0, \ldots, y_{n-1})^T$ with

$$y_k = \frac{1}{n} \sum_{j=0}^{n-1} x_j e^{+2\pi i j k/n}, \text{ for } 0 \leq k < n.$$

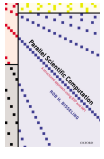
  Same as DFT formula, except for the scaling $1/n$ and the sign of the exponent.



Roots of unity



- $\omega_8 = e^{-2\pi i/8} = e^{-\pi i/4} = \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i.$
- ${\omega_n}^n = e^{-2\pi i n/n} = e^{-2\pi i} = 1.$
- ${\omega_n}^{n/2} = e^{-2\pi i(n/2)/n} = e^{-\pi i} = -1.$
- ${\omega_n}^2 = e^{-4\pi i/n} = e^{-2\pi i/(n/2)} = \omega_{n/2}.$



Matrix–vector multiplication

- Define the $n \times n$ Fourier matrix F_n by

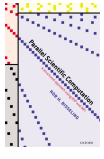
$$(F_n)_{jk} = {\omega_n}^{jk}, \quad \text{for } 0 \leq j, k < n.$$

- Hence $F_n\mathbf{x} = DFT(\mathbf{x})$:

$$(F_n\mathbf{x})_j = \sum_{k=0}^{n-1}(F_n)_{jk}x_k = \sum_{k=0}^{n-1} x_k{\omega_n}^{jk} = (DFT(\mathbf{x}))_j.$$

- Because $\omega_4 = e^{-2\pi i/4} = e^{-\pi i/2} = -i$:

$$F_4 = \begin{bmatrix} {\omega_4}^0 & {\omega_4}^0 & {\omega_4}^0 & {\omega_4}^0 \\ {\omega_4}^0 & {\omega_4}^1 & {\omega_4}^2 & {\omega_4}^3 \\ {\omega_4}^0 & {\omega_4}^2 & {\omega_4}^4 & {\omega_4}^6 \\ {\omega_4}^0 & {\omega_4}^3 & {\omega_4}^6 & {\omega_4}^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$



Cost of straightforward DFT

- Complex multiplication

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

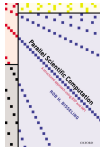
  requires 1 real addition, 1 real subtraction, 4 real multiplications, hence a total of 6 flops.

- Complex addition

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

  requires 2 real additions.

- To compute y_k, we need n complex multiplications and $n-1$ complex additions, so $6n + 2(n-1) = 8n - 2$ flops.
- To compute the n components of $\mathbf{y}$, we need $8n^2 - 2n$ flops.



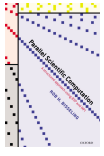
Splitting into even and odd components

$$y_k = \sum_{j=0}^{n-1} x_j \omega_n^{jk} = \sum_{j=0}^{n/2-1} x_{2j} \omega_n^{2jk} + \sum_{j=0}^{n/2-1} x_{2j+1} \omega_n^{(2j+1)k}.$$

Using $\omega_n^2 = \omega_{n/2}$ gives

$$y_k = \sum_{j=0}^{n/2-1} x_{2j} \omega_{n/2}^{jk} + \omega_n^k \sum_{j=0}^{n/2-1} x_{2j+1} \omega_{n/2}^{jk}, \quad \text{for } 0 \leq k < n.$$

- Each sum is a DFT of length $n/2$, with $0 \leq k < n/2$.
- Thus, we can compute the first half of the DFT by a DFT on the even components of $\mathbf{x}$ and a DFT on the odd components.
- Cost is $2 \cdot [8(n/2)^2 - 2(n/2)] + 8(n/2) = 4n^2 + 2n$ flops.



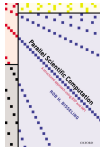
Computing the second half of the DFT

Let $n/2 \leq k < n$. Substituting $k = k' + n/2$ into

$$y_k = \sum_{j=0}^{n/2-1} x_{2j}\omega_{n/2}^{jk} + \omega_n^k \sum_{j=0}^{n/2-1} x_{2j+1}\omega_{n/2}^{jk}$$

gives $0 \leq k' < n/2$ and

$$\begin{aligned} y_{k'+n/2} &= \sum_{j=0}^{n/2-1} x_{2j}\omega_{n/2}^{j(k'+n/2)} + \omega_n^{k'+n/2} \sum_{j=0}^{n/2-1} x_{2j+1}\omega_{n/2}^{j(k'+n/2)} \\ &= \sum_{j=0}^{n/2-1} x_{2j}\omega_{n/2}^{jk'} - \omega_n^{k'} \sum_{j=0}^{n/2-1} x_{2j+1}\omega_{n/2}^{jk'}, \end{aligned}$$

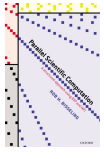


because $\omega_{n/2}^{n/2} = 1$ and $\omega_n^{n/2} = -1$. Now drop the primes.

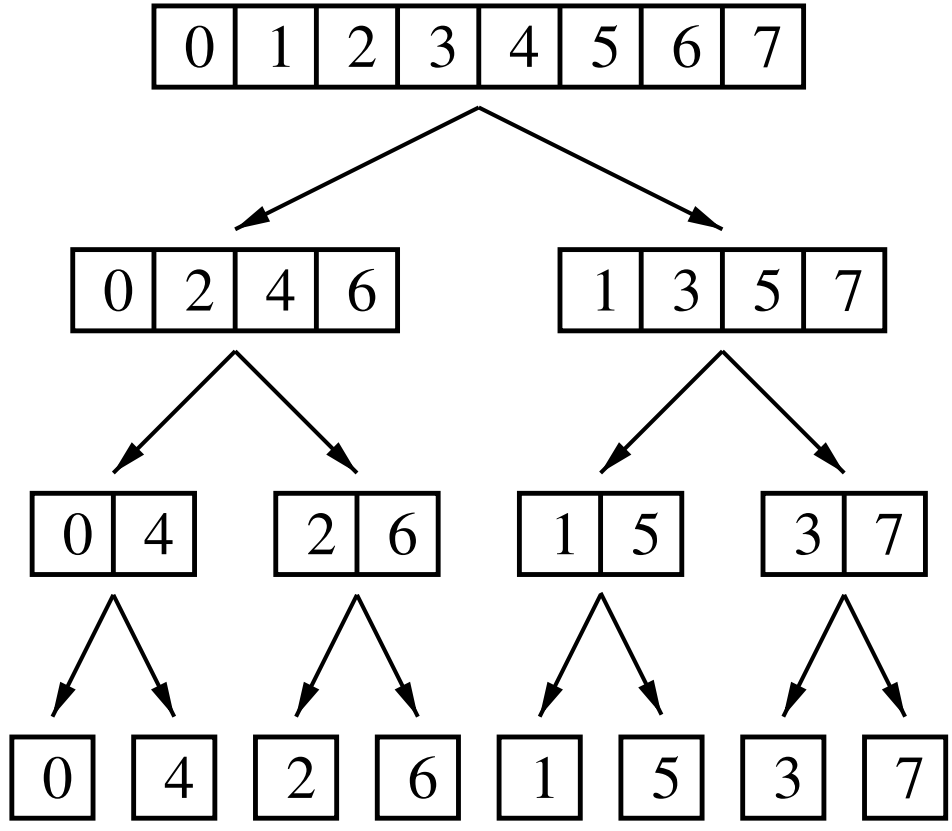
Cost reduction of one split

$$y_{k+n/2} = \sum_{j=0}^{n/2-1} x_{2j}\omega_{n/2}^{jk} \; - \; \omega_n^k \sum_{j=0}^{n/2-1} x_{2j+1}\omega_{n/2}^{jk}, \quad \text{for } 0 \leq k < n/2.$$

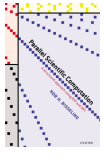
- This is the same formula as for the first half, except for the subtraction.
- Thus, we can compute the second half of the DFT almost without extra work, performing just $n/2$ complex subtractions, i.e., n flops.
- The total cost for the whole DFT with one split is $4n^2 + 3n$ flops, thus saving about half the flops from the original $8n^2 - 2n$.



Recursive computation of DFT



The problem is split repeatedly, until the problem size is 1.

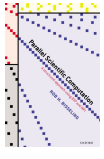


Recursive fast Fourier transform (FFT) algorithm

input: $\mathbf{x}$: vector of length n.
output: $\mathbf{y}$: vector of length n, $\mathbf{y} = F_n\mathbf{x}$.
call: $\mathbf{y} := \mathrm{FFT}(\mathbf{x}, n)$.

if $n \bmod 2 = 0$ **then**
$\mathbf{x}^{\mathrm{e}} := x(0\colon 2\colon n-1)$; $\mathbf{y}^{\mathrm{e}} := \mathrm{FFT}(\mathbf{x}^{\mathrm{e}}, n/2)$;
$\mathbf{x}^{\mathrm{o}} := x(1\colon 2\colon n-1)$; $\mathbf{y}^{\mathrm{o}} := \mathrm{FFT}(\mathbf{x}^{\mathrm{o}}, n/2)$;
for $k := 0$ **to** $n/2-1$ **do**
$\tau := \omega_n^k y_k^{\mathrm{o}}$;
$y_k := y_k^{\mathrm{e}} + \tau$;
$y_{k+n/2} := y_k^{\mathrm{e}} - \tau$;
else $\mathbf{y} := \mathrm{DFT}(\mathbf{x}, n)$;

$$y_k = \sum_{j=0}^{n/2-1} x_{2j}\omega_{n/2}^{jk} + \omega_n^k \sum_{j=0}^{n/2-1} x_{2j+1}\omega_{n/2}^{jk}, \quad \text{for } 0 \le k < n/2.$$

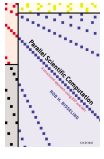


Cost of fast Fourier transform

- Loop has complex multiplication, addition, subtraction, together $6 + 2 + 2 = 10$ flops.
- $n/2$ iterations of loop, hence a total of $n/2 \cdot 10 = 5n$ flops.
- Perform 2 FFT($n/2$) operations and $5n$ flops for FFT(n):

$$\begin{aligned} T(n) &= 2T(\frac{n}{2}) + 5n \\ &= 2\left(2T(\frac{n}{4}) + 5\frac{n}{2}\right) + 5n = 4T(\frac{n}{4}) + 2 \cdot 5n \\ &= \cdots = nT(1) + (\log_2 n) \cdot 5n = 5n \log_2 n. \end{aligned}$$

- Much faster than $8n^2$ time for direct computation of DFT.
- For $n = 32768$ (0.74 s audio), an FFT can be done real-time on a 3.3 Mflop/s PC, but it would take 43 min using the straightforward DFT. Gain factor: 3500.



Summary

- The fast Fourier transform (FFT) idea was discovered by Gauss (1805), rediscovered by Danielson and Lanczos (1942), and is commonly attributed to Cooley and Tukey (1965), who rediscovered it in the digital era.
- The FFT is the computational workhorse in many applications, from weather forecasting to signal and image processing. Without the FFT, modern medicine would be impossible.
- The cost of an FFT of length n is $5n \log_2 n$ flops.
- We have derived a recursive FFT algorithm, i.e., an algorithm that calls itself with a smaller problem size.