Exercise 1: 1D Fourier Transform pairs

Set the time domain and the frequency domain

\[ N = 1024; \]
\[ dt = 1/N; \]
\[ t = -0.5:dt:0.5-dt; \]
\[ \omega = -N/2:N/2-1; \]

and plot the following functions with the corresponding (discrete) Fourier transforms:

\[ \omega_0 = 50; \sigma = .01; a = 0.05; bf = 0.005; \]
\[ f_0 = \text{zeros}(1,N); f_0(\text{find(abs(t-\omega_0/N)<dt/2)))=1; \]
\[ f_1 = \exp(2*\pi*i*\omega_0*t); \]
\[ f_2 = \cos(2*\pi*\omega_0*t); \]
\[ f_3 = -\text{abs}(t)+1; \]
\[ f_4 = \text{zeros}(1,N); f_4(\text{find(abs(t)<a)))=1; \]
\[ f_5 = \exp(-t.^2/(2*\sigma)); \]
\[ f_6 = (1.\exp((-t^2)/(bf+1))-1.\exp((t+0.2)/(bf+1))); \]

Change the parameters \( \omega_0, \sigma, a, \) and \( bf \). What do you observe? Give explanations.

Hints for the plots:

\[ \text{figure; subplot(1,2,1); plot(t, ...); title('The function') } \]
\[ F = \text{fftshift(fft(fftshift(...)))); title('The DFT of the function') } \]

Exercise 2: 2D Fourier Transform pairs

The same as above but on a 2D domain. Now we take the couples \((x, y)\) and \((\omega_x, \omega_y)\) instead of \(t\) and \(\omega\), respectively. The 2D domains:

\[ N = 256; \]
\[ dx = 1/N; \]
\[ xv = -0.5:dx:0.5-dx; yv = xv; \]
\[ \omega_\text{axv} = -N/2:N/2-1; \omega_\text{ayv} = \omega_\text{axv}; \]
\[ [x\ y] = \text{meshgrid}(xv,yv); \]
\[ [\text{omegax omegay}] = \text{meshgrid}(\text{omegaxv,omegayv}); \]
\[ r = \sqrt{x^2+y^2}; \% \text{you may need it to compute the functions with circular symmetric profiles} \]

For example, define the functions

\[ \omega_0x = 5; \omega_0y = -10; \sigma = .01; a = 0.05; bf = 0.005; \% \text{the parameters} \]
\[ f_{0,2} = \text{zeros}(N,N); f_{0,2}(\text{find}(\text{abs}(x-\omega_0x/N)<dt/2 \& \text{abs}(y-\omega_0y/N)<dt/2))=1; \]
\[ f_{1,2} = \exp(2\pi i*(\omega_0x*x+\omega_0y*y)); \]
\[ f_{2,2} = \cos(2\pi*(\omega_0x*x+\omega_0y*y)); \]
\[ f_{3,2} = \ldots; \]
\[ f_{4,2} = \text{zeros}(N,N); f_{4,2}(\text{rv})=1; f_{4,2}(\text{find}(\text{abs}(r)<a))=1; f_{4,2}=\text{reshape}(f_{4,2},N,N); \]
\[ f_{5,2} = \ldots; \]
\[ f_{6,2} = (1./(\exp((r-0.2)/bf)+1)-1./\exp((r+0.2)/bf)+1)); \]

The 2D DFT is \( F_2 = \text{fftshift} (\text{fft2} (\text{fftshift}(\ldots)))) \).

Do not forget to change the parameters!

Hint for plots: instead of the \texttt{plot} function, use the commands

\[ \text{mesh}(xv,yv,\ldots) \% \text{here replace the dots with the function} \]
\[ \text{mesh}(\text{omegaxv,omegayv,\ldots} F_2) \% \text{here replace the dots with abs or real or imag or log10(abs} \]
\[ \text{or the command imagesc} \]

**Exercise 3: the shift theorem**

Take \( f_3 \) from exercise 1 and compute its Fourier transform, \( F_3 \). Multiply \( F_3 \) by a linear phase term \( \exp(i2\pi\omega t_0) \) with \( t_0=0.2 \). Apply Inverse Fourier Transform to the resulting function. What do you notice? Give explanations.

Hint: to compute the inverse DFT use \( \text{fftshift}(\text{ifft}(\text{fftshift}(\ldots)))) \) (replace the dots with the function)

**Exercise 4: the convolution theorem**

Take \( f_4 \) from exercise 1 with \( a = 0.05 \). Compute the DFT and plot it:

\[ F_4 = \text{fftshift}(\text{fft}(\text{fftshift}(f_4))); \]
\[ \text{figure;plot(omega,real(F_4))} \]

Now construct a boxcar function, \( W(\omega) \) in the frequency domain, for instance:

\[ BW=200; \]
\[ W=\text{zeros}(1,N); W(\text{find}(\text{abs}(\omega)<BW))=1; \]
\[ \text{figure;plot(omega,real(W))} \]

Multiply \( F_4 \) by \( W \) and denote the product by \( F_4W \). Transform back \( F_4W \) the to the time domain. What do you observe? Can you explain this phenomenon at the hand of the Convolution theorem?

**Exercise 5 (extra)**

Let \( f \) be \( 2\pi \)-periodic given by \( f(t) = t - \pi \) for \( t \in [0,2\pi) \). Show that:

a) \( \gamma_k = i/k \)

b) \[ \frac{s^2}{\nu} = \sum_{k=1}^{\infty} \frac{1}{k^2} \] (Hint: use Parseval’s formula)