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Estimates for Kloosterman sums for totally real number fields

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1. Introduction

In [12], Kuznetsov gave a sum formula in which Fourier coefficients of real analytic modular forms on the upper half plane are related to Kloosterman sums; see also [13].

This formula has been used in various ways. In [13], it is applied to the classical Kloosterman sums $S(n, m; k) = \sum_x^* e^{2\pi i (nx + m\bar{x})/k}$, where $n, m, k \in \mathbb{Z}$, $n, m \neq 0$, $k > 0$; in the sum, x runs over the integers $0 < x \leqq k$ that are coprime to k, and satisfies $x\bar{x} \equiv 1 \bmod k$. Kuznetsov shows that

$$(1) \qquad \sum_{1 \leqq k \leqq X} \frac{S(n, m; k)}{k} = O(X^{1/6} (\log X)^{1/3}) \quad (X \to \infty),$$

see Theorem 3 in [13]. The main theme of this paper is a generalization of this result to the Hilbert modular group.

For any number field F, there are Kloosterman sums

$$(2) \qquad S(r, r'; c) = \sum_{d \bmod (c)}^* e^{2\pi i \operatorname{Tr}_{F/\mathbb{Q}}((rd + r'a)/c)},$$

where $c \neq 0$ is an element of the ring of integers $\mathcal{O}$ of F, and where r and r' are non-zero elements of $\mathcal{O}'$, the inverse different of F. These Kloosterman sums satisfy a bound of Weil-Salié type, see [4], Theorem 10.

In this paper, we consider these Kloosterman sums for totally real number fields F of degree d. We denote the different embeddings $\sigma_1, \ldots, \sigma_d$: $F \to \mathbb{R}$. We show that for $d \geqq 2$:

$$(3) \qquad \sum_{c \in \mathcal{O}, 0 < |c^{\sigma_j}| \leqq X} \frac{S(r, r'; c)}{|N_{F/\mathbb{Q}}(c)|} \ll X^{(d-1)/2 + \varepsilon} \quad \text{as } X \to \infty,$$

under the assumption of absence of eigenvalues with exceptional coordinates. For $d = 1$, the method yields an $O(X^{\frac{1}{6}+\varepsilon})$ estimate. Theorem 4.7.1 gives a more complete statement. It shows that the presence of eigenvalues of exceptional type may add terms that are larger than the estimate given here and it also gives the influence of the parameters r and r'. However, if we input the best known bounds for exceptional eigenvalues in Corollary 4.7.2, our result implies cancellation of Kloosterman sums for any totally real number field F. That is, the bound given in Corollary 4.7.2 is strictly smaller than the bound obtained by using Weil's estimate (see Remark after Corollary 4.7.2).

Our proof is based on the bound of Weil-Salié type mentioned above, and uses a new sum formula of Kuznetsov type. Actually, the sum formula turns out to be useful only on the region $|c^{\sigma_j}| \geqq 1$ for all j. For the other regions, a better result is obtained by a more direct estimate, that does not use cancellation between Kloosterman sums.

Joyner, [10], gives an estimate of sums of Kloosterman sums for real quadratic number fields of class number one. His estimate for this special case is similar, but seemingly stronger than the one in Theorem 4.7.1 of this paper. However, there are gaps in proofs in [10]. (See Comparison after Corollary 4.7.2.)

Kuznetsov's sum formula is concerned with modular forms on the group $\mathrm{SL}_2(\mathbb{R})$. It has been extended in various ways. Its extension in [17] treats automorphic forms on Lie groups of real rank one. In [4], this extension is used to study sums of Kloosterman sums for this class of groups, whereas [1] gives a formula which includes the contribution of all K-types, in the case of $\mathrm{SL}_2(\mathbb{R})$.

In these cases, the sum formula has the following form:

$$\int_{\mathscr{S}} k(\nu)\, d\sigma(\nu) = \Delta(k) + \sum_{\gamma} S(\gamma)\tilde{k}(\xi_\gamma). \tag{4}$$

Here $d\sigma$ is a measure with support $\mathscr{S} \subset i[0, \infty) \cup (0, \infty)$; this measure can be described in terms of Fourier coefficients of automorphic forms for a discrete subgroup Γ of the Lie group G under consideration. The γ run over a subset of Γ, the $S(\gamma)$ are generalized Kloosterman sums, and $\xi_\gamma \in G$ is determined by γ. The term $\Delta(k)$ is given by an integral over the line $\mathrm{Re}\,\nu = 0$. The function $\tilde{k}$ depends on the test function k by an integral transformation.

One difference of this paper with the method in [4] is that there one uses an approximate inversion of the transformation $k \mapsto \tilde{k}$. That approach, together with a Weil type estimate, suffices to get an estimate on averages of Kloosterman sums, which for the modular group coincides with Kuznetsov's estimate. In the case when Γ is SL_2 over the ring of integers of an imaginary quadratic number field, in the absence of exceptional eigenvalues, it yields the result stated by Sarnak in [20], p. 308.

In the present paper, we generalize Kuznetsov's sum formula to the case when Γ is a subgroup of finite index in the Hilbert modular group. The main difference with the approach in [4] is that here we shall invert the integral transforms exactly, more in the spirit of [13] and [1]. In order to do an exact inversion, it will now be necessary to use a version of the sum formula that includes the contribution of non-trivial K-types. This formula will have the same structure as the ones in [12], [13], [17], [4], [5], but the various terms will be somewhat more complicated since they will include the contribution of the discrete series, as in [1]. This will allow to get an arbitrary smooth function of compact support as $\tilde{k}$ in (4). (This is not guaranteed for the Kuznetsov formula in, for instance, [5].)

We formulate the new version of the sum formula in Section 2, and prove it in Section 5. We apply the sum formula to the case of congruence subgroups $\Gamma_0(I)$ of Hecke type, for which the Kloosterman sums are precisely those defined in (2). In Section 3, we use the Weil-Salié bound to give an estimate of the Kloosterman term $\sum_{\gamma} S(\gamma)\tilde{k}(\xi_\gamma)$ for suitably chosen test functions k. This yields an estimate of the measure $d\sigma$. In Section 4, another choice of test function allows us to use this information in the other direction, and leads to estimates of sums of Kloosterman sums.

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2. Description of the extended sum formula

Let F be a totally real number field, and let $\mathcal{O}$ be its ring of integers. We consider the algebraic group $\mathbf{G} = R_{F/\mathbb{Q}}(\mathrm{SL}_2)$ over $\mathbb{Q}$ obtained by restriction of scalars applied to SL_2 over F.

Let $\sigma_1, \ldots, \sigma_d$ be the embeddings $F \to \mathbb{R}$. We have

$$(5) \qquad G := \mathbf{G}_{\mathbb{R}} \cong \mathrm{SL}_2(\mathbb{R})^d, \quad \mathbf{G}_{\mathbb{Q}} \cong \{(x^{\sigma_1}, \ldots, x^{\sigma_d}) \colon x \in \mathrm{SL}_2(F)\}.$$

G contains $K := \prod_{j=1}^{d} \mathrm{SO}_2(\mathbb{R})$ as a maximal compact subgroup.

The image of $\mathrm{SL}_2(\mathcal{O}) \subset \mathrm{SL}_2(F)$ corresponds to $\mathbf{G}_{\mathbb{Z}}$. This is a discrete subgroup of $\mathbf{G}_{\mathbb{R}}$ with finite covolume. It is called the *Hilbert modular group*, see [6], §3. In this section, $\Gamma \subset \mathbf{G}_{\mathbb{Z}}$ is a subgroup of finite index.

For this situation, we shall prove a Kuznetsov type sum formula that is more general than the one considered in [23] and [18]. It has the structure indicated in (4); a precise formulation will have to wait till we have more notation available, see Subsection 2.7. The main variable is the *test function* k. It is a holomorphic function on a region in $\mathbb{C}^d$. The function $\tilde{k}$ depends on k via a Bessel transformation, to be described in Definition 2.5.4.

The sum formula relates three terms. The measure $d\sigma$, to be defined in (19), gives information on Fourier coefficients and on the eigenvalues of the Casimir operators at the real places of automorphic forms for Γ. The Kloosterman term $\sum_{\gamma} S(\gamma)\tilde{k}(\xi_\gamma)$, to be defined more precisely in (22), is a sum containing (generalized) Kloosterman sums associated to the group Γ. In (29), we define the term $\Delta(k)$ by an explicit expression.

The present version of the sum formula has two advantages in comparison with those in [23] and [18]: It contains information concerning all automorphic forms, not only those with trivial K-type, and the transformation $k \mapsto \tilde{k}$ is more versatile.

We restrict ourselves to automorphic forms of even weight. So we shall consider only functions on G that satisfy $f(gm) = f(g)$ for all m in the center

$$M := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}^d$$

of G.

Overview. The Subsections 2.1–2.6 form a preparation for the statement of the sum formula in Subsection 2.7. After fixing some notations in Subsection 2.1, we discuss automorphic forms for Γ in Subsection 2.2. We define generalized Kloosterman sums in Subsection 2.3, the delta term in Subsection 2.6, and discuss the class of test functions for the sum formula in Subsection 2.5.

We prove the sum formula in Section 5.

2.1. Notations and conventions. We consider F as embedded in $\mathbb{R}^d$ by

$$\xi \mapsto (\xi^{\sigma_1}, \dots, \xi^{\sigma_d}),$$

and similarly $\mathbf{G}_{\mathbb{Q}} \subset G$.

Definition 2.1.1. For $x, y \in \mathbb{R}^d$ we define $xy \in \mathbb{R}^d$ by $(xy)_j := x_j y_j$, and $|x| \in \mathbb{R}^d$ by $|x|_j = |x_j|$.

For $x \in \mathbb{R}^d$ we put $S(x) := \sum_{j=1}^{d} x_j$. This extends the trace $\mathrm{Tr}_{F/\mathbb{Q}}: F \to \mathbb{Q}$. Similarly, $N(y) := \prod_{j=1}^{d} y_j$ extends the norm of F over $\mathbb{Q}$ to $N: (\mathbb{R}^*)^d \to \mathbb{R}^*$.

Functions of product type. The test functions on G that we use are often of product type: $f(g) = \prod_{j=1}^{d} f_j(g_j)$ for $g = (g_1, \dots, g_d) \in G$, with f_j a function on the j-th factor $\mathrm{SL}_2(\mathbb{R})$. We use the notation $f = \bigtimes_{j=1}^{d} f_j$. We use the same concept of product type, and the $\times$-notation, for functions on $\mathbb{R}^d$ and other products.

Subgroups of G. We put, respectively, for $y \in \mathbb{R}^d_{>0}$, $x \in \mathbb{R}^d$, and $\vartheta \in (\mathbb{R}/2\pi\mathbb{Z})^d$:

$$(6) \qquad a[y] := \left(\begin{pmatrix} \sqrt{y_1} & 0 \\ 0 & 1/\sqrt{y_1} \end{pmatrix}, \dots, \begin{pmatrix} \sqrt{y_d} & 0 \\ 0 & 1/\sqrt{y_d} \end{pmatrix} \right) \in G,$$

$$n[x] := \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & x_d \\ 0 & 1 \end{pmatrix} \right) \in G,$$

$$k[\vartheta] := \left(\begin{pmatrix} \cos\vartheta_1 & \sin\vartheta_1 \\ -\sin\vartheta_1 & \cos\vartheta_1 \end{pmatrix}, \dots, \begin{pmatrix} \cos\vartheta_d & \sin\vartheta_d \\ -\sin\vartheta_d & \cos\vartheta_d \end{pmatrix} \right) \in G.$$

In this way, we obtain three subgroups of G: the identity component of a maximal $\mathbb{R}$-split torus $A := \{a[y]: y \in \mathbb{R}^d_{>0}\}$, the unipotent subgroup $N := \{n[x]: x \in \mathbb{R}^d\}$ and the maximal compact subgroup $K := \{k[\vartheta] : \vartheta \in \mathbb{R}^d\} = SO_2(\mathbb{R})^d$. We normalize the Haar measures on these three groups by respectively $da := \frac{dy_1}{y_1} \cdots \frac{dy_d}{y_d}$ for $a = a[y]$, $dn := \frac{dx_1}{\pi} \cdots \frac{dx_d}{\pi}$ for $n = n[x]$, and $dk := \frac{d\vartheta_1}{2\pi} \cdots \frac{d\vartheta_d}{2\pi}$ for $k = k[\vartheta]$.

$M = \{k[\vartheta] : \vartheta \in (\pi\mathbb{Z})^d\}$ is the center of G. We call $P := NAM$ the *standard parabolic subgroup* of G.

Let C_j be the Casimir operator of the j-th factor in $G = \prod_{j=1}^{d} SL_2(\mathbb{R})$. We normalize it such that it corresponds to the differential operator $C_j = -y_j^2 \partial_{y_j}^2 - y_j^2 \partial_{x_j}^2 + y_j \partial_{x_j} \partial_{\vartheta_j}$ in the coordinates $g = n[x]a[y]k[\vartheta]$.

Roots. As in [23], §1, we use the roots α_j defined by $a[y]^{\alpha_j} := y_j$ and let $\rho = \sum_{j=1}^{d} \frac{1}{2} \alpha_j$.

The identity component A_0 of the $\mathbb{Q}$-split torus $\mathbf{A}_0$ in [23], §1 consists of the $a[y]$ with $y_1 = \cdots = y_d$. The group ${}^0A$ is characterized as the kernel of $a \mapsto a^\rho$.

We choose $dg := a^{-2\rho}\, dn\, da\, dk$ as the Haar measure on $G = NAK$.

Characters of K. The character $\mathfrak{d}_q : k[\vartheta] \mapsto e^{iS(q\vartheta)}$ of K is well defined for each $q \in \mathbb{Z}^d$. A function on G has *weight q* if it transforms on the right according to this character. As we restrict ourselves to even functions, we consider only weights $q \in (2\mathbb{Z})^d$.

Cusps. From [6], Corollary 3.5_1, it follows that Γ has a finite number of cusp classes. Let $\mathscr{P}$ be a set of representatives of those classes. For each $\kappa \in \mathscr{P}$ we fix $g_\kappa \in \mathbf{G}_\mathbb{Q}$ such that $\kappa = g_\kappa \cdot \infty$.

For each $\kappa \in \mathscr{P}$ there is a parabolic group $P^\kappa := g_\kappa P g_\kappa^{-1}$, with decomposition $P^\kappa := N^\kappa A^\kappa M$, $N^\kappa := g_\kappa N g_\kappa^{-1}$, $A^\kappa := g_\kappa A g_\kappa^{-1}$. We put $A_0^\kappa := g_\kappa A_0 g_\kappa^{-1}$, ${}^0A^\kappa := g_\kappa {}^0A g_\kappa^{-1}$, and $a_\kappa^\rho := (g_\kappa^{-1} a g_\kappa)^\rho$. We use conjugation by g_κ to transport the Haar measures of N and A to N^κ and A^κ.

Let $\kappa \in \mathscr{P}$. For each $g \in G$, we have a unique decomposition $g = n_\kappa(g) g_\kappa a_\kappa(g) k_\kappa(g)$, with $n_\kappa(g) \in N^\kappa$, $a_\kappa(g) \in A$, and $k_\kappa(g) \in K$. The seemingly unnatural choice to take $n_\kappa(g)$ in N^κ, instead of in N, is convenient later on.

Discrete subgroups. We define $\Gamma_{P^\kappa} := \Gamma \cap P^\kappa$ and $\Gamma_{N^\kappa} := \Gamma \cap N^\kappa$. We refer to the discussion in [6], §2, §3 for the following facts: There is a lattice $\mathfrak{t}_\kappa$ in $\mathbb{R}^d$ such that $\Gamma_{N^\kappa} = \{g_\kappa n[\xi] g_\kappa^{-1} : \xi \in \mathfrak{t}_\kappa\}$. There is also a lattice Λ_κ in the hyperplane $u_1 + \cdots + u_d = 0$ in $\mathbb{R}^d$ such that all elements of Γ_{P^κ} have the form $n g_\kappa a[y] g_\kappa^{-1} m$, with $n \in N^\kappa$, $m \in M$, and $(\log y_1, \ldots, \log y_d) \in \Lambda_\kappa$.

Characters. All characters of N^κ are $\chi_r : g_\kappa n[x] g_\kappa^{-1} \mapsto e^{2\pi i S(rx)}$, with $r \in \mathbb{R}^d$. The characters of $\Gamma_{N^\kappa} \backslash N^\kappa$ are obtained by taking $r \in \mathfrak{t}_\kappa' := \{r \in \mathbb{R}^d : S(rx) \in \mathbb{Z}$ for all $x \in \mathfrak{t}_\kappa\}$. Proposition AI.19 in [6] implies that if $r \in \mathfrak{t}_\kappa'$ is non-zero, then all its components are non-zero. We identify χ_r with the character $n[x] \mapsto e^{2\pi i S(rx)}$ of N.

Let $a = g_\kappa a[y]g_\kappa^{-1} \in A^\kappa$. Then $aN^\kappa a^{-1} = N^\kappa$, and $\chi_r(ana^{-1}) = \chi_{yr}(n)$. So it is sensible to define $a \cdot r := yr \in \mathbb{R}^d$. If we write $\gamma \in \Gamma_{P^\kappa}$ in the form $n_\gamma a_\gamma m_\gamma$, with $n_\gamma = n_\kappa(\gamma g_\kappa) \in N^\kappa$, $a_\gamma = g_\kappa a_\kappa(\gamma g_\kappa)g_\kappa^{-1} \in A^\kappa$, $m_\gamma = k_\kappa(\gamma g_\kappa) \in M$, then $\gamma n \gamma^{-1} = a_\gamma n a_\gamma^{-1}$ for all $n \in N^\kappa$, and $a_\gamma \cdot \mathfrak{t}'_\kappa = \mathfrak{t}'_\kappa$. If $r \in \mathfrak{t}'_\kappa$, $r \neq 0$, then $a_\gamma \cdot r = r$ implies $a_\gamma = 1$.

2.2. Automorphic forms.

Definition 2.2.1. Let $q \in (2\mathbb{Z})^d$ and $\lambda \in \mathbb{C}^d$. An *automorphic form* for Γ is a function $f \in C^\infty(G)$ satisfying

i) *Transformation behavior.* $f(\gamma g k) = f(g)\mathfrak{d}_q(k)$ for all $\gamma \in \Gamma$, $g \in G$, and $k \in K$.

ii) *Eigenfunction of the Casimir operators.* $C_j f = \lambda_j f$ for $j = 1, \ldots, d$.

iii) *Polynomial growth.* If $F = \mathbb{Q}$, then $f(n_\kappa g_\kappa a[y]g_\kappa^{-1}) = O(y^b)$ as $y \to \infty$ for some $b \in \mathbb{R}$, uniformly in $n_\kappa \in N^\kappa$, for each $\kappa \in \mathscr{P}$.

We call q the *weight* and λ the *eigenvalue* of f. Often we write $\lambda_j = \frac{1}{4} - \nu_j^2$, with $\operatorname{Re} \nu_j \geqq 0$, and call $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{C}^d$ the *spectral parameter*.

Let $\mathfrak{H}$ be the upper half plane. If F is a holomorphic automorphic form on $\mathfrak{H}^d$ for Γ, as in Definition 4.5 of [6], of weight q, then

$$f(n_\infty[x]a_\infty[y]k) := \prod_{j=1}^{d} (y_j^{q_j/2}) F(z)\mathfrak{d}_q(k),$$

with $z = (x_1 + iy_1, \ldots, x_d + iy_d)$, is an automorphic form as defined above, with weight q and eigenvalue λ given by $\lambda_j = \frac{1}{2}q_j - \frac{1}{4}q_j^2$.

Fourier expansion. For each $\kappa \in \mathscr{P}$ an automorphic form f is invariant under Γ_{N^κ} on the left. So there is an absolutely convergent Fourier expansion

$$(7) \qquad f(ng_\kappa a) = \sum_{r \in \mathfrak{t}'_\kappa} \chi_r(n) F_\kappa(r, f; a) \quad \text{for } n \in N^\kappa, a \in A.$$

Proposition 2.2.2. *Let $d > 1$. For each automorphic form f and each $\kappa \in \mathscr{P}$, there exists $b \in \mathbb{R}$ such that $f(ng_\kappa a[y]) = O(N(y)^b)$ for $y_j \geqq 1$, $n \in N_\kappa$.*

Remark. This is the so-called *Götzky-Koecher principle*, see Proposition 4.9 in [6]. It shows that condition iii) in Definition 2.2.1 is automatically satisfied if the totally real number field F is not the field of rational numbers.

The proof is the same as in the holomorphic case discussed in [6]. As this proof shows us the structure of the Fourier terms $F_\kappa(r, f)$, we repeat it in the present context.

Proof. Condition ii) imposes d linear second order differential equations on the functions $F_\kappa(r, f)$ on A^κ, each differential equation involving only one component $a^{g_\kappa \alpha_j}$.

For $r \neq 0$, explicit computations lead to the Whittaker differential equation at each place (see, e.g., [3], 4.2.5, 4.2.8), and a decomposition

$$F_\kappa(r, f; a[y]) = \prod_{j=1}^{d} h_j(y_j),$$

where h_j is a linear combination

$$h_j(y) = \tilde{c}_j W_{\mathrm{sign}(r_j)q_j/2, \nu_j}(4\pi|r_j|y) + c_j W_{-\mathrm{sign}(r_j)q_j/2, \nu_j}(-4\pi|r_j|y)$$

of Whittaker functions, with a convention concerning the branch to be used in the second term. The h_j are unique up to a multiplicative factor. In particular

$$c_j := \lim_{y \to \infty} h_j(y) y^{\frac{1}{2}\mathrm{sign}(r_j)q_j} e^{-2\pi|r_j|y}$$

exists. If $d = 1$, then condition iii) forces $c_1 = 0$. The Fourier term of order zero has also the form $F_\kappa(0, f; a[y]) = \prod_{j=1}^{d} h_j(y_j)$. Here h_j is a linear combination of $y \mapsto y^{\nu_j + 1/2}$ and $y \mapsto y^{-\nu_j + 1/2}$ if $\lambda_j \neq \frac{1}{4}$, and of $y \mapsto y^{1/2}$ and $y \mapsto y^{1/2} \log y$ if $\lambda_j = \frac{1}{4}$. So this Fourier term satisfies the condition of polynomial growth.

Condition i) applied to $\gamma \in \Gamma_{P^\kappa}$, with $\gamma = n_\gamma g_\kappa a_\gamma g_\kappa^{-1} m_\gamma$, $n_\gamma \in N^\kappa$, $a_\gamma \in A$, $m_\gamma \in M$, gives

$$F_\kappa(a_\gamma \cdot r, f; a) = \chi_r(n_\gamma) F_\kappa(r, f; a_\gamma a). \tag{8}$$

Let us fix $r \in \mathfrak{t}'_\kappa$, $r \neq 0$. The absolute convergence of the Fourier series implies the convergence of the subsum

$$\sum_{\gamma \in (\Gamma_{P^\kappa} \cap N^\kappa M) \backslash \Gamma_{P^\kappa}} |\chi_{a_\gamma \cdot r}(n) F_\kappa(a_\gamma \cdot r, f; a)| = \sum_{\gamma \in (\Gamma_{P^\kappa} \cap N^\kappa M) \backslash \Gamma_{P^\kappa}} |F_\kappa(r, f; a_\gamma a)|.$$

Suppose $c_\ell \neq 0$ for a non-zero $F_\kappa(r, f)$, $r \neq 0$. There exists $\gamma \in \Gamma_{P^\kappa}$ such that $a_\gamma = a[y]$ satisfies $y_\ell > 1$, and $y_j < 1$ for $j \neq \ell$. The sum $\sum_{n=1}^{\infty} |F_\kappa(r, f; a[y]^n)|$ converges, so its terms go to zero. For all j, there are $a_j, b_j \in \mathbb{R}$ such that $\lim_{y \downarrow 0} h_j(y) y^{a_j} (\log y)^{b_j}$ exists and is non-zero. (For general values of ν_j, we have $a_j = -\frac{1}{2} \mp \nu_j$, $b_j = 0$.) Thus we have obtained that

$$\lim_{n \to \infty} y_\ell^{-\frac{1}{2}\mathrm{sign}(r_\ell)q_\ell n} e^{2\pi|r_\ell|y_\ell^n} \prod_{j \neq \ell} \left(y_j^{na_j} (n \log y_j)^{b_j}\right) = 0.$$

This is impossible. Hence each h_ℓ is a multiple of a rapidly decreasing Whittaker function:

$$F_\kappa(r, f; a[y]) = (\text{constant}) \cdot \prod_{j=1}^{d} W_{\mathrm{sign}(r_j)q_j/2, \nu_j}(4\pi|r_j|y_j) \quad \text{for each } r \in \mathfrak{t}'_\kappa.$$

A reasoning as in [3], Lemma 4.3.7, together with the known form of $F_\kappa(0,f)$, gives polynomial growth of f at the cusp κ. □

Notation. Define, for $\kappa \in \mathscr{P}$, $r \in \mathfrak{t}'_\kappa \setminus \{0\}$, $n \in N^\kappa$, $y \in \mathbb{R}^d_{>0}$, $k \in K$, $\nu \in \mathbb{C}^d$:

$$\mathrm{W}^{r,\nu}_{\kappa,q}(ng_\kappa a[y]k) := \chi_r(n)\mathfrak{d}_q(k)\prod_{j=1}^{d} W_{\mathrm{sign}(r_j)q_j/2,\nu_j}(4\pi|r_j|y_j). \tag{9}$$

In this way, the Fourier expansion at κ of the automorphic form f of weight q has the form

$$f(g) = F_\kappa(0,f;a_\kappa(g))\mathfrak{d}_q(k_\kappa(g)) + \sum_{r\in\mathfrak{t}'_\kappa\setminus\{0\}} a_\kappa(r,f)\mathrm{W}^{r,\nu}_{\kappa,q}(g). \tag{10}$$

Maass operators. The Lie algebra of G acts on automorphic forms by differentiation on the right. In [3], §2.2, §4.5, we see that there are differential operators $\mathbf{E}_j^\pm$ in the complexification of the universal enveloping algebra of G sending automorphic forms of weight q to automorphic forms of weight $q \pm 2\varepsilon_j$, where ε_j is the j-th unit vector in $\mathbb{R}^d$. These operators preserve the eigenvalue. Table 4.1 on page 63 of [3] shows that for $r \in \mathfrak{t}'_\kappa$, $r \neq 0$:

$$a_\kappa(r,\mathbf{E}_j^\pm f) = a_\kappa(r,f)\cdot\begin{cases} -2 & \text{if } \pm r_j > 0,\\ 2\left(\frac{1}{4}(q_j \pm 1)^2 - \nu_j^2\right) & \text{if } \pm r_j < 0.\end{cases} \tag{11}$$

Eisenstein series. For each $\kappa \in \mathscr{P}$, $q \in (2\mathbb{Z})^d$, there is an Eisenstein series

$$E_q(P^\kappa,\nu,i\mu,g) := \sum_{\Gamma_{P^\kappa}\setminus\Gamma} a_\kappa(\gamma g)^{p+2\nu p+i\mu}\mathfrak{d}_q(k_\kappa(\gamma g)). \tag{12}$$

Here $\nu \in \mathbb{C}$, and μ is an element of a lattice $\mathscr{L}_\kappa$ in the hyperplane $S(x) = 0$ in $\mathbb{R}^d$. The series converges for $\operatorname{Re}\nu > \frac{1}{2}$, and has a meromorphic continuation in ν. In this way $\nu \mapsto E_q(P^\kappa,\nu,i\mu)$ is a meromorphic family of automorphic forms of weight q, with eigenvalue $\nu \mapsto \left(\frac{1}{4} - (\nu + i\mu_j)^2\right)_j$. The Eisenstein series satisfy

$$\mathbf{E}_j^\pm E_q(P^\kappa,\nu,i\mu) = (1 + 2\nu + 2i\mu_j \pm q_j)E_{q\pm 2\varepsilon_j}(P^\kappa,\nu,i\mu). \tag{13}$$

Spectral decomposition. Let $L^2(\Gamma\backslash G,q)$ be the closed subspace of $L^2(\Gamma\backslash G)$ consisting of the elements transforming on the right according to the character $k[\vartheta] \mapsto e^{iS(q\vartheta)}$ of K, with q running through $(2\mathbb{Z})^d$. The space $L^2(\Gamma\backslash G,q)$ of weight $q \in (2\mathbb{Z})^d$ is the orthogonal direct sum of subspaces $L^2_c(\Gamma\backslash G,q)$ and $L^2_d(\Gamma\backslash G,q)$. The subspace $L^2_d(\Gamma\backslash G,q)$ has a countable orthonormal basis $\mathscr{U}_q$ consisting of square integrable automorphic forms of weight q. The orthogonal complement, $L^2_c(\Gamma\backslash G,q)$, is described by integrals of Eisenstein series. For bounded functions f, f_1 in $L^2(\Gamma\backslash G, q)$, the projections f^c and f_1^c onto the space $L_c^2(\Gamma\backslash G, q)$ satisfy

$$(14)\quad \langle f^c, f_1^c\rangle = \sum_{\lambda\in\mathscr{P}} c_\lambda \sum_{\mu\in\mathscr{L}_\lambda} \int_{-\infty}^{\infty} \langle f, E_q(P^\lambda, iy, i\mu)\rangle \overline{\langle f_1, E_q(P^\lambda, iy, i\mu)\rangle}\, dy,$$

with suitable constants c_λ. So for f, f_1 as above, $\langle f, f_1\rangle = \langle f^c, f_1^c\rangle + \sum_{\psi\in\mathscr{U}_q} \langle f, \psi\rangle\overline{\langle f_1, \psi\rangle}$.

Representations. Let $L^2(\Gamma\backslash G)^+$ be the closure of $\sum_{q\in 2\mathbb{Z}} L^2(\Gamma\backslash G, q)$, and similarly for $L_c^2(\Gamma\backslash G)^+$. This is an invariant subspace of $L^2(\Gamma\backslash G)^+$ for the action of G by right translation. The orthogonal complement $L_d^2(\Gamma\backslash G)^+$ is the closure of $\bigoplus_{\varpi} V_\varpi$, where V_ϖ runs through an orthogonal family of closed irreducible subspaces for the G-action. Each ϖ has the form $\varpi = \bigotimes_j \varpi_j$, with ϖ_j an even unitary irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. Table 1 lists the possible isomorphism classes for each ϖ_j. For each ϖ we define the spectral parameter $\nu_\varpi = (\nu_{\varpi,1}, \ldots, \nu_{\varpi,d})$, with $\nu_{\varpi,j}$ as in the last column of the table. The eigenvalue λ_ϖ is given by $\lambda_{\varpi,j} = \frac{1}{4} - \nu_{\varpi,j}^2$.

The constant functions give rise to $\varpi = 1 := \bigotimes_j 1$. It occurs with multiplicity one. Using Proposition AI.19 in [6], one can show that if V_ϖ does not consist of the constant functions, then $\varpi_j \neq 1$ for all j.

Table 1. ***Irreducible unitary even representations of the Lie group*** $\mathbf{SL_2(\mathbb{R})}$.
All characters of $\mathrm{SO}_2(\mathbb{R})$ occur at most once; the characters that occur are listed under *weights*. The last column gives the spectral parameter ν, with $\mathrm{Re}\,\nu \geqq 0$, such that $\frac{1}{4} - \nu^2$ is the eigenvalue of the Casimir operator. See [14], Chap. VI, §6.

notation	name	weights	ν
1	trivial representation	0	$\frac{1}{2}$
$H(s)$, $s \in i[0,\infty)$	unitary principal series	$q \in 2\mathbb{Z}$	$\frac{s}{2}$
$H(s)$, $s \in (0,1)$	complementary series	$q \in 2\mathbb{Z}$	$\frac{s}{2}$
D_b^+, $b \geqq 2, b \in 2\mathbb{Z}$	holomorphic discrete series	$q \geqq b$, $q \in 2\mathbb{Z}$	$\frac{b-1}{2}$
D_b^-, $b \geqq 2, b \in 2\mathbb{Z}$	antiholomorphic discrete series	$q \leqq -b$, $q \in 2\mathbb{Z}$	$\frac{b-1}{2}$

For each ϖ the subspace $V_{\varpi,q}$ of weight $q \in (2\mathbb{Z})^d$ has dimension at most 1. We could choose bases $\mathscr{U}_q$ by taking a unit vector in $V_{\varpi,q}$ for each ϖ with $V_{\varpi,q} \neq \{0\}$. But it is more convenient to take an orthogonal system $\{\psi_{\varpi,q}\}$ with $\psi_{\varpi,q} \in V_{\varpi,q}$ that satisfies

$$\mathbf{E}_j^{\pm}\psi_{\varpi,q} = (1 + 2\nu_{\varpi,j} \pm q_j)\psi_{\varpi,q\pm 2e_j},$$

just like the Eisenstein series. The norms are $\|\psi_{\varpi,q}\| = \prod_{j=1}^{d} \mathrm{n}(q_j, \nu_{\varpi,j})^{1/2}$, where

$$(15)\qquad \mathrm{n}(q,\nu) := \begin{cases} 1 & \text{if } \operatorname{Re}\nu = 0, \\ \dfrac{\Gamma(1/2-\nu-q/2)}{\Gamma(1/2+\nu-q/2)} = \dfrac{\Gamma(1/2-\nu+q/2)}{\Gamma(1/2+\nu+q/2)} & \text{if } 0 < \nu < \dfrac{1}{2} \\ \left(\dfrac{|q|-b}{2}\right)! \Big/ \left(\dfrac{b+|q|}{2}-1\right)! & \text{if } \nu = \dfrac{b-1}{2},\ b \in 2\mathbb{N}. \end{cases}$$

(See [14], Chap. VI, §6.)

Proposition 2.2.3. *Let $j \in \{1,\dots,d\}$. Suppose that ϖ satisfies $\varpi_j = D_b^{\pm}$ for some j, with $b \in 2\mathbb{Z}$, $b \geqq 2$. Then $a_\kappa(r,\psi_{\varpi,q}) = 0$ for all $\psi_{\varpi,q} \in V_\varpi$, for all $r \in \mathfrak{t}_\kappa' \setminus \{0\}$ such that $\mp r_j > 0$.*

Remark. This proposition generalizes the fact that the non-zero Fourier coefficients of a holomorphic automorphic form on $\mathfrak{H}^d$ have an order r satisfying $r_j \geqq 0$ for all j.

Proof. Let $f \in V_{\varpi,q}$ be non-zero. Equation (11) implies that

$$a_\kappa(r, (\mathbf{E}_j^{\mp})^m f) \neq 0 \quad \text{for all } m \geqq 0,$$

if and only if $a_\kappa(r,f) \neq 0$. But for m large enough, the weight $q_j \mp 2m$ does not occur in $D_b^{\pm}$. □

Fourier coefficients. Formula (11) shows that the Fourier coefficients of a given order r are essentially a property of ϖ, not of the individual automorphic forms in V_ϖ.

For the formulation of the sum formula it is convenient to introduce the following functions:

$$(16)\qquad d_\kappa^r(q,\nu) := \frac{1}{\operatorname{vol}(\Gamma_{N^\kappa}\backslash N^\kappa)} \prod_{j=1}^{d} \frac{(-1)^{q_j/2}(2\pi|r_j|)^{-1/2}}{\Gamma\left(\frac{1}{2}+\nu_j+\frac{1}{2}q_j\operatorname{sign}(r_j)\right)}.$$

Note that $d_\kappa^r(q,\nu) \neq 0$ for all weights q that occur in an irreducible unitary representation with spectral parameter ν. This d_κ^r is related to the function d_r in (4) of [5], but not exactly equal. A computation based on (11) shows that the equations

$$(17)\qquad a_\kappa(r,\psi_{\varpi,q}) = c_\kappa^r(\varpi) d_\kappa^r(q,\nu_\varpi),$$

$$(18)\qquad a_\kappa(r, E_q(P^\lambda,\nu,i\mu)) = D_\lambda^{\kappa,r}(\nu,i\mu) d_k^r(q,\nu+i\mu)$$

determine $c_\kappa^r(\varpi)$ and $D_\lambda^{\kappa,r}(\nu,i\mu)$, independently of q.

Definition 2.2.4. We define

$$Y := \left(i[0,\infty) \cup \left(0,\frac{1}{2}\right) \cup \left\{\frac{b-1}{2} : b \geqq 2, b \in 2\mathbb{Z}\right\}\right)^d.$$

Consider $\kappa, \kappa' \in \mathscr{P}$, $r \in \mathfrak{t}_\kappa' \setminus \{0\}$, $r' \in \mathfrak{t}_{\kappa'}' \setminus \{0\}$. We define the measure $d\sigma_{r,r'}^{\kappa,\kappa'}$ on Y by

$$(19) \quad \int_Y \eta(\nu)\, d\sigma_{r,r'}^{\kappa,\kappa'}(\nu) := \sum_{\varpi \neq 1} \eta(\nu_\varpi)\overline{c_\kappa^r(\varpi)}c_{\kappa'}^{r'}(\varpi)$$
$$+ \sum_{\lambda \in \mathscr{P}} c_\lambda \sum_{\mu \in \mathscr{L}_\lambda} \int_{-\infty}^{\infty} \eta(iy + i\mu)\overline{D_\lambda^{\kappa,r}(iy, i\mu)} D_\lambda^{\kappa',r'}(iy, i\mu)\, dy.$$

We identify the complex number iy with $(iy, iy, \ldots, iy) \in \mathbb{C}^d$.

The constant functions do not occur in the sum over ϖ, as its Fourier coefficients of non-zero order vanish. The next result shows that there may be many ϖ that do not contribute to the measure $d\sigma_{r,r'}^{\kappa,\kappa'}$.

Proposition 2.2.5. *Let κ, κ', r, r' be as in Definition 2.2.4. Put $E_{r,r'} := \{j : r_j r_j' < 0\}$. All ϖ that satisfy $\varpi_j = D_b^{\pm}$ for some $j \in E_{r,r'}$ do not contribute to the measure $d\sigma_{r,r'}^{\kappa,\kappa'}$. The support of $d\sigma_{r,r'}^{\kappa,\kappa'}$ is contained in $\left\{\nu \in Y : \mathrm{Re}\, \nu_j < \frac{1}{2} \text{ for all } j \in E_{r,r'}\right\}$.*

Proof. See Proposition 2.2.3. □

Notation. For κ, κ', r, r' as above we define $\mathbf{e} \in \{1, -1\}^d$ by $\mathbf{e}_j = \mathrm{sign}(r_j r_j')$. We put $Y^{\mathbf{e}} := \left\{\nu \in Y : \mathrm{Re}\, \nu_j < \frac{1}{2} \text{ for all } j \in E_{r,r'}\right\}$. In this way $\mathrm{Supp}(d\sigma_{r,r'}^{\kappa,\kappa'}) \subset Y^{\mathbf{e}}$.

Remark. In arithmetical situations, there are restrictions on the complementary series representations that can occur as a factor of an automorphic representation. For instance, Gelbart and Jacquet, see [7], Theorem (9.3), prove that no ν_j can be in the interval $\left[\frac{1}{4}, \frac{1}{2}\right)$. In [15] there is an even stronger restriction for the case $d = 1$. These results have been obtained by L-function methods. In [5], the sum formula is used to prove that $\nu_j \in \left(\frac{1}{4}, \frac{1}{2}\right)$ is impossible; in Subsection 3.3 we shall adapt that argument to the present situation.

We call $\frac{1}{4} - \nu^2 := \left(\frac{1}{4} - \nu_j^2\right)_j$ an *exceptional eigenvalue*, if $\nu_j \in \left(0, \frac{1}{2}\right)$ for some $j = 1, \ldots, d$. Such a ν_j we call an *exceptional coordinate.*

If $d = 1$, it can be shown that only finitely many exceptional eigenvalues can occur for a given Γ. In the present more general situation, we do not have this information. One coordinate could stay small, whereas other coordinates tend to infinity in some sequence of eigenvalues.

2.3. Kloosterman term. Kloosterman sums can be viewed as number theoretic objects. They arise also in the theory of automorphic forms, when one writes down the Fourier expansion of Poincaré series. Then one has to sum over the intersection of Γ with the big cell in the Bruhat decomposition.

Bruhat decomposition. Let $s_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The group $\mathrm{SL}_2(F) \cong \mathbf{G}_\mathbb{Q}$ is the disjoint union of $\mathbf{G}_\mathbb{Q} \cap P$ and the big cell $\mathrm{C} := (P \cap \mathbf{G}_\mathbb{Q})s_0(N \cap \mathbf{G}_\mathbb{Q})$.

Each $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}_{\mathbb{Q}}$ with $c \neq 0$ can be written uniquely in the form $\xi = n'(\xi)m(\xi)a_\xi s_0 n''(\xi)$, with $n'(\xi) = n[a/c]$, $n''(\xi) = n[d/c]$, $a_\xi = a[y]$ with $y_j = |c^{\sigma_j}|^{-2}$, and $m(\xi) \in M$. Note that a_ξ and $m(\xi)$ are not necessarily in $\mathbf{G}_{\mathbb{Q}}$, but

$$m(\xi)a_\xi = a_\xi m(\xi) = \begin{pmatrix} 1/c & 0 \\ 0 & c \end{pmatrix} \in \mathbf{G}_{\mathbb{Q}}.$$

Definition 2.3.1. Let $\kappa, \kappa' \in \mathscr{P}$. We define ${}^{\kappa'}\Gamma^{\kappa} := \Gamma \cap g_{\kappa'} C g_{\kappa}^{-1}$ and

$${}^{\kappa'}\mathscr{C}^{\kappa} := \{c \in F^*: g_{\kappa'}^{-1}\gamma g_\kappa = \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix} \text{ for some } \gamma \in {}^{\kappa'}\Gamma^{\kappa}\}.$$

For each $c \in {}^{\kappa'}\mathscr{C}^{\kappa}$ we put ${}^{\kappa'}\Gamma^{\kappa}(c) := \left\{\gamma \in {}^{\kappa'}\Gamma^{\kappa}: g_{\kappa'}^{-1}\gamma g_\kappa = \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix}\right\}$.

Note that $a_\xi = a[c^{-2}]$ for $\xi \in g_{\kappa'}^{-1}\,{}^{\kappa'}\Gamma^{\kappa}(c) g_\kappa$.

Proposition 2.3.2. *Let $\kappa, \kappa' \in \mathscr{P}$. For each $c \in {}^{\kappa'}\mathscr{C}^{\kappa}$ there is a finite set ${}^{\kappa'}\mathscr{S}^{\kappa}(c) \subset \mathbf{G}_{\mathbb{Q}}$ such that ${}^{\kappa'}\Gamma^{\kappa}(c) = \bigsqcup_{\gamma \in {}^{\kappa'}\mathscr{S}^{\kappa}(c)} \Gamma_{N^{\kappa'}}\gamma\Gamma_{N^{\kappa}}$. The set ${}^{\kappa'}\mathscr{S}^{\kappa} := \bigcup_{c \in {}^{\kappa'}\mathscr{C}^{\kappa}} {}^{\kappa'}\mathscr{S}^{\kappa}(c)$ is a system of representatives for $\Gamma_{N^{\kappa'}} \backslash {}^{\kappa'}\Gamma^{\kappa} / \Gamma_{N^{\kappa}}$.*

$\Gamma = \Gamma_{P^\kappa} \sqcup {}^{\kappa'}\Gamma^{\kappa}$ (disjoint union) if $\kappa = \kappa'$, and $\Gamma = {}^{\kappa'}\Gamma^{\kappa}$ otherwise.

Proof. Well known. □

Definition 2.3.3. For $\kappa, \kappa' \in \mathscr{P}$, $r \in \mathfrak{t}'_\kappa \backslash \{0\}$, $r' \in \mathfrak{t}'_{\kappa'} \backslash \{0\}$, we define the generalized *Kloosterman sum*:

$$S(\kappa, r; \kappa', r'; c) := \sum_{\gamma \in {}^{\kappa'}\mathscr{S}^{\kappa}(c)} \chi_r\big(n''(g_{\kappa'}^{-1}\gamma g_\kappa)\big)\chi_{r'}\big(n'(g_{\kappa'}^{-1}\gamma g_\kappa)\big) \tag{20}$$

$$= \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in g_{\kappa'}^{-1}\,{}^{\kappa'}\mathscr{S}^{\kappa}(c) g_\kappa} e^{2\pi i S(rd/c) + 2\pi i S(r'a/c)}. \tag{21}$$

This definition does not depend on the choice of the system of representatives ${}^{\kappa'}\mathscr{S}^{\kappa}(c)$.

The definition amounts to $S(\kappa, r; \kappa', r'; c) = \sum_g e^{2\pi i S(r(d/c)) + 2\pi i S(r'(a/c))}$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs through $g_{\kappa'}^{-1}\,{}^{\kappa'}\mathscr{S}^{\kappa}(c) g_\kappa$. (The interpretation of $S\left(r\frac{d}{c}\right)$ is $\sum_{j=1}^{d} r_j (d/c)^{\sigma_j}$.)

Definition 2.3.4. Let $\kappa, \kappa' \in \mathscr{P}$, $r \in \mathfrak{t}'_\kappa \backslash \{0\}$, $r' \in \mathfrak{t}'_{\kappa'} \backslash \{0\}$. For each function f on $\mathbb{R}^d_{>0}$ we define

$$K^{\kappa,\kappa'}_{r,r'}(f) := \sum_{c \in {}^{\kappa'}\mathscr{C}^{\kappa}} \frac{S(\kappa, r; \kappa', r'; c)}{|N(c)|} f\left(\left|\frac{rr'}{c^2}\right|\right). \tag{22}$$

The interpretation of the argument of f is $\left|\frac{rr'}{c^2}\right|_j = \frac{|r_j r'_j|}{(c^{\sigma_j})^2}$.

The *Kloosterman term* in the sum formula is $K^{\kappa,\kappa'}_{-r,-r'}(f)$ for a certain test function f. The set ${}^{\kappa'}\mathscr{C}^{\kappa}$ is discrete in $\mathbb{R}^d$. This gives the absolute convergence of $K^{\kappa,\kappa'}_{r,r'}(f)$ for compactly supported f. In Proposition 5.1.2 we show the convergence for a larger class of functions.

2.4. Congruence case. Our main interest in this paper is in the following subgroups of finite index in the Hilbert modular group.

Definition 2.4.1. Let I be a non-zero ideal of $\mathcal{O}$. We define

$$\Gamma_0(I) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \colon c \in I \right\}.$$

This group has a cusp ∞ corresponding to the standard parabolic subgroup

$$P^{\infty} = P = NAM.$$

We have ${}^{\infty}\mathscr{C}^{\infty} = I \backslash \{0\}$, $\Delta_N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \colon b \in \mathcal{O} \right\}$, $\Gamma_P = \left\{ \begin{pmatrix} \varepsilon & b \\ 0 & \varepsilon^{-1} \end{pmatrix} \colon \varepsilon \in \mathcal{O}^*, b \in \mathcal{O} \right\}$. We have $t'_{\infty} = \mathcal{O}' = \{x \in F \colon \mathrm{Tr}_{F/\mathbb{Q}}(xy) \in \mathbb{Z} \text{ for all } y \in \mathcal{O}\}$. The inverse of this fractional ideal in F is the different of F.

For each $c \in I \backslash \{0\}$ and $d \in \mathcal{O}$ which is relatively prime to c, we can choose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(I)$. If d runs through representatives modulo c, then the corresponding matrices form a set ${}^{\infty}\mathscr{S}^{\infty}(c)$. The corresponding Kloosterman sum is

$$S(\infty, r; \infty, r'; c) = \sum_{d \bmod c}{}^{*} e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}}((rd + r'a)/c)},$$

with $ad \equiv 1 \bmod c$. In [4], we denoted this Kloosterman sum by $S[r, r'; c]$; here we shall write $S(r, r'; c)$.

Now, the norm $N(c)$ in (22) is the usual norm $N_{F/\mathbb{Q}}$. Note that $|N(c)|$ is equal to the norm of the ideal $(c) = c\,\mathcal{O}$.

2.5. Test functions. The sum formula will depend on parameters $\kappa, \kappa' \in \mathscr{P}$, $r \in t'_{\kappa} \backslash \{0\}$, $r' \in t'_{\kappa'} \backslash \{0\}$. The class of test functions will depend on the vector $(\mathrm{sign}(r_1 r'_1), \ldots, \mathrm{sign}(r_d r'_d))$. That is the vector $\mathbf{e}$ in the definition below.

Definition 2.5.1. Let $\tau \in \left(\frac{1}{2}, 1\right)$, and $\mathbf{e} \in \mathbb{R}^d$ with $\mathbf{e}_j \in \{1, -1\}$. We define $\mathscr{K}^{\mathbf{e}}$ as the set of functions $k = \bigtimes_{j=1}^{d} k_j$, where each k_j is an even function on the set

$$\begin{cases} \{\nu \in \mathbb{C} \colon |\mathrm{Re}\,\nu| \leqq \tau\} \cup \left(\frac{1}{2} + \mathbb{Z}\right) & \text{if } \mathbf{e}_j = 1 \\ \{\nu \in \mathbb{C} \colon |\mathrm{Re}\,\nu| \leqq \tau\} & \text{if } \mathbf{e}_j = -1, \end{cases}$$

that is holomorphic on $\{\nu \in \mathbb{C}: |\mathrm{Re}\,\nu| \leqq \tau\}$, and that satisfies the conditions

$$k_j(\nu) \ll (1 + |\mathrm{Im}\,\nu|)^{-a}$$

on $|\mathrm{Re}\,\nu| \leqq \tau$ for some $a > 2$, and

$$\begin{cases} \sum_{b \geqq 2, b \in 2\mathbb{Z}} \frac{b-1}{2} \left| k_j\left(\frac{b-1}{2}\right)\right| < \infty & \text{if } e_j = 1, \\ h_j\left(\pm \frac{1}{2}\right) = 0 & \text{if } e_j = -1. \end{cases}$$

The dependence on the parameter τ is not visible in the notation.

The function k_j is an element of the class ${}_1F^a_{0,\tau}$ in Definition 14.2.7 in [1] if $e_j = 1$, and an element of ${}_{-1}F^a_{0,\tau}$ otherwise.

If $k \in \mathscr{K}^e$ satisfies $k_j(\nu) \ll (1 + |\mathrm{Im}\,\nu|)^{-a}$ for some $a > \frac{13}{2}$ on the strip and $k_j\left(\frac{b-1}{2}\right) = 0$ for all $b \geq 2$, $b \in 2\mathbb{Z}$, then $\nu \mapsto \frac{k(\nu)}{\cos \pi\nu}$ is the j-th factor of a test function in the sum formula in [5], Definition 5.1.

Definition 2.5.2. Let $e \in \{1, -1\}^d$. We define a measure $d\eta^e$ on Y^e by

$$\int_{Y^e} f(\nu)\, d\eta^e(\nu) := \prod_{j=1}^{d} \left(\frac{i}{2} \int_{\mathrm{Re}\,\nu_j = 0} f_j(\nu_j)\nu_j \tan \pi\nu_j \, d\nu_j + \left(\frac{1+e_j}{2}\right) \sum_{b \geqq 2, b \in 2\mathbb{Z}} \frac{b-1}{2} f_j\left(\frac{b-1}{2}\right)\right)$$

for $f = \bigtimes_{j=1}^{d} f_j \in C_c(Y^e)$.

Definition 2.5.3. Let $e \in \{1, -1\}^d$. We define the *Bessel kernel* $\mathscr{B}_e := \bigtimes_{j=1}^{d} \mathscr{B}_{e_j,j}$ on $(\mathbb{R}_{>0} \times \mathbb{C})^d$ by

$$(23) \qquad \mathscr{B}_{\pm 1,j}(t_j, \nu_j) := \frac{1}{\sin \pi\nu_j} \left(J^{\pm 1}_{-2\nu_j}(4\pi\sqrt{t_j}) - J^{\pm 1}_{2\nu_j}(4\pi\sqrt{t_j})\right),$$

$$\text{where } J^{\varepsilon}_u(y) = \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{k!\,\Gamma(u+1+k)} \left(\frac{y}{2}\right)^{u+2k} = \begin{cases} J_u(y) & \text{if } \varepsilon = 1, \\ I_u(y) & \text{if } \varepsilon = -1. \end{cases}$$

The function $\nu \mapsto \mathscr{B}_e(t, \nu)$ is even and holomorphic on $\mathbb{C}^d$ for each $t \in \mathbb{R}^d_{>0}$. It satisfies $\overline{\mathscr{B}_e(t, \bar{\nu})} = \mathscr{B}_e(t, \nu)$.

Definition 2.5.4. For each $k \in \mathscr{K}^e$, we define the *Bessel transform* $\mathsf{B}_e k$ on $\mathbb{R}^d_{>0}$ by

$$\mathsf{B}_e k(t) := \int_{Y^e} k(\nu) \mathscr{B}_e(t, \nu)\, d\eta^e(\nu).$$

The estimate $J^{\pm 1}_u(y) \ll_{y_0} |\Gamma(u+1)|^{-1} y^{\mathrm{Re}\,u}$ uniformly for $0 < y \leqq y_0$ (based on the power series expansion), shows that the integral defining $\mathsf{B}_e k(t)$ converges absolutely.

Let us write $\mathsf{B}_{\mathrm{e}}k = \bigtimes_{j=1}^{d} \beta_{e_j} k_j$. The estimate shows that we have the following absolutely convergent integral representations:

$$\beta_{\pm 1} k_j(t) = \frac{i}{2} \int_{\mathrm{Re}\, v=0} k_j(v) \left(J_{-2v}^{\pm 1}(4\pi\sqrt{t}) - J_{2v}^{\pm 1}(4\pi\sqrt{t})\right) \frac{v\, dv}{\cos \pi v}$$

$$+ \left(\frac{1 \pm 1}{2}\right) \sum_{b \geqq 2, b \in 2\mathbf{Z}} k_j\left(\frac{b-1}{2}\right)(-1)^{b/2}(b-1) J_{b-1}(4\pi\sqrt{t}) \tag{24}$$

$$= -i \int_{\mathrm{Re}\, v=\alpha} k_j(v) J_{2v}^{\pm 1}(4\pi\sqrt{t}) \frac{v\, dv}{\cos \pi v} \quad \text{for } 0 \leqq \alpha < \frac{1}{2}$$

$$+ \left(\frac{1 \pm 1}{2}\right) \sum_{b \geqq 2, b \in 2\mathbf{Z}} k_j\left(\frac{b-1}{2}\right)(-1)^{b/2}(b-1) J_{b-1}(4\pi\sqrt{t}) \tag{25}$$

$$= -i \int_{\mathrm{Re}\, v=\tau} k_j(v) J_{2v}^{\pm 1}(4\pi\sqrt{t}) \frac{v\, dv}{\cos \pi v}$$

$$+ \left(\frac{1 \pm 1}{2}\right) \sum_{b \geqq 4, b \in 2\mathbf{Z}} k_j\left(\frac{b-1}{2}\right)(-1)^{b/2}(b-1) J_{b-1}(4\pi\sqrt{t}). \tag{26}$$

Note that the term with $b = 2$ has been canceled by the residue at $\frac{1}{2}$ in the transition from (25) to (26). The relation with the definitions in §14.2 of [1] is $\beta_{\pm 1} k_j(t) = (b_0^{\pm})^{\leftarrow} k_j(4\pi\sqrt{t})$.

Proposition 2.5.5. *Let* $k = \bigtimes_{j=1}^{d} k_j \in \mathscr{K}^{\mathrm{e}}$. *Then*

$$\mathsf{B}_{\mathrm{e}} k(t) \ll \prod_{j=1}^{d} \min(t_j^{\tau}, 1). \tag{27}$$

Proof. The estimate of $J_u^{\pm 1}(y) \ll_{y_0} |\Gamma(u+1)|^{-1} y^{\mathrm{Re}\, u}$ for $0 < y \leqq y_0$ applied to (26) gives $\beta_{\pm 1} k_j(t) = O(t^{\tau})$ as $t \downarrow 0$.

From (7) in [24], 6.2, we derive the estimate $J_u(y) \ll e^{\pi |\mathrm{Im}\, u|}/\mathrm{Re}\, u$ for $\mathrm{Re}\, u > 0$. This implies that $\beta_1 k_j(t) \ll 1$ for all $t > 0$.

Use (25) with $0 < \alpha < \frac{1}{2} a - 1$, and note that

$$\frac{1}{\sin \pi u}\left(I_{-u}(y) - I_u(y)\right) = \frac{2}{\pi} K_u(y) \ll y^{-\mathrm{Re}\, u} \left|\Gamma\left(u + \frac{1}{2}\right)\right|,$$

for $\mathrm{Re}\, u > 0$, see (41) in Lemma 11.1 of [5]. Thus we obtain the boundedness of $\beta_{-1} k_j(t)$. □

Proposition 2.5.6. *Let* $\mathrm{e} \in \{1, -1\}^d$. *For each* $f \in C_c^{\infty}(\mathbb{R}_{>0})^d$ *we define* $\mathsf{B}_{\mathrm{e}}^{\leftarrow} f$ *on* $\mathbb{C}^d$ *by*

$$\mathsf{B}_{\mathrm{e}}^{\leftarrow} f(v) := 2^{-d} \int_{\mathbb{R}_{>0}^d} f(t) \mathscr{B}_{\mathrm{e}}(t, v)\, d^{*}t, \tag{28}$$

with $d^*t := \frac{dt_1}{t_1}\cdots\frac{dt_d}{t_d}$. *The function* $\mathsf{B}_{\mathbf{e}}^{-}f$ *is even and holomorphic on* $\mathbb{C}^d$, *and determines an element of* $\mathscr{K}^{\mathbf{e}}$. *Moreover*, $\mathsf{B}_{\mathbf{e}}(\mathsf{B}_{\mathbf{e}}^{-}f) = f$.

Proof. See [1], Lemma 14.2.3 and Proposition 14.2.8. □

2.6. Delta term.

Definition 2.6.1. Let $\kappa, \kappa' \in \mathscr{P}$, $r \in \mathfrak{t}_\kappa'\backslash\{0\}$, $r' \in \mathfrak{t}_{\kappa'}'\backslash\{0\}$. We define $\alpha(\kappa, r; \kappa', r') := 0$ if $\kappa \neq \kappa'$, and $\alpha(\kappa, r; \kappa, r') := \sum_\gamma \chi_{r'}(n_\kappa(\gamma g_\kappa))^{-1}$, where the sum is over representatives γ of those classes in $\Gamma_{N^\kappa}\backslash\Gamma_{P^\kappa}$ for which $\chi_r(n) = \chi_{r'}(\gamma n \gamma^{-1})$ for all $n \in N^\kappa$. This condition amounts to $r = a_\gamma \cdot r'$, with $a_\gamma = g_\kappa a_\kappa(\gamma g_\kappa) g_\kappa^{-1}$.

Note that if $\alpha(\kappa, r; \kappa', r') \neq 0$, then $\operatorname{sign}(r_j) = \operatorname{sign}(r_j')$ for all $j = 1, \ldots, d$.

Notation. $\mathbf{p} := (1, 1, \ldots, 1) \in \{1, -1\}^d$.

Definition 2.6.2. Take κ, κ', r, r' as above. We define, for $k \in \mathscr{K}^{\mathbf{e}}$, the *delta term* by

$$\Delta_{r,r'}^{\kappa,\kappa'}(k) := \operatorname{vol}(\Gamma_{N^\kappa}\backslash N^\kappa)\alpha(\kappa, r; \kappa', r') \int_{Y^{\mathbf{p}}} k(\nu)\, d\eta^{\mathbf{p}}(\nu). \tag{29}$$

See Definition 2.5.2 for the measure $d\eta^{\mathbf{p}}$. The convergence follows easily from the conditions in Definition 2.5.1.

Proposition 2.6.3. *Let* $\mathbf{e}_j = \operatorname{sign}(r_j r_j')$. *Then* $\Delta_{r,r'}^{\kappa,\kappa'}(\mathsf{B}_{\mathbf{e}}^{-}f) = 0$ *for all* $f \in C_c^\infty(\mathbb{R}_{>0})^d$.

Proof. See [1], Lemma 14.2.16. □

2.7. Sum formula. Finally, we are ready to state the *Kuznetsov sum formula* for the present situation:

Theorem 2.7.1. *Let* $\kappa, \kappa' \in \mathscr{P}$, $r \in \mathfrak{t}_\kappa'\backslash\{0\}$, $r' \in \mathfrak{t}_{\kappa'}'\backslash\{0\}$. *Define* $\mathbf{e} \in \{1, -1\}^d$ *by* $\mathbf{e}_j := \operatorname{sign}(r_j r_j')$. *Let* $k \in \mathscr{K}^{\mathbf{e}}$.

The function k *is integrable for the measure* $d\sigma_{r,r'}^{\kappa,\kappa'}$. *The delta term* $\Delta_{r,r'}^{\kappa,\kappa'}(k)$ *and the Kloosterman term* $K_{r,r'}^{\kappa,\kappa'}(\mathsf{B}_{\mathbf{e}}k)$ *converge absolutely, and the following relation holds:*

$$\int_{Y^{\mathbf{e}}} k(\nu)\, d\sigma_{r,r'}^{\kappa,\kappa'}(\nu) = \Delta_{r,r'}^{\kappa,\kappa'}(k) + K_{-r,-r'}^{\kappa,\kappa'}(\mathsf{B}_{\mathbf{e}}k). \tag{30}$$

Remark. κ and κ' denote cusps. The lattice $\mathfrak{t}_\kappa$ corresponds to the subgroup $\Gamma_{N^\kappa} = \Gamma \cap N^\kappa$ of translations for κ. The element $r \in \mathfrak{t}_\kappa'$ determines a character χ_r of $\Gamma_{N^\kappa}\backslash N^\kappa$, and similarly for $r' \in \mathfrak{t}_{\kappa'}'$. See Section 2.1.

The set of test functions $\mathscr{K}^{\mathbf{e}}$ and the Bessel transformation $\mathsf{B}_{\mathbf{e}}$ have been discussed in Section 2.5. See (19), (29), and (22) respectively, for the definitions of $d\sigma_{r,r'}^{\kappa,\kappa'}$, $\Delta_{r,r'}^{\kappa,\kappa'}$, and $K_{r,r'}^{\kappa,\kappa'}$.

Proof. We postpone the proof until Section 5.

2.8. Dependence on choices. The three terms in the sum formula depend on the choice of the elements g_κ transforming the cusp ∞ to $\kappa \in \mathscr{P}$. Let us consider the effect of replacing g_κ by $\tilde{g}_\kappa = g_\kappa p$, with $p = \begin{pmatrix} u & v \\ 0 & 1/u \end{pmatrix} \in P \cap \mathbf{G}_\mathbf{Q}$, and similarly $\tilde{g}_{\kappa'} = g_{\kappa'}\tilde{p}$, $\tilde{p} = \begin{pmatrix} u' & v' \\ 0 & 1/u' \end{pmatrix}$. Note that $v, v' \in F$ and $u, u' \in F^*$.

The new set $\tilde{\mathfrak{t}}_\kappa$ is isomorphic to $\mathfrak{t}_\kappa$ by $\mathfrak{t}_\kappa \to \tilde{\mathfrak{t}}_\kappa$: $\xi \mapsto u^{-2}\xi$; hence $r \mapsto u^2 r$ gives an isomorphism $\mathfrak{t}'_\kappa \to \tilde{\mathfrak{t}}'_\kappa$. So χ_r: $g_\kappa n[x] g_\kappa^{-1} \mapsto e^{2\pi i S(rx)}$ and $\tilde{\chi}_{u^2 r}$: $\tilde{g}_\kappa n[x] \tilde{g}_\kappa^{-1} \mapsto e^{2\pi i S(u^2 rx)}$ are two ways of describing the same character of N^κ.

The set $\widetilde{{}^{\kappa'}\Gamma^\kappa}$ does not change, but there is a bijection ${}^{\kappa'}\mathscr{C}^\kappa \to {}^{\kappa'}\tilde{\mathscr{C}}^\kappa$: $c \mapsto uu'c$. Moreover ${}^{\kappa'}\Gamma^\kappa(uu'c) = {}^{\kappa'}\Gamma^\kappa(c)$, and we can take ${}^{\kappa'}\mathscr{S}^\kappa(uu'c) = {}^{\kappa'}\mathscr{S}^\kappa(c)$. We use (21), and note that $\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$ running through $\tilde{g}_{\kappa'}^{-1}\widetilde{{}^{\kappa'}\mathscr{S}^\kappa}(uu'c)\tilde{g}_\kappa$ can be written as $\begin{pmatrix} au/u' - cv'u & * \\ uu'c & cvu' + du'/u \end{pmatrix}$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ running through $g_{\kappa'}^{-1}{}^{\kappa'}\mathscr{S}^\kappa(c) g_\kappa$. This leads to the following relations:

$$\text{(31)} \qquad \tilde{S}(\kappa, u^2 r; \kappa', (u')^2 r'; uu'c) = e^{2\pi i (S(urv) - S(u'r'v'))} S(\kappa, r; \kappa', r'; c),$$

$$\tilde{K}^{\kappa,\kappa'}_{u^2 r, (u')^2 r'}(f) = \frac{e^{2\pi i (S(urv) - S(u'r'v'))}}{|N(uu')|} K^{\kappa,\kappa'}_{r,r'}(f).$$

Under the transformation $r \mapsto u^2 r$, the signs of the coordinates do not change, so the vector $\mathbf{e}$ and the Bessel transformation $\mathsf{B}_\mathbf{e}$ are not affected.

If the delta term is non-zero, then $\kappa = \kappa'$, and hence $u = u'$, $v' = v$. We write $\gamma \in \Gamma_{P^\kappa}$ as $\gamma = g_\kappa n[x] a[y] m g_\kappa^{-1} = \tilde{g}_\kappa n[\tilde{x}] a[\tilde{y}] \tilde{m} \tilde{g}_\kappa^{-1}$, with $m, \tilde{m} \in M$. A comparison shows that $\tilde{y} = y$ and $\tilde{x} = u^{-2}x - vu^{-1} + vu^{-1}y$. The condition on the $\gamma \in \Gamma_{P^\kappa}$ occurring in the definition of $\alpha(\kappa, r; \kappa', r')$ is the equality of the characters $\chi_r = \tilde{\chi}_{u^2 r}$ and $n \mapsto \chi_{r'}(\gamma n \gamma^{-1}) = \tilde{\chi}_{u^2 r'}(\gamma n \gamma^{-1})$ of N^κ. Both formulations lead to $yr' = r$. For such $\gamma \in \Gamma_{P^\kappa}$, we have $n_\kappa(\gamma g_\kappa) = g_\kappa n[x] g_\kappa^{-1}$, and $\tilde{n}_\kappa(\gamma \tilde{g}_\kappa) = \tilde{g}_\kappa n[\tilde{x}] \tilde{g}_\kappa^{-1}$. Hence

$$\begin{aligned} \tilde{\chi}_{u^2 r'}\big(\tilde{n}_\kappa(\gamma \tilde{g}_\kappa)\big) &= e^{2\pi i S(r'x - urr' + urr'y)} \\ &= e^{2\pi i S(urr) - 2\pi i S(urr')} \chi_{r'}\big(n_\kappa(\gamma g_\kappa)\big). \end{aligned}$$

The measure on N^κ is the measure on N, transported to N^κ by conjugation by g_κ, respectively $\tilde{g}_\kappa$. This leads to $\widetilde{\mathrm{vol}}(\Gamma_{N^\kappa} \backslash N^\kappa) = |N(u)|^{-2} \mathrm{vol}(\Gamma_{N^\kappa} \backslash N^\kappa)$. Thus we find the transformation behavior of the delta term:

$$\begin{aligned} \text{(32)} \qquad \tilde{\Delta}^{\kappa,\kappa'}_{u^2 r, (u')^2 r'}(k) &= |N(u)|^{-2} e^{-2\pi i S(urv) + 2\pi i S(ur'v)} \Delta^{\kappa,\kappa'}_{r,r'}(k) \\ &= |N(uu')|^{-1} e^{-2\pi i S(urv) + 2\pi i S(u'r'v')} \Delta^{\kappa,\kappa'}_{r,r'}(k). \end{aligned}$$

Let us turn to the Fourier coefficients of an automorphic form f. From (9), (10), and (16), we derive the following relations:

$$\tilde{W}^{u^2r,v}_{\kappa,q}(g) = e^{-2\pi i S(urv)} W^{r,v}_{\kappa,q}(g),$$

$$\tilde{a}(u^2r,f) = e^{2\pi i S(urv)} a_\kappa(r,f),$$

$$\tilde{d}^{u^2r}_\kappa(q,v) = |N(u)| d^r_\kappa(q,v).$$

The choice of g_κ does not affect the Haar measure on G, so the $\psi_{\varpi,q}$ stay unchanged. Equation (17) implies that $\tilde{c}^{u^2r}_\kappa(\varpi) = |N(u)|^{-1} e^{2\pi i S(urv)} c^r_\kappa(\varpi)$. We have an analogous result for $D^{\kappa,r}_\lambda(v,i\mu)$, see (18). Thus, we obtain from (19) the transformation behavior of the measure on the spectral side of the sum formula:

$$d\tilde{\sigma}^{\kappa,\kappa'}_{u^2r,(u')^2r'} = |N(uu')|^{-1} e^{-2\pi i S(urv)+2\pi i S(u'r'v')}\, d\sigma^{\kappa,\kappa'}_{r,r'}. \tag{33}$$

One might object that the Eisenstein series $E_q(P^\lambda, v, i\mu)$ depends on the choice of g_λ. But the resulting factor is compensated by the change in the constant c_λ, see (14).

We compare (33) and (32) with (31) (but here with opposite r and r'), to conclude that all terms in the sum formula (30) depend on u and u' in the same way.

3. Estimation of the measure

The main goal of this section is to estimate the measure of the set $Y(X) = \{v \in Y : |v_j| \leqq X_j, 1 \leqq j \leqq d\}$, for $X \in \mathbb{R}^d_{>1/4}$ with respect to $d\sigma^{\kappa,\kappa'}_{r,r'}$. We cannot do this unless we have additional information concerning the Kloosterman term. Hence we shall restrict ourselves to the case $\Gamma = \Gamma_0(I)$, see Subsection 2.4.

3.1. General result. Definition 2.2.4 implies the following result. It enables us to restrict ourselves to estimation of the measure $d\sigma^{\kappa,\kappa}_{r,r}$.

Lemma 3.1.1. *If f is integrable for the measures $d\sigma^{\kappa,\kappa}_{r,r}$ and $d\sigma^{\kappa',\kappa'}_{r',r'}$, then it is also integrable for $d\sigma^{\kappa,\kappa'}_{r,r'}$, and*

$$\left| \int_{Y^e} |f(v)|\, |d\sigma^{\kappa,\kappa'}_{r,r'}(v)| \right| \leqq \left(\int_{Y^p} |f(v)|\, d\sigma^{\kappa,\kappa}_{r,r}(v) \right)^{1/2} \left(\int_{Y^p} |f(v)|\, d\sigma^{\kappa',\kappa'}_{r',r'}(v) \right)^{1/2}.$$

3.2. Estimation of the Kloosterman term. Now we restrict ourselves to the case $\Gamma = \Gamma_0(I)$, $\kappa = \kappa' = \infty$, discussed in Subsection 2.4.

In this section and in the next one, we omit $\kappa = \infty$ and $\kappa' = \infty$ from the notation.

Lemma 3.2.1. *Let $f : (\mathbb{R}^*)^d \to \mathbb{C}$ satisfy*

$$|f(y)| \leqq C(f) \prod_{j=1}^{d} \min\{1, |y_j|^\tau\},$$

for some constant $C(f)$ and some $\tau > 1/4$ and let $r \in \mathcal{O}' \backslash \{0\}$ be fixed. Then

$$K_{r,r}(f) \ll_{F,\varepsilon} C(f)|N(r)|^{1/2+\varepsilon}.$$

Proof. We use the Weil-Salié estimate (see [4], Theorem 10):

$$|S(r,r;c)| \ll_F N_{r,r}(c)^{1/2}|N(c)|^{1/2+\varepsilon}.$$

To define the factor $N_{r,r}(c)$, we use the following decompositions into prime ideals:

$$(r) = \prod_P P^{v_P(r)}, \quad (c) = \prod_P P^{v_P(c)}, \quad \mathcal{O}' = \prod_P P^{-d_P}.$$

Then $N_{r,r}(c) = \prod_P N(P)^{\min(v_P(r), v_P(c)-d_P)}$. We obtain the following estimate for the Kloosterman term:

$$K_{r,r}(f) \ll C(f) \sum_{c \neq 0} N_{r,r}(c)^{1/2}|N(c)|^{-1/2+\varepsilon} \prod_{j=1}^{d} \min\{1, |(r/c)^{\sigma_j}|^{2\tau}\}$$

$$\ll C(f) \sum_{(c) \neq 0} N_{r,r}(c)^{1/2}|N(c)|^{-1/2+\varepsilon} \sum_{u \in \mathcal{O}^*} \prod_{j=1}^{d} \min\{1, |(r/uc)^{\sigma_j}|^{2\tau}\}.$$

For each $c \neq 0$, we apply Lemma 8.1 of [5], with $a = 0$, $b = 2\tau$, $y_j = (c/r)^{\sigma_j}$ and we get

$$(34) \quad K_{r,r}(f) \ll C(f) \sum_{(c) \neq 0} \frac{N_{r,r}(c)^{1/2}}{|N(c)|^{1/2-\varepsilon}} (1 + |\log|N(c/r)||\,|d^{-1}) \min\{1, |N(r/c)|^{2\tau}\}.$$

We now write $(r) = R_+ R_-{}^{-1}$ where $R_+ = \prod_{P, v_P(r) \geqq 0} P^{v_P(r)}$ and $R_- = \prod_{P, v_P(r) \leqq 0} P^{-v_P(r)}$. The fact that $r \in \mathcal{O}'$ implies $N(R_-) \leqq N(\mathcal{O}')^{-1} \ll_F 1$.

Any (c) in the sum can be written uniquely $(c) = KJ$ where $K = K_{c,r} := ((c), R_+)$. Therefore, for each c, $L = \dfrac{R_+}{K_{c,r}}$ and $J = \dfrac{(c)}{K_{c,r}}$ are prime to each other.

We replace in the sum $N_{r,r}(c)$ by the larger quantity $N(K_{c,r})$. For each ideal K dividing R_+, we shall sum over the ideals (c) such that $K_{c,r} = K$. We write $L = \dfrac{R_+}{K}$, and use that $\left|N\left(\dfrac{r}{c}\right)\right| = \dfrac{N(R_-)^{-1}N(L)}{N(J)} \ll \dfrac{N(L)}{N(J)}$, to obtain the following:

$$K_{r,r}(f) \ll C(f) \sum_{K|R_+} \sum_{(J,L)=1} \frac{N(K)^{\varepsilon}}{|N(J)|^{1/2-\varepsilon}} \left(1 + \left|\log \frac{N(J)}{N(L)}\right|^{d-1}\right) \min\left(1, \frac{N(L)^{2\tau}}{N(J)^{2\tau}}\right).$$

For each fixed ideal K dividing R_+, the inner sum S_K in the expression above can be majorized by the same sum taken over all ideals J. Now, the number of ideals in $\mathcal{O}$ with norm n is $O(n^\varepsilon)$ (see [19], Lemma 4.2, p. 152). Here and in the rest of the proof we may and will take the same ε as above.

Thus we have

$$S_K \ll N(K)^\varepsilon \sum_{n=1}^{\infty} \frac{n^\varepsilon}{n^{1/2-\varepsilon}} \left(1 + \left|\log \frac{n}{N(L)}\right|^{d-1}\right) \min\left(1, \frac{N(L)^{2\tau}}{n^{2\tau}}\right)$$

$$\ll N(K)^\varepsilon N(L)^\varepsilon \sum_{n=1}^{N(L)} n^{-1/2+2\varepsilon} + N(K)^\varepsilon \sum_{n>N(L)} n^{-1/2+2\varepsilon-2\tau} \left(\frac{n}{N(L)}\right)^\varepsilon N(L)^{2\tau}.$$

The first summand is estimated by $N(K)^\varepsilon N(L)^{1/2+3\varepsilon}$, while the second summand is dominated by

$$N(K)^\varepsilon N(L)^{2\tau-\varepsilon} \int_{x \geqq N(L)} x^{3\varepsilon-\frac{1}{2}-2\tau}\, dx.$$

Since $\tau > 1/4$, this integral is convergent for $\varepsilon > 0$ small enough, hence the second summand is $O\big(N(K)^\varepsilon N(L)^{1/2+2\varepsilon}\big)$. Therefore we get

$$S_K \ll |N(r)|^\varepsilon N(L)^{1/2+3\varepsilon}.$$

We now estimate the total number $k(M)$ of ideals K dividing the ideal M in terms of $N(M)$. We have

$$k(M) = \prod_{P,\, v_P(M) \geqq 0} \left(1 + \frac{\log N(P)^{v_P(M)}}{\log N(P)}\right).$$

Let $\varepsilon > 0$ be given. If $N(P) \geqq \varepsilon^{-1}$, then $x^{-\varepsilon}\big(1 + \log x/N(P)\big) \leqq 1$ for $x \geqq 1$. Moreover, there exists C_ε such that $x^{-\varepsilon}\big(1 + \log x/N(P)\big) \leqq C_\varepsilon$ for all $x \geqq 1$ and all prime ideals P. Thus we find that $k(M) \leqq C_\varepsilon^n N(M)^\varepsilon$, where n is the number of P with $N(P) \leqq \varepsilon^{-1}$. So the number of ideals K dividing R_+ is $O\big(N(R_+)^\varepsilon\big) = O\big(|N(r)|^\varepsilon\big)$, for any $\varepsilon > 0$. (We use again that $N(R_-) \ll_F 1$.)

Thus we obtain

$$K_{r,r}(f) \ll C(f)|N(r)|^\varepsilon \sum_{K|R_+} N(K)^{\frac{1}{2}+3\varepsilon}$$

$$\ll C(f)|N(r)|^\varepsilon N(R_+)^{\frac{1}{2}+4\varepsilon} \ll C(f)|N(r)|^{\frac{1}{2}+5\varepsilon}. \quad \square$$

3.3. Use of the sum formula. We now proceed to estimate the $d\sigma_{r,r}$-measure of the set

$$Y(X) = \{v \in Y\colon |v_j| \leqq X_j,\ 1 \leqq j \leqq d\}.$$

with $X \in \left(\frac{1}{4}, \infty\right)^d$. We apply the sum formula to a convenient test function and we estimate the right hand side of the sum formula. The same method led us in [5] to a Selberg type estimate for the coordinates of the eigenvalues. If an eigenvalue $\frac{1}{4} - \nu^2 = \left(\frac{1}{4} - \nu_j^2\right)_j$ is exceptional, then $\nu_j \in \left(0, \frac{1}{2}\right)$ for some j. By a Selberg type estimate, we mean the statement that $\nu_j \notin \left(\frac{1}{4}, \frac{1}{2}\right]$ for such exceptional coordinates. Selberg, [21], showed this in the case $F = \mathbb{Q}$.

In [5], we restricted ourselves to trivial K-types. In that situation, the weight is zero at all infinite places, so the automorphic representations have no factors of discrete series type (characterized by $\nu_j \in \frac{1}{2} + \mathbb{Z}_{\geqq 0}$). In the present case, a Selberg type estimate means the following:

$$\text{(35)} \qquad \operatorname{Supp}(d\sigma_{r,r'}) \subset \prod_{j=1}^{d} \left(i[0,\infty) \cup \left(0, \frac{1}{4}\right] \cup \left(\frac{1}{2} + \mathbb{Z}_{\geq 0}\right)\right).$$

Stronger results than (35) have been reached by other methods, see [7], Theorem (9.3), and [15] (Kim and Shahidi have recently improved this bound to $\frac{1}{6} - \frac{1}{51}$).

The computations we need to prove Proposition 3.3.1 give a proof of (35) without much additional effort.

Plan. We use a partition $\{1, \ldots, d\} = E_c \sqcup E_d \sqcup E_e$ (disjoint union), and look for an estimate of the measure of $d\sigma_{r,r}$ on the set

$$\text{(36)} \qquad \tilde{Y}(X) := \left\{ \nu \in Y : \begin{array}{ll} \nu_j \in i[0, X_j] \cup \left(0, \frac{1}{2}\right] & (j \in E_c), \\ \nu_j \in \frac{3}{2} + \mathbb{Z}_{\geqq 0}, \nu_j \leqq X_j & (j \in E_d), \\ \nu_j \in (\alpha, \beta) & (j \in E_e), \end{array} \right\}$$

with $(\alpha, \beta) \subset \left(\frac{1}{4}, \frac{1}{2}\right)$ fixed. When we vary the partition, the sets $\tilde{Y}(X)$ are not disjoint, but the union is equal to $Y(X)$. For $E_e \neq \emptyset$, we shall show that $\tilde{Y}(X)$ does not intersect the support of $d\sigma_{r,r}$, and for $E_e = \emptyset$, we shall estimate the mass of $\tilde{Y}(X)$.

Choice of the test function. We take $\frac{1}{2} < \tau < \frac{3}{4}$. For each factor, we choose k_j as an approximation of the characteristic function of the corresponding component of $\tilde{Y}(X)$. We employ the following functions:

$$h_p(\nu) = \begin{cases} e^{p(\nu^2-1/4)} & \text{if } |\mathrm{Re}\,\nu| \leq \tau, \\ 0 & \text{if } \nu \in \frac{1}{2}+\mathbb{Z}, |\nu| > \tau, \end{cases}$$

$$g_T(\nu) = e^{T(\nu^2-1/4)} \cos \pi\nu,$$

$$\varphi_q(\nu) = \begin{cases} 1 & \text{if } \frac{3}{2} \leqq |\nu| \leqq q,\ \nu \in 1/2+\mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

We make the following choices for the factors of the test function $k = \times k_j$:

$$\begin{aligned} &\text{If } j \in E_c: \quad k_j = h_p \quad \text{with } p = X_j^{-2}, \\ &\text{if } j \in E_d: \quad k_j = \varphi_q \quad \text{with } q = X_j, \\ &\text{if } j \in E_e: \quad k_j = g_T \quad \text{with } T \text{ large.} \end{aligned}$$

It is easy to verify that k is a function in the class $\mathscr{K}^{\mathbf{p}}$ (see Definition 2.5.1).

Delta term. We have, by Equation (29), that

$$\Delta_{r,r}(k) = \mathrm{vol}(\Gamma_N \backslash N)\alpha(\infty, r; \infty, r) \prod_{j \in E} \delta(k_j),$$

where $\alpha(\infty, r; \infty, r) = 2$ and

$$\delta(k_j) = \frac{i}{2} \int_{\mathrm{Re}\,\nu=0} k_j(\nu_j)\nu_j \tan \pi\nu_j \, d\nu_j + \sum_{b \geq 2, b \in 2\mathbb{Z}} \frac{b-1}{2} k_j\left(\frac{b-1}{2}\right).$$

We have, as $p \downarrow 0$,

$$\begin{aligned} \delta(h_p) &= \int_0^\infty h_p(iu) u \tanh \pi u \, du + \frac{1}{2} e^0 \\ &= e^{-p/4} \int_0^\infty e^{-pu^2} u\left(1 + O(e^{-2\pi u})\right) du + \frac{1}{2} \\ &= p^{-1} e^{-p/4} \int_0^\infty e^{-u^2} u \, du + O(1) = \frac{1}{2} p^{-1} + O(1), \end{aligned}$$

and

$$\delta(\varphi_q) = \sum_{b \geqq 4, b \in 2\mathbb{Z}} \frac{b-1}{2} \varphi_q\left(\frac{b-1}{2}\right) = \sum_{1 \leq \frac{b-1}{2} \leq q} \frac{b-1}{2} = O(q^2) \quad \text{as } q \to \infty,$$

$$\delta(g_T) \ll e^{-T/4} \int_0^\infty e^{-Tu^2} u e^{\pi u} \, du \ll T^{-1} e^{-T/4} \qquad \text{as } T \to \infty.$$

Thus, we get

$$\Delta_{r,r}(k) \ll \prod_{j\in E_r\cup E_d} X_j^2 \prod_{j\in E_e} T^{-1}e^{-T/4}.$$

Bessel transform. We have, by Equation (26),

$$\beta_+ h_p(t) = -i \int_{\mathrm{Re}\, \nu=\tau} e^{(\nu^2-1/4)p} J_{2\nu}(4\pi\sqrt{t})\frac{\nu\, d\nu}{\cos \pi\nu}.$$

If $t \leqq 1$, we use

$$|J_{2\nu}(y)| \ll_{y_0} y^{2\,\mathrm{Re}\,\nu}|\Gamma(2\nu+1)|^{-1} \ll y^{2\tau}e^{\pi|\mathrm{Im}\,\nu|}(1+|\mathrm{Im}\,\nu|)^{-2\tau-1/2},$$

$0 < y \leqq y_0$ for each $y_0 > 0$, and hence

$$\begin{aligned}\beta_+ h_p(t) &\ll \int_{\mathrm{Re}\,\nu=\tau} e^{p(\mathrm{Re}(\nu^2)-1/4)}t^\tau(1+|\mathrm{Im}\,\nu|)^{-2\tau-1/2}\frac{|\nu|e^{\pi|\mathrm{Im}\,\nu|}}{|\cos\pi\nu|}|d\nu|\\ &\ll t^\tau e^{p\tau^2-p/4}\int_{-\infty}^{\infty} e^{-pu^2}(1+|u|)^{-2\tau-1/2+1}\,du\\ &\ll t^\tau p^{\tau-3/4}\int_0^\infty e^{-u^2}(\sqrt{p}+u)^{1/2-2\tau}\,du \ll t^\tau p^{\tau-3/4}.\end{aligned}$$

On the other hand, we use $|J_{2\nu}(y)| \ll e^{\pi|\mathrm{Im}\,\nu|}$ for $\mathrm{Re}\,\nu = 0$, see (7) in 6.2 of [24], and $J_1(y) \ll y^{-1/2}$ as $y \to \infty$, see 7.21 of [24]. If $t \geqq 1$, with (24), we obtain as $p \downarrow 0$:

$$\beta_+ h_p(t) \ll e^{-p/4}\int_0^\infty e^{-pu^2}(1+u)\,du + t^{-1/4} \ll p^{-1} + t^{-1/4} \ll p^{-1}.$$

Putting both estimates together, we get

$$\beta_+ h_p(t) \ll p^{-1}\min(1, t^\tau). \tag{37}$$

Next, we use (25) and proceed in a similar way as with h_p, and find for $t \leqq 1$

$$\begin{aligned}\beta_+ g_T(t) &\ll \int_{-\infty}^{\infty} e^{T(\alpha^2-u^2-1/4)}t^\alpha(1+|\mathrm{Im}\,\nu|)^{-2\alpha-1/2+1}e^{\pi|u|}\,du\\ &\ll t^\alpha T^{-1/2}e^{T(\alpha^2-1/4)}.\end{aligned}$$

On the other hand, for $t \geqq 1$, we have

$$\begin{aligned}\beta_+ g_T(t) &\ll e^{T(\alpha^2-1/4)}\int_0^\infty e^{-u^2T+\pi u}(1+u)\,du\\ &\ll T^{-1/2}e^{T(\alpha^2-1/4)}.\end{aligned}$$

Therefore,

$$\beta_+ g_T(t) \ll T^{-1/2} e^{T(\alpha^2-1/4)} \min(1, t^\alpha).$$

We have

$$\beta_+ \varphi_q(t) = \sum_{1 \leqq \frac{b-1}{2} \leqq q} (-1)^{b/2}(b-1) J_{b-1}(4\pi\sqrt{t}).$$

From the power series expansion, we have $|J_{b-1}(y)| \ll_{y_0} y^{b-1}\Gamma(b)^{-1}$, $0 < y \leqq y_0$ for each $y_0 > 0$. If $t \leqq 1$, we have $\beta_+\varphi_q(t) \ll t^{3/2}$. If $t \geqq 1$, we see from [2], (3.3.3) or [5], (40), that $J_u(t) \ll u^{-1}$ for $u > 0$ hence $\beta_+\varphi_q(t) \ll O(q)$, thus $\beta_+\varphi_q(t) \ll q \min(t^{3/2}, 1)$.

We put $C(k) := (T^{-1/2} e^{T(\alpha^2-1/4)})^{|E_c|} \prod_{j \in E_c \cup E_d} X_j^2$. We have obtained the estimate

$$\mathrm{B}k(t) \ll C(k) \prod_{j=1}^{d} \min(t_j^\alpha, 1). \tag{38}$$

Kloosterman term. By Lemma 3.2.1, we have $K_{r,r}(\mathrm{B}k) \ll C(k)|N(r)|^{1/2+\varepsilon}$.

Spectral term. Theorem 2.7.1 implies the inequality

$$\int_Y k(v)\, d\sigma_{r,r}(v) \ll |N(r)|^{1/2+\varepsilon} C(k).$$

Note that $r \in \mathcal{O}' \setminus \{0\}$, so $N(r)$ stays away from zero.

The test function k is non-negative on the set Y. Moreover, for $v \in \tilde{Y}(X)$, we have, for T sufficiently large (depending on α and β):

$$k(v) \geqq (e^{T(\alpha^2-1/4)} \cos \pi\alpha)^{|E_c|} \prod_{j \in E_c} e^{-(X_j^2+1/4)/X_j^2}.$$

From this we obtain the following estimate:

$$\begin{aligned}\int_{\tilde{Y}(X)} d\sigma_{r,r}(v) &\leqq \prod_{j \in E_c} (e^{(X_j^2+1/4)/X_j^2})(e^{-T(\alpha^2-1/4)}(\cos \pi\alpha)^{-1})^{|E_c|} \int_Y k(v)\, d\sigma_{r,r}(v) \\ &\ll |N(r)|^{1/2+\varepsilon}(T^{-1/2}(\cos \pi\alpha)^{-1})^{|E_c|} \prod_{j \in E_c} (X_j^2 \cdot 1) \prod_{j \in E_d} (X_j^2) \\ &\ll_\alpha |N(r)|^{1/2+\varepsilon} T^{-|E_c|/2} \prod_{j \in E_c \cup E_d} X_j^2.\end{aligned}$$

The integral $\int_{\tilde{Y}(X)} d\sigma_{r,r}(v)$ does not depend on the large quantity T. Hence it vanishes if $E_c \neq \emptyset$, for each choice of X, α and β. This implies (35).

The support of the measure $d\sigma_{r,r}$ is contained in the union of the sets $\tilde{Y}(X)$, where we let E_c run over the subsets of $\{1,\ldots,d\}$, with E_d as its complement. We obtain the following result:

Proposition 3.3.1. *Let $\Gamma = \Gamma_0(I)$ with I a non-zero ideal of $\mathcal{O}$. Put*

$$Y(X) = \{\nu \in Y \colon |\nu_j| \leqq X_j\}, \quad \text{with} \quad X \in \left(\frac{1}{4}, \infty\right)^d.$$

Then we have, for $r \in \mathcal{O}'\backslash\{0\}$, the estimate

$$\int_{Y(X)} d\sigma_{r,r}(\nu) \ll_{\varepsilon,F} |N(r)|^{1/2+\varepsilon} N(X)^2,$$

for each $\varepsilon > 0$.

Lemma 3.1.1 implies:

Corollary 3.3.2. *Under the conditions of Proposition* 3.3.1, *with $r, r' \in \mathcal{O}'\backslash\{0\}$:*

$$\int_{Y(X)} |d\sigma_{r,r'}(\nu)| \ll_{\varepsilon,F} |N(rr')|^{1/4+\varepsilon} N(X)^2,$$

for each $\varepsilon > 0$ and $X \in \left(\frac{1}{4}, \infty\right)^d$.

In the sequel, we want to use the measure of $d\sigma_{r,r'}$ on sets contained in

$$Z(E) := \prod_{j \in E} \left(0, \frac{1}{4}\right] \times \prod_{j \notin E} \left(i[0,\infty) \cup \left(\frac{1}{2} + \mathbb{Z}_{\geq 0}\right)\right), \tag{39}$$

with $E \subset \{1,\ldots,d\}$. This can be arranged by applying Corollary 3.3.2 with X replaced by $\tilde{X}$ such that $\tilde{X}_j = \min(X_j, 1/4)$ if $j \in E$, and $\tilde{X}_j = X_j$ otherwise. As $Z(E) \cap Y(X) \subset Y(\tilde{X})$, we find

$$\int_{\nu \in Z(E), |\nu_j| \leqq X_j} |d\sigma_{r,r'}(\nu)| \ll_{\varepsilon,F} |N(rr')|^{1/4+\varepsilon} \prod_{j \notin E} X_j^2, \tag{40}$$

for each $\varepsilon > 0$, and $X \in \left(\frac{1}{4}, \infty\right)^d$.

4. Estimates of sums of Kloosterman sums

We keep the assumption $\Gamma = \Gamma_0(I)$, with I a non-zero ideal of $\mathcal{O}$. We consider the Kloosterman sums $S(r, r'; c) = S(\infty, r; \infty, r'; c)$ with $c \in I\backslash\{0\}$, and $r, r' \in \mathcal{O}'\backslash\{0\}$.

In this section, we do not show the dependence on r and r' in the notation, but keep track of their influence on the estimates.

4.1. Sums of Kloosterman sums. We define, for $x \in (0,\infty)^d$,

$$\Lambda(x) := \sum_{c \in I \setminus \{0\},\, x_j/2 < |c^{\sigma_j}| \leqq x_j} \frac{S(r,r';c)}{|N(c)|}, \tag{41}$$

$$\tilde{\Lambda}(x) := \sum_{c \in I \setminus \{0\},\, |c^{\sigma_j}| \leqq x_j} \frac{S(r,r';c)}{|N(c)|}. \tag{42}$$

We clearly have $\Lambda(x) = \tilde{\Lambda}(x) = 0$ if $N(x) < 1$, since the sums are empty in this case.

First, we give an estimate by replacing all Kloosterman sums by their absolute value. Our aim is to take cancellation between Kloosterman sums into account by approximating $\Lambda(x)$ by the Kloosterman term in the sum formula, for a suitable test function.

We wanted to estimate the sums in terms of $|N(r)|$ and $N(x) = \prod_j x_j$, but it turned out better to separate the small and large factors in the norms:

Definition 4.1.1. For $x \in (0,\infty)^d$, we put $N_+(x) := \prod_{j=1}^{d} \max(x_j, 1)$, and

$$N_-(x) := N(x)/N_+(x) = \prod_{j=1}^{d} \min(x_j, 1).$$

For $c \in F$, we denote $|c| := (|c^{\sigma_j}|)_j \in (0,\infty)^d$. So $|N(c)| = N_+(|c|) \cdot N_-(|c|)$.

Lemma 4.1.2. i) *Let $a, b \in [0,\infty)^d$, with $N(b) > 1$, and $a_j < b_j$ for $j = 1, \ldots, d$. Then, for each $\varepsilon \in \left(0, \frac{1}{2}\right)$,*

$$\sum_{c,\, a_j \leqq |c^{\sigma_j}| \leqq b_j} |N(c)|^{-1} |S(r,r';c)| \ll_{F,\varepsilon} |N(rr')|^{1/2} N(a)^{-1/2+\varepsilon} N_+(b-a).$$

ii) $\Lambda(x) \ll_{F,\varepsilon} |N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} N_+(x)$ *for $N(x) \geqq 1$, for each $0 < \varepsilon < \frac{1}{2}$.*

iii) $\tilde{\Lambda}(x) \ll_F N_+(x)$.

Proof. For part i), we use the Weil-Salié estimate from [4], Theorem 10:

$$|S(r,r';c)| \ll |N(rr')|^{1/2} |N(c)|^{1/2+\varepsilon}, \tag{43}$$

for each $\varepsilon > 0$. We use also that the number of $c \in \mathcal{O}$ with $a_j \leqq |c^{\sigma_j}| \leqq b_j$ for all j is $O\left(\prod_j \max(b_j - a_j, 1)\right) = O(N_+(b-a))$. This gives parts i) and ii). We use the trivial estimate $|S(r,r';c)| \leqq |N(c)|$ and count the number of c with $|c^{\sigma_j}| \leqq x_j$ to obtain part iii). □

Now we give a more careful estimate of $\tilde{\Lambda}(x)$, which still does not use cancellation between Kloosterman sums.

Lemma 4.1.3. *For each $x \in (0,\infty)^d$, we have*

$$|\tilde{\Lambda}(x)| \leqq \sum_{c \neq 0, |c^{\sigma_j}| \leq x_j} \frac{|S(r,r';c)|}{|N(c)|} \ll_{\varepsilon,F} |N(rr')|^{1/2} N(x)^{\frac{1}{2}+\varepsilon} N_+(x)^{\varepsilon},$$

for each $\varepsilon > 0$.

Proof. Let $x \in (0,\infty)^d$ be given. We use auxiliary sums $\Lambda_E(x)$, depending on a subset $E \subset \{j \in \{1,\dots,d\}: x_j > 1\}$. We define $\tilde{x} \in (0,\infty)$ by

$$\tilde{x}_j := \begin{cases} x_j & \text{if } j \in E, \\ \min(x_j, 1) & \text{if } j \notin E. \end{cases}$$

We define

$$D_{x,E} := \{c \in I\backslash\{0\}: 1 \leq |c^{\sigma_j}| \leqq x_j, \text{ if } j \in E, 0 \leqq |c^{\sigma_j}| \leqq \tilde{x}_j \text{ if } j \notin E\},$$

$$\Lambda_E(x) := \sum_{c \in D_{x,E}} |N(c)|^{-1} |S(r,r';c)|.$$

From $|\tilde{\Lambda}(x)| \leqq \sum_E \Lambda_E(x)$, we see that it suffices to prove that

$$\Lambda_E(x) \ll_{\varepsilon,F} |N(rr')|^{1/2} N(\tilde{x})^{\frac{1}{2}+\varepsilon} N_+(x)^{\varepsilon} \tag{44}$$

for each $\varepsilon > 0$.

We fix E. For $n \in \mathbb{N}$ with $n \leqq N(x)$, we denote $D_{n,x} := \{c \in D_{x,E}: |N(c)| = n\}$. By the Weil-Salié estimate, we have

$$\Lambda_E(x) \ll |N(rr')|^{1/2} \sum_{c \in D_{x,E}} |N(c)|^{-\frac{1}{2}+\varepsilon} = |N(rr')|^{1/2} \sum_{1 \leqq n \leqq N(\tilde{x})} n^{-\frac{1}{2}+\varepsilon} |D_{n,x}|.$$

Clearly, the cardinality $|D_{n,x}|$ of $D_{n,x}$ is bounded by the product of the number of ideals I with norm $N(I) = n$ times the number of $c \in D_{n,E}$ such that $(c) = (c_0)$, for a fixed c_0 in $D_{n,x}$.

Given $c_0 \in D_{n,x}$, we wish to estimate the number of units μ such that $c = \mu c_0 \in D_{n,x}$. By definition, $\mu c_0 \in D_{n,E}$ if and only if $1 \leqq |\mu^{\sigma_j}|\,|c_0^{\sigma_j}| \leqq x_j$ if $j \in E$, and

$$0 < |\mu^{\sigma_j}|\,|c_0^{\sigma_j}| \leqq \min(x_j, 1)$$

if $j \notin E$. If $j \notin E$, then

$$\frac{1}{\prod_{i \in E} x_i} \leqq \frac{1}{\prod_{i \in E} |\mu^{\sigma_i}|\,|c_0^{\sigma_i}|} \leqq \frac{n}{\prod_{i \in E} |\mu^{\sigma_i}|\,|c_0^{\sigma_i}|}$$

$$= \frac{\prod_{i=1}^{d} |(\mu c_0)^{\sigma_i}|}{\prod_{i \in E} |(\mu c_0)^{\sigma_i}|} \leqq |\mu^{\sigma_j}|\,|c_0^{\sigma_j}|.$$

Hence, if $\mu c_0 \in D_{n,E}$, then $N_+(x)^{-1} \leqq |\mu^{\sigma_j}|\,|c_0^{\sigma_j}| \leqq \min(x_j, 1)$ if $j \notin E$ and

$$1 \leqq |\mu^{\sigma_j}|\,|c_0^{\sigma_j}| \leqq x_j \leqq N_+(x),$$

if $j \in E$. We have

$$-\log|c_0^{\sigma_j}| \leqq \log|\mu^{\sigma_j}| \leqq \log N_+(x) - \log|c_0^{\sigma_j}|, \qquad \text{if } j \in E,$$
$$-\log|c_0^{\sigma_j}| - \log N_+(x) \leqq \log|\mu^{\sigma_j}| \leqq -\log|c_0^{\sigma_j}|, \quad \text{if } j \notin E.$$

On the other hand, $\Lambda = \{(\log|\mu^{\sigma_1}|, \ldots, \log|\mu^{\sigma_d}|) \colon \mu \in \mathcal{O}^*\}$ is a lattice of dimension $d-1$ contained in the hyperplane $\sum_{i=1}^{d} x_i = 0$ in $\mathbb{R}^d$. The map $\mu \mapsto (\log|\mu^{\sigma_j}|)_j$ is a homomorphism with finite kernel, depending on F. The set $\{\mu \in \mathcal{O}^* \colon \mu c_0 \in D_{n,E}\}$ is mapped to the set $\{\lambda \in \Lambda \colon a_j \leqq \lambda_j \leqq b_j\}$, where

$$a_j = \begin{cases} -\log|c_0^{\sigma_j}| & \text{if } j \in E, \\ -\log|c_0^{\sigma_j}| - \log N_+(x) & \text{if } j \notin E, \end{cases}$$

$$b_j = \begin{cases} \log N_+(x) - \log|c_0^{\sigma_j}| & \text{if } j \in E, \\ -\log|c_0^{\sigma_j}| & \text{if } j \notin E. \end{cases}$$

In both cases we have $b_j - a_j = \log N_+(x)$.

Let $C = \prod_{j=1}^{d} [a_j, b_j] \subset \mathbb{R}^d$. The cardinality of $\Lambda \cap C$ is estimated by

$$\bigl(1 + \log N_+(x)\bigr)^d = O\bigl(N_+(x)^{\varepsilon}\bigr).$$

Estimating the number of ideals with norm n by $O(n^{\varepsilon})$, we have obtained:

$$|D_{n,x}| = O\bigl(n^{\varepsilon} N_+(x)^{\varepsilon}\bigr),$$
$$\Lambda_E(x) \ll |N(rr')|^{1/2} \sum_{1 \leqq n \leqq N(\tilde{x})} n^{-\frac{1}{2}+2\varepsilon} N_+(x)^{\varepsilon}$$
$$\ll |N(rr')|^{1/2} N(\tilde{x})^{\frac{1}{2}+2\varepsilon} N_+(x)^{\varepsilon}.$$

This concludes the proof of (44). □

4.2. Smooth bounds. To estimate sums of Kloosterman sums with the help of the sum formula, we start from a test function that has suitable properties in the Kloosterman term, and see what estimates that implies in the spectral term. For these estimates we use [2] as far as possible. We arrange the notations with the comparison with [2] in mind.

The test function is built from a function $\psi \in C_c^{\infty}(0, \infty)$ with the same properties as the function τ in Lemma 4.1 of [2]:

$$0 \leqq \psi \leqq 1, \quad \int_0^{\infty} |\psi'(y)|\,dy = 2, \quad Y = \int_0^{\infty} |\psi''(y)|\,dy > 15.$$

The large parameter Y governs the steepness of ψ. We put the additional conditions on ψ that $\mathrm{Supp}(\psi) \subset (2 - 1/Y, 4)$ and that $\sum_{n=-\infty}^{\infty} \psi(2^n y) = 1$ for all $y \in \mathbb{R}$. Hence $\psi = 1$ on $[2, 4 - 2/Y]$.

We define

$$\Lambda_\psi(x) := \sum_{c \in I \setminus \{0\}} \frac{S(r, r'; c)}{|N(c)|} \prod_{j=1}^{d} \psi(2x_j / |c^{\sigma_j}|). \tag{45}$$

This is an approximation of $\Lambda(x)$. The sharp bounds $\frac{1}{2} x_j < |c^{\sigma_j}| \leqq x_j$ have been made smooth. This enables us to use the sum formula to estimate $\Lambda_\psi(x)$.

Lemma 4.2.1. *Let $x \in (0, \infty)^d$ with $N(x) \geqq 1$. Then, for each $\varepsilon > 0$ we have*

$$|\Lambda(x) - \Lambda_\psi(x)| \ll_{\varepsilon, F} \begin{cases} |N(rr')|^{1/2} N(x)^{1/2+\varepsilon} \max(x_{\min}^{-1}, Y^{-1}) & \text{if } N(x) = N_+(x), \\ |N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} N_+(x) & \text{if } N(x) \neq N_+(x), \end{cases}$$

where $x_{\min} := \min_j (x_j)$.

Proof. We shall estimate this difference trivially. We use repeatedly part i) of Lemma 4.1.2.

Put $\eta = \frac{1}{1 - 1/2Y}$; so $\eta = 1 + O(1/Y) = O(1)$ is slightly larger than 1. The difference $|\Lambda(x) - \Lambda_\psi(x)|$ is bounded by a sum $\sum_E D_E$, where E runs over the nonempty subsets of $\{1, \ldots, d\}$ and $D_E = \sum_c |N(c)|^{-1} |S(r, r'; c)|$ is determined by the conditions $|c^{\sigma_j}| \in \left(\frac{1}{2} x_j, \frac{1}{2} \eta x_j\right) \cup (x_j, \eta x_j)$ if $j \in E$, and $|c^{\sigma_j}| \in \left(\frac{1}{2} \eta x_j, x_j\right)$ if $j \notin E$. By Lemma 4.1.2 we find

$$|D_E| \ll |N(rr')|^{1/2} \left(\eta^{d - |E|} N(x)\right)^{-1/2+\varepsilon} \prod_{j \in E} \max((\eta - 1) x_j, 1) \prod_{j \notin E} \max(x_j, 1).$$

Suppose first that $x_j \geqq Y$ for all j. Then $\max((\eta - 1) x_j, 1) \ll Y^{-1} x_j$ for all j. Hence $D_E \ll |N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} N_+(x) Y^{-|E|} \leqq |N(rr')|^{1/2} N(x)^{1/2+\varepsilon} Y^{-1}$ for each non-empty E.

Now suppose that $x_{\min} < Y$. We have

$$\begin{aligned} |D_E| &\ll |N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} \prod_{i \in E} \max(x_i Y^{-1}, 1) \prod_{j \notin E} \max(x_j, 1) \\ &\ll |N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} N_+(x) \prod_{i \in E} \frac{\max(x_i Y^{-1}, 1)}{\max(x_i, 1)}. \end{aligned}$$

The largest contribution occurs when $E = \{i\}$ consists of one element. If $x_i \leqq 1$, then $|D_{\{i\}}| \leqq |N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} N_+(x)$. Otherwise,

$$|D_{\{i\}}| \ll |N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} N_+(x) \max(Y^{-1}, x_i^{-1}).$$

Therefore, if some coordinate x_j is smaller than 1, the contribution of $|D_E|$ is bounded by $|N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} N_+(x)$. Otherwise

$$\sum_E |D_E| \ll |N(rr')|^{1/2} N(x)^{1/2+\varepsilon} \max(Y^{-1}, x_{\min}^{-1}). \quad \square$$

4.3. Choice of a test function. For any $f \in C_c^\infty(0,\infty)^d$, Proposition 2.5.6 provides us with $k = \mathrm{B}_{\mathbf{e}}^- f \in \mathscr{K}^{\mathbf{e}}$, which can be used as a test function in the sum formula, Theorem 2.7.1. We choose f in such a way that the Kloosterman term, see (22), satisfies $K_{r,r'}(f) = \Lambda_\psi(x)$. Hence we take $f = \bigtimes_{j=1}^d f_j$ with $f_j(y) = \psi\left(\frac{2x_j\sqrt{y}}{\sqrt{|(rr')^{\sigma_j}|}}\right)$.

Then, $k = \mathrm{B}_{\mathbf{e}}^- f = \bigtimes_j k_j \in \mathscr{K}^{\mathbf{e}}$. Proposition 2.5.6 and equation (23) show that

$$k_j(v) = \frac{1}{\sin \pi v} \int_0^\infty \left(J_{-2v}^{\mathbf{e}_j}(y) - J_{2v}^{\mathbf{e}_j}(y)\right) \psi\left(\frac{x_j y}{2\pi\sqrt{|(rr')^{\sigma_j}|}}\right) \frac{dy}{y}. \tag{46}$$

Let us take $X \in (0,\infty)^d$, with

$$X_j := 2\pi\sqrt{|(rr')^{\sigma_j}|}/x_j, \tag{47}$$

for $j = 1, \ldots, d$ and $\tilde{f}_j(y) = \psi(y/X_j)$. Then $\tilde{f}_j(4\pi y) = f_j(y^2)$. Note that $\tilde{f}_j$ corresponds to the function denoted by f_{X_j} in Lemma 4.1 in [2], and $k_j(v) = \frac{4}{\pi}(b_0^{\mathbf{e}_j})^{-1} \tilde{f}_j(v)$ in the notations of [2], Proposition 2.9.

4.4. Local estimates. Here we give some estimates concerning the individual factors k_j of the test function chosen in the previous subsection.

Lemma 4.4.1. *Let $1 \leqq j \leqq d$ and let μ be a measure supported in*

$$i[0,\infty) \cup \left(\frac{1}{2} + \mathbb{Z}_{\geqq 0}\right)$$

that satisfies $\int_{|v| \leqq T} |d\mu(v)| \ll AT^2$ *as* $T \to \infty$, *for some* $A \geqq 0$. *Then*

$$\int k_j(v)\, d\mu(v) \ll AC(X_j, Y),$$

where

$$C(X,Y) := \begin{cases} Y^{1/2} + |\log X| & \text{if } X \leqq 1, \\ X(\log Y)^2 + Y^{1/2} & \text{if } 1 \leqq X \leqq Y, \\ X(\log X)^2 & \text{if } X \geqq Y. \end{cases} \tag{48}$$

Proof. The lemma is proved in the same way as Lemma 4.2 in [2], taking the maximum of the estimates for $\mathbf{e}_j = 1$ and $\mathbf{e}_j = -1$. □

In the situation of [2], the interval $\left(0, \frac{1}{2}\right)$ contained no exceptional eigenvalues, except those coming from powers of the Dedekind eta function. Here, the contribution of that interval is more complicated. We cannot restrict ourselves to measures consisting of a finite linear combination of point masses. On the other hand, we can use the fact that no coordinate of $\nu \in \mathrm{Supp}(d\sigma_{r,r'})$ is an element of $\left(\frac{1}{4}, \frac{1}{2}\right)$, see (35).

Lemma 4.4.2. *Let $1 \leqq j \leqq d$. Suppose that μ is a bounded measure on $\left(0, \frac{1}{2}\right)$, with support contained in $(0, \beta] \subset \left(0, \frac{1}{2}\right)$, and $\int_{(0,\beta]} |d\mu| \leqq A$ for some $A \geqq 0$. Then $\int k_j(\nu)\, d\mu(\nu) \ll A\, D(X_j, \beta)$, where*

$$D(X, \beta) := \begin{cases} X^{-2\beta}(1 + |\log X|) & \text{if } X \leqq 1, \\ X^{-1}(1 + \log X) & \text{if } X \geqq 1. \end{cases}$$

Proof. We estimate $k_j(\nu)$ on $(0, \beta]$, with $\beta < \frac{1}{2}$.

Near $\nu = 0$, we do not consider the Bessel functions in (46) separately, but use:

$$\text{(49)} \qquad \frac{1}{\sin \pi\nu}\left(J^{\mathbf{e}_j}_{-2\nu}(y) - J^{\mathbf{e}_j}_{2\nu}(y)\right) = \begin{cases} ie^{\pi i\nu} H^{(1)}_{2\nu}(y) - ie^{-\pi i\nu} H^{(2)}_{2\nu}(y) & \text{if } \mathbf{e}_j = 1, \\ \dfrac{4}{\pi} \cos \pi\nu K_{2\nu}(y) & \text{if } \mathbf{e}_j = -1. \end{cases}$$

(See, e.g., [24], (1)–(2) of 3.61 and (6) of 3.7.)

Case $\mathbf{e}_j = -1$. For $X_j \geqq 1$, we use $K_{2\nu}(y) \ll \left|\Gamma\left(2\nu + \frac{3}{2}\right)\right| y^{-2\,\mathrm{Re}\,\nu - 1}$ obtained by partial integration of Basset's formula (see [5], proof of Lemma 11.1). Hence $k_j(\nu) \ll X_j^{-2\nu-1}$, and $\int k_j(\nu)\, d\mu(\nu) \ll A X_j^{-2\nu-1} \ll A$, when $X_j \geqq 1$.

Set $F_{X_j}(u) = \int_0^\infty \tilde{f}_j(y) e^{-uy} \frac{dy}{y}$, where $\tilde{f}_j(y) = \psi(y/X_j)$. The factor $F_{X_j}(u)$ appears, when we insert into (46) an integral representation of the factor with Bessel functions, and interchange the order of integration. The quantity $F_{X_j}(u)$ is holomorphic in u and satisfies (see [2], p. 303):

$$\text{(50)} \qquad F_{X_j}(u) \ll \min\left(1, \frac{1}{X_j|u|}\right) e^{-X_j \mathrm{Re}\, u} \quad \text{for } \mathrm{Re}\, u \geqq 0.$$

Let $X_j \leqq 1$. We use the following integral representation:

$$\frac{4}{\pi} \cos \pi\nu K_{2\nu}(y) = \frac{2}{\pi} \cos \pi\nu \int_{-\infty}^{\infty} e^{-y\cosh z + 2\nu z}\, dz,$$

see (7) in 6.22 of [24]. To estimate $k_j(\nu)$, we note first that the contribution of $z \geqq 0$ is the larger one. We break up the integral over $[0, \infty)$ at the point $T = \log(X_j^{-1} + \sqrt{X_j^{-2} - 1})$

determined by $X_j \cosh T = 1$. So $T = O(\log X_j^{-1})$ as $X_j \downarrow 0$. Below T, we use the estimate $F_{X_j}(\cosh z) \ll e^{-\cosh z} \ll 1$. On $z \geqq T$, we have $F_{X_j}(\cosh z) \ll e^{-X_j \cosh z}/X_j \cosh z$. Here we go over to the variable $y = X_j \cosh z$.

$$\int_{-\infty}^{\infty} F_{X_j}(\cosh z) e^{2vz}\, dz \ll \int_0^T e^{2vz}\, dz + \int_T^{\infty} \frac{e^{-X_j \cosh z} e^{2vz}}{X_j \cosh z}\, dz$$

$$\ll T e^{2vT} + X_j^{-2v} \int_1^{\infty} e^{-y} y^{2v-1} \frac{dy}{\sqrt{y^2 - X_j^2}} \ll X_j^{-2v} |\log X_j| + X_j^{-2v}$$

$$\ll X_j^{-2v}(1 + |\log X_j|).$$

For $\mathbf{e}_j = -1$, the statement of the lemma follows.

Case $\mathbf{e}_j = 1$. We use

$$\pm i e^{\pm \pi i v} H_{2v}^{(1 \text{ or } 2)}(t) = \frac{e^{\pm \pi i v}}{\pi} \int_{L_\pm} e^{t \sinh z - 2vz}\, dz,$$

where $\pm = +$ (respectively $-$), corresponds to $H^{(1)}$ (respectively $H^{(2)}$). The path $L_\pm$ consists of the straight lines $(-\infty, 0]$, $\pm i[0, \pi]$, and $\pm \pi i + [0, \infty)$, see (2), (3) in 6.21 of [24]. Thus, it suffices to estimate

$$\sum_{\pm} \frac{e^{\pm \pi i v}}{\pi} \int_{L_\pm} F_{X_j}(-\sinh z) e^{-2vz}\, dz.$$

We estimate the contribution of the vertical part with the help of (50):

$$\sum_{\pm} \frac{1}{\pi} \int_0^{\pi} |F_{X_j}(\mp i \sin y)|\, |e^{\mp 2ivy}|\, dy \ll \int_0^{\pi/2} \min\left(1, (X_j \sin y)^{-1}\right) dy.$$

This is $O(1)$ if $X_j \leq 1$. For $X_j \geq 1$, we find an estimate by

$$\int_0^{1/X_j} dy + \int_{1/X_j}^{\pi/2} (X_j y)^{-1}\, dy = O(X_j^{-1} \log X_j).$$

We combine the contribution of the horizontal parts of $L_\pm$ into

$$\sum_{\pm} \frac{e^{\pm \pi i v}}{\pi} \int_0^{\infty} F_{X_j}(\sinh z)(e^{2vz} + e^{\mp 2\pi i v} e^{-2vz})\, dz$$

$$= \frac{2 \cos \pi v}{\pi} \int_0^{\infty} F_{X_j}(\sinh z)(e^{2vz} + e^{-2vz})\, dz$$

$$\ll \int_0^{\infty} \min\left(1, (X_j \sinh z)^{-1}\right) e^{-X_j \sinh z + 2vz}\, dz$$

$$\ll \int_0^U e^{2vz}\, dz + X_j^{-1} \int_U^{\infty} e^{-X_j \sinh z + 2vz} \frac{dz}{\sinh z},$$

where $U = \log(X_j^{-1} + \sqrt{X_j^{-2} + 1})$. The integral from 0 to U is $O(U) = O(X_j^{-1})$ for $U \leqq \log(1+\sqrt{2})$, and $O(Ue^{2\nu U}) = O(X_j^{-2\nu}|\log X_j|)$ for $U \geqq \log(1+\sqrt{2})$. (Note that U is strictly decreasing in X_j. At $X_j = 1$ its value is $\log(1+\sqrt{2})$.) In the other integral we use $y = X_j \sinh z$ as a new variable, and obtain the following:

$$\int_1^\infty e^{-y}(X_j^{-1}y + \sqrt{1 + X_j^{-2}y^2})^{2\nu} y^{-1} \frac{dy}{\sqrt{X_j^2 + y^2}}$$

$$\leqq X_j^{-2\nu} \int_1^\infty e^{-y} y^{2\nu}(1 + \sqrt{1 + X_j^2 y^{-2}})^{2\nu} \frac{dy}{y \max(1, X_j)}$$

$$\ll \max(X_j^{-2\nu}, 1) \cdot \min(1, X_j^{-1}) = \max(X_j^{-2\nu}, X_j^{-1}).$$

Taking all estimates together, we obtain the lemma in the case $\mathbf{e}_j = 1$. □

4.5. Global estimates. Theorem 2.7.1, when applied to the test function k chosen above, and Proposition 2.6.3 give the equality

$$\Lambda_\psi(x) = \int k(\nu)\, d\sigma_{-r,-r'}(\nu).$$

For $E \subset \{1, \ldots, d\}$, we put $S_E(k) := \int_{Z(E)} k(\nu) d\sigma_{-r,-r'}(\nu)$, see (39). Hence we have $\Lambda_\psi(x) = \sum_E S_E(k)$.

If $E \neq \emptyset$, the term $S_E(k)$ is given by a sum over the set

$$\Sigma_{-r,-r'}(E) := \mathrm{Supp}(d\sigma_{-r,-r'}) \cap Z(E).$$

The term $S_0(k)$ is more complicated, as it contains also the contribution of the continuous spectrum. If $E \neq \emptyset$, and $\nu \in \Sigma_{-r,-r'}(E)$, then the coordinates ν_j with $j \in E$ are confined to the finite interval $\left(0, \frac{1}{4}\right]$. This is no reason to suppose that the set $\Sigma_{-r,-r'}(E)$ should be finite. A strong generalization of the Selberg conjecture would be that all these sets, except $\Sigma_{-r,-r'}(\emptyset)$, are empty.

To keep track of the influence of eigenvalues with exceptional coordinates, we define $\beta(E) \in \left[0, \frac{1}{4}\right]^d$ by

$$\beta(E)_j := \begin{cases} 0, & j \notin E \text{ or } \Sigma_{-r,-r'}(E) = \emptyset, \\ \sup\limits_{\nu \in \Sigma_{-r,-r'}(E)} \nu_j, & j \in E \text{ and } \Sigma_{-r,-r'}(E) \neq \emptyset. \end{cases} \tag{51}$$

So $\beta(\emptyset) = (0, \ldots, 0)$. Result (35) implies that all $\beta_j(E)$ are at most $\frac{1}{4}$.

Lemma 4.5.1. *For $E \subset \{1, \ldots, d\}$, let $C_E(X, Y) := \prod_{j \in E} D(X_j, \beta(E)_j) \cdot \prod_{j \notin E} C(X_j, Y)$. Then, for any $\varepsilon > 0$*

$$S_E(k) \ll_{\varepsilon, F} |N(rr')|^{1/4+\varepsilon} C_E(X, Y).$$

Remark. See (48) for C, and Lemma 4.4.2 for D.

Proof. We write $A = |N(rr')|^{1/4+\varepsilon}$, $\tilde{X}_j = X_j$ if $j \notin E$, and $\tilde{X}_j = \frac{1}{4}$ if $j \in E$. We denote $Z(E)_j := \left(0, \frac{1}{4}\right]$ if $j \in E$, and $Z(E)_j := i[0,\infty) \cup \left(\frac{1}{2} + \mathbb{Z}_{\geq 0}\right)$ if $j \notin E$. So $Z(E) = \prod_{j=1}^{d} Z(E)_j$.

On each $Z(E)_\ell$ we define a measure $d\tau_\ell$ by

$$(52) \qquad \int_{Z(E)_\ell} \varphi(\nu_\ell)\, d\tau_\ell(\nu_\ell) := \int_{\nu \in Z(E), |\nu_j| \leq \tilde{X}_j \text{ for } j<\ell} \varphi(\nu_\ell) k_{\ell+1}(\nu_{\ell+1}) \cdots k_d(\nu_d)\, d\sigma_{-r,-r'}(\nu),$$

for continuous compactly supported functions φ on $Z(E)_\ell$. To see that this indeed defines a measure, we have to show that the integral in the right hand side of (52) makes sense, and that the result is bounded by $\|\varphi\|_\infty$ times a constant depending on Supp(φ). Lemma 3.1.1 shows that we can restrict ourselves to the case $r = r'$, in which the measure is positive. Theorem 2.7.1 gives the integrability of $|k| = \bigtimes_j |k_j|$. The factors of k can be changed independently of each other. In particular, at the places $j \leq \ell$, we can use the functions h_p and φ_q as employed in Subsection 3.3. The characteristic function of any compact set in one factor can be majorized by a positive linear combination of such functions. So the integral in (52) makes sense, as the integrand is continuous and bounded by an integrable function. Moreover, the integral is bounded by $\|\varphi\|_\infty$ times the integral of a positive function depending only on Supp(φ).

Let $\tilde{C}_j := C(X_j, Y)$ if $j \notin E$, and $\tilde{C}_j := D(X_j, \beta(E)_j)$ if $j \in E$. We shall prove by induction that the measure $d\tau_\ell$ satisfies

$$(53) \qquad \int_{\nu_\ell \in Z(E)_\ell, |\nu_\ell| \leq \tilde{X}_\ell} d\tau_\ell(\nu_\ell) \ll A\tilde{X}_\ell^2 \prod_{j=1}^{\ell-1} \tilde{X}_j^2 \cdot \prod_{j=\ell+1}^{d} \tilde{C}_j.$$

The induction runs from $\ell = d$ down to $\ell = 1$. For $\ell = d$, the statement in (53) amounts to

$$\int_{\nu \in Z(E), |\nu_j| \leq \tilde{X}_j} |d\sigma_{-r,-r'}(\nu)| \ll AN(\tilde{X})^2,$$

see (40).

If $\ell < d$, we have

$$\int_{\nu_\ell \in Z(E)_\ell, |\nu_\ell| \leq \tilde{X}_\ell} d\tau_\ell(\nu_\ell) = \int_{\nu_{\ell+1} \in Z(E)_{\ell+1}} k_{\ell+1}(\nu_{\ell+1})\, d\tau_{\ell+1}(\nu_{\ell+1}).$$

We use Lemma 4.2.2 if $j \in E$, and Lemma 4.4.1 otherwise, and we obtain (using the inductive hypothesis)

$$\int_{\nu_{\ell+1} \in Z(E)_{\ell+1}} k_{\ell+1}(\nu_{\ell+1})\, d\tau_{\ell+1}(\nu_{\ell+1}) \ll A\tilde{C}_{\ell+1} \prod_{j=1}^{\ell} \tilde{X}_j^2 \cdot \prod_{j=\ell+2}^{d} \tilde{C}_j.$$

This proves (53).

One more application of Lemma 4.4.1 or 4.4.2 gives

$$\int_{Z(E)_1} k_1(\nu_1)\, d\tau_1(\nu_1) \ll A \prod_{j=1}^{d} \tilde{C}_j,$$

which is the statement in the lemma. □

4.6. Choice of Y. In the previous subsection, we have written the sum with smooth bounds $\Lambda_\psi(x)$ as $\sum_E S_E(k)$, where E runs over the subsets of $\{1,\dots,d\}$. The terms with $E \neq \emptyset$ give the contribution of eigenvalues with exceptional coordinates. These terms might vanish. It seems sensible to choose Y in such a way that $S_\emptyset(k) + |\Lambda(x) - \Lambda_\psi(x)|$ is estimated optimally. The estimate of $\Lambda(x)$ that we shall obtain cannot be better than this.

If some $x_j < 1$, we have by Lemma 4.2.1

$$|\Lambda(x) - \Lambda_\psi(x)| \ll |N(rr')|^{1/2} N(x)^{-1/2+\varepsilon} N_+(x).$$

In the next subsection, we shall use Lemma 4.1.3 if $x_j < 1$ for some j.

Assumption on x. In this subsection, we proceed with $x \in [1,\infty)^d$; hence $N(x) = N_+(x)$.

In the next computations, we use an "absorbing ε", to avoid introducing many small quantities.

The Lemmas 4.2.1 and 4.5.1 give the following estimates:

$$|\Lambda(x) - \Lambda_\psi(x)| \ll |N(rr')|^{1/2} N(x)^{1/2+\varepsilon} x_{\min}^{-1} + |N(rr')|^{1/2} N(x)^{1/2+\varepsilon} Y^{-1},$$

$$S_\emptyset(k) \ll |N(rr')|^{1/4+\varepsilon} \prod_{j=1}^{d} C(X_j, Y),$$

where $X_j = 2\pi\sqrt{|(rr')^{\sigma_j}|}/x_j$, as defined in (47), and C is as in (48). For $X_j \geqq 1$ we use $C(X_j, Y) \ll Y^{1/2} + X_j(\log X_j)^2$, otherwise the choice of Y becomes too complicated. We put

$$(54) \qquad L(X) := \prod_{j,\, X_j \leqq 1} (1 + |\log X_j|) \prod_{j,\, X_j \geqq 1} (1 + X_j (\log X_j)^2),$$

$$(55) \qquad b_j := 2\pi |(rr')^{\sigma_j}|^{1/2},$$

$$(56) \qquad A = A(x) := \{j \colon x_j \leqq b_j\}.$$

So $\prod_{j=1}^{d} C(X_j, Y) \ll Y^{\frac{d}{2}} L(X)$, and $X_j = \dfrac{b_j}{x_j}$.

We estimate $L(X)$ by

$$L(X) \ll \prod_{j \notin A} \left(\frac{x_j}{b_j}\right)^{\varepsilon} \prod_{j \in A} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon} \ll N(x)^{\varepsilon} \prod_{j \notin A} b_j^{-\varepsilon} \prod_{j \in A} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon}$$

$$\ll N_{-}(b)^{-\varepsilon} N(x)^{\varepsilon} \prod_{j \in A} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon}.$$

In this way,

$$S_0(k) \ll Y^{d/2} N(b)^{\frac{1}{2}+\varepsilon} N_{-}(b)^{-\varepsilon} N(x)^{\varepsilon} \prod_{j \in A} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon}.$$

So, we have the estimate

$$(57) \quad S_0(k) + |\Lambda(x) - \Lambda_{\psi}(x)| \ll N(b) N(x)^{1/2+\varepsilon} x_{\min}^{-1}$$
$$+ N(b) N(x)^{1/2+\varepsilon} Y^{-1} + Y^{d/2} N(b)^{\frac{1}{2}+\varepsilon} N_{-}(b)^{-\varepsilon} N(x)^{\varepsilon} \prod_{j \in A} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon}.$$

We shall try to give an optimal estimation of (57). The final estimate cannot be better than $N(b)N(x)^{1/2+\varepsilon} x_{\min}^{-1}$. We wish to choose Y such that the other two terms in (57) are very close to each other.

This suggests $Y^{\frac{d+2}{2}} = N(b)^{1/2} N(x)^{1/2} \prod_{j \in A} \left(\frac{b_j}{x_j}\right)^{-1}$. But we have to take into account the condition $Y > 15$ in Subsection 4.2, and the above value for Y could be smaller than 1. To grant this condition, we take

$$(58) \qquad Y = 15 d_{F/\mathbb{Q}}^{\frac{1}{d+2}} N(b)^{\frac{1}{d+2}} N(x)^{\frac{1}{d+2}},$$

where $d_{F/\mathbb{Q}}$ is the discriminant of the field F. Since $|N(r)| \geqq d_{F/\mathbb{Q}}^{-1}$ for any $r \in \mathcal{O}' \setminus \{0\}$, we have $d_{F/\mathbb{Q}} N(b) \geqq 1$. Thus, Y is at least 15.

With this choice of Y, the second term in (57) is smaller than the last one. The final estimate is

$$(59) \qquad |\Lambda(x) - \Lambda_{\psi}(x)| + S_0(k) \ll N(b) N(x)^{\frac{1}{2}+\varepsilon} x_{\min}^{-1}$$
$$+ N(b)^{\frac{d+1}{d+2}+\varepsilon} N_{-}(b)^{-\varepsilon} \prod_{j \in A(x)} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon} \cdot N(x)^{\frac{d}{2(d+2)}+\varepsilon}.$$

The contribution of the exceptional spectrum to $\Lambda_{\psi}(x)$ is given by the remaining sum $\sum_{E \neq 0} S_E(k)$. By Lemma 4.5.1 we have

$$(60) \qquad S_E(k) \ll |N(rr')|^{\frac{1}{4}+\varepsilon} \prod_{j \in E} D(X_j, \beta(E)_j) \prod_{j \notin E} C(X_j, Y).$$

We obtain the following estimate:

$$S_E(k) \ll N(b)^{\frac{1}{2}+\varepsilon} Y^{\frac{d-|E|}{2}} \prod_{j\in E\setminus A} \left(\frac{b_j}{x_j}\right)^{-2\beta(E)_j-\varepsilon} \prod_{j\in E\cap A} \left(\frac{x_j}{b_j}\right)^{1-\varepsilon} \prod_{j\in A\setminus E} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon} \prod_{j\notin A\cup E} \left(\frac{b_j}{x_j}\right)^{-\varepsilon}$$

$$\ll N(b)^{\frac{1}{2}+\frac{d-|E|}{2(d+2)}+\varepsilon} N(x)^{\frac{d-|E|}{2(d+2)}} \prod_{j\in E\setminus A} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon} \prod_{j\in E\cap A} \left(\frac{x_j}{b_j}\right)^{1-\varepsilon} \prod_{j\in A\setminus E} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon} \cdot N(x)^{\varepsilon} \prod_{j\notin A\cup E} b_j^{-\varepsilon}.$$

We define, $\alpha_E := \dfrac{d-|E|}{2(d+2)}$ and

$$F_E(b,x) := N(b)^{\frac{1}{2}+\alpha_E+\varepsilon} N_-(b)^{-\varepsilon} N(x)^{\alpha_E+\varepsilon} \cdot \prod_{j\in E\setminus A} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon} \prod_{j\in E\cap A} \left(\frac{x_j}{b_j}\right)^{1-\varepsilon} \prod_{j\in A\setminus E} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon}. \tag{61}$$

In this way, we have $S_E \ll F_E(b,x)$, and

$$\Lambda(x) \ll N(b) N(x)^{\frac{1}{2}+\varepsilon} x_{\min}^{-1} + N(b)^{\frac{d+1}{d+2}+\varepsilon} N_-(b)^{-\varepsilon} \prod_{j\in A(x)} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon} \cdot N(x)^{\frac{d}{2(d+2)}+\varepsilon} + \sum_{E\neq\emptyset} F_E(b,x).$$

Proposition 4.6.1. *Let $x\in(1,\infty)^d$, and let $r,r'\in\mathcal{O}'\setminus\{0\}$. Then*

$$\Lambda(x) \ll_{\varepsilon,F} N(|rr'|)^{\frac{1}{2}} \frac{N(x)^{\frac{1}{2}+\varepsilon}}{x_{\min}} + C_{r,r'} N(x)^{\frac{d}{2(d+2)}+\varepsilon} \prod_{j\in A(x)} \left(\frac{\sqrt{|(rr')^{\sigma_j}|}}{x_j}\right)^{1+\varepsilon}$$

$$+ \sum_{E\neq\emptyset} \tilde{C}_{r,r'}(E) N(x)^{\frac{d-|E|}{2(d+2)}+\varepsilon} \prod_{j\in E\setminus A(x)} \left(\frac{x_j}{\sqrt{|(rr')^{\sigma_j}|}}\right)^{2\beta(E)_j+\varepsilon}$$

$$\cdot \prod_{j\in E\cap A(x)} \left(\frac{x_j}{\sqrt{|(rr')^{\sigma_j}|}}\right)^{1-\varepsilon} \prod_{j\in A(x)\setminus E} \left(\frac{\sqrt{|(rr')^{\sigma_j}|}}{x_j}\right)^{1+\varepsilon},$$

with $\beta(E)_j$ defined as in (51), *and*

$$x_{\min} = \min_j(x_j),$$

$$b_j = 2\pi\sqrt{|(rr')^{\sigma_j}|},$$

$$A(x) = \{j\colon x_j \leqq b_j\},$$

$$C_{r,r'} = N(|rr'|)^{\frac{d+1}{2(d+2)}+\varepsilon} N_-(|rr'|)^{-\varepsilon},$$

$$\tilde{C}_{r,r'}(E) = N(|rr'|)^{\frac{1}{4}+\frac{d-|E|}{4(d+2)}+\varepsilon} N_-(|rr'|)^{-\varepsilon}.$$

Remark. If no eigenvalue has exceptional coordinates at precisely the indices specified by $E \neq \emptyset$, then the summand corresponding to that set E can be omitted.

Case $d = 1$. The estimate in the proposition is not optimal for $d = 1$. Let us look at this case, in the absence of exceptional eigenvalues. Now $F = \mathbb{Q}$, so $b \geqq 2\pi$. We have

$$|\Lambda(x) - \Lambda_\psi(x)| \ll |rr'|^{1/2} x^{1/2+\varepsilon} \max(x^{-1}, Y^{-1}),$$

$$S_0(k) \ll |rr'|^{1/4+\varepsilon} C(X, Y),$$

where $X = 2\pi\sqrt{|rr'|}/x$, and $C(X, Y) \ll \begin{cases} Y^{1/2} + \log X & \text{if } X \leqq 1, \\ Y^{1/2} + X(\log X)^2 & \text{if } X \geqq 1. \end{cases}$

i) For $x \geqq b$, we choose $Y = 15b^{1/3}x^{1/3}$. Since $Y \gg x$ we get

$$|\Lambda(x) - \Lambda_\psi(x)| + S_0(k) \ll bx^{1/2+\varepsilon}b^{-1/3}x^{-1/3} + b^{1/2+2\varepsilon}(b^{1/6}x^{1/6} + x^\varepsilon)$$
$$\ll b^{2/3}x^{1/6+\varepsilon}.$$

So

$$\Lambda(x) \ll |rr'|^{1/3} x^{1/6+\varepsilon}. \tag{62}$$

ii) For $b^{1/3} \leqq x \leqq b$, we choose $Y = 15b^{-1/3}x$. Then we have $15 \leqq Y \ll x$. Using that $x^{-1} \leqq b^{-1/3}$, we get

$$\Lambda(x) \ll bx^{1/2+\varepsilon}b^{1/3}x^{-1} + b^{1/2+2\varepsilon}(b^{-1/6}x^{1/2} + b^{1+\varepsilon}x^{-1-\varepsilon}) \tag{63}$$
$$\ll b^{7/6+\varepsilon} \ll |rr'|^{7/12+\varepsilon}.$$

Note. In the case of general d, we have not used this choice of Y, as it may be smaller than 1. Here this is the optimal choice.

4.7. Estimation of sums of Kloosterman sums. In this subsection, we will prove the main result in the paper. We estimate, for sufficiently large $x \in (0, \infty)^d$, the average

$$\tilde{\Lambda}(x) = \sum_{c \in I\setminus\{0\}, |c^{\sigma_j}| \leqq x_j} \frac{S(r, r'; c)}{|N(c)|}$$

of Kloosterman sums for totally real fields, which generalizes Kuznetsov's estimate in the case $d = 1$. In Kuznetsov's case, the absolute value $|c|$ stays away from 0. For $d \geqq 2$, some coordinates of c may tend to zero. It turns out that the sum formula of Kuznetsov type in

Theorem 2.7.1 gives good results if all coordinates of c are not too small; see Proposition 4.6.1. For all other c, we shall use Lemma 4.1.3.

We want to consider $\tilde{\Lambda}(x)$ for sufficiently large $x \in (1,\infty)^d$, for fixed $r, r' \in \mathcal{O}'\backslash\{0\}$. Our concept of "sufficiently large" will depend on r and r'. We will restrict ourselves to $x_j \geqq b_j = 2\pi\sqrt{|(rr')^{\sigma_j}|}$ for all $j \in \{1,\ldots,d\}$.

We start with the decomposition

$$\tilde{\Lambda}(x) = \sum_{c\in I\backslash\{0\},|c^{\sigma_j}|\leqq x_j} \frac{S(r,r';c)}{|N(c)|} = \sum_n \Lambda(2^{-n}x), \tag{64}$$

where n runs over $\mathbb{N}^d_{\geqq 0}$, and $(2^{-n}x)_j = 2^{-n_j}x_j$. See (41) for the sum $\Lambda(\cdot)$. Note that all but finitely many $\Lambda(2^{-n}x)$ vanish, as c runs through a subset of a lattice.

We split up the sum over n in (64), and shall choose for each term a suitable way to estimate it. This decomposition is parametrized by subsets $J \subset \{1,\ldots,d\}$. Set $N_j := \log_2 x_j$. We put

$$\tilde{\Lambda}(x;J) := \sum_{n\in N_{x,J}} \Lambda(2^{-n}x), \tag{65}$$

$$N_{x,J} := \{n \in \mathbb{N}^d_{\geqq 0} : n_j \leqq N_j \text{ if } j\in J, n_j > N_j \text{ otherwise}\}. \tag{66}$$

That is, $n \in N_{x,J}$ if and only if $2^{-n_j}x_j \geqq 1$ for any $j \in J$ and $2^{-n_j}x_j < 1$ for $j \notin J$.

Hence

$$\tilde{\Lambda}(x) = \sum_J \tilde{\Lambda}(x;J). \tag{67}$$

To estimate $\tilde{\Lambda}(x;J)$, we shall use Lemma 4.1.3 for each $J \neq \{1,\ldots,d\}$, whereas for $J = \{1,\ldots,d\}$, we shall use Proposition 4.6.1, which we have proved with the help of the sum formula.

Let $J \neq \{1,\ldots,d\}$. We apply Lemma 4.1.3 with x replaced by $\tilde{x}$, given by $\tilde{x}_j := x_j$ if $j \in J$, and $\tilde{x}_j := 2^{-[N_j]}x_j$ if $j \notin J$. Note that in the latter case $\tilde{x}_j \in [1,2)$.

In this way, we obtain

$$|\tilde{\Lambda}(x;J)| \leqq \sum_{c\in I\backslash\{0\},|c^{\sigma_j}|\leqq \tilde{x}_j} \frac{|S(r,r';c)|}{|N(c)|} \ll |N(rr')|^{\frac{1}{2}} N(\tilde{x})^{\frac{1}{2}+\varepsilon} N_+(\tilde{x})^{\varepsilon}$$

$$\ll |N(rr')|^{\frac{1}{2}} \prod_{j\in J} x_j^{\frac{1}{2}+\varepsilon}.$$

Clearly, the largest contribution is given by the sets J of cardinality $d-1$. If $J=\{1,\dots,d\}\backslash\{\ell\}$, then

$$\prod_{j\in J} x_j = \frac{N(x)}{x_\ell} \leqq \frac{N(x)}{x_{\min}},$$

where $x_{\min}=\min(x_j)$. In this way, we have

$$\sum_{J\subsetneqq\{1,\dots,d\}} \tilde{\Lambda}(x;J) \ll |N(rr')|^{\frac{1}{2}}\left(\frac{N(x)}{x_{\min}}\right)^{\frac{1}{2}+\varepsilon}. \tag{68}$$

Now, we start with the region corresponding to $J=J(d):=\{1,\dots,d\}$. We have

$$\tilde{\Lambda}\big(x;J(d)\big)=\sum_{0\leqq n_j\leqq N_j}\Lambda(2^{-n}x). \tag{69}$$

By Proposition 4.6.1, we may estimate the sum in (69) by $\Sigma_1+\Sigma_2+\Sigma_3$, where

$$\Sigma_1 := \sum_{0\leqq n_j\leqq N_j} N(b)N(2^{-n}x)^{\frac{1}{2}+\varepsilon}\min_j(2^{-n_j}x_j)^{-1},$$

$$\Sigma_2 := \sum_{0\leqq n_j\leqq N_j} N(b)^{\frac{d+1}{d+2}+\varepsilon}N_-(b)^{-\varepsilon}N(2^{-n}x)^{\frac{d}{2(d+2)}+\varepsilon} \cdot \prod_{j\in A(2^{-n}x)}\left(\frac{2^{n_j}b_j}{x_j}\right)^{1+\varepsilon}, \tag{70}$$

$$\Sigma_3 := \sum_{E\neq 0}\sum_{0\leqq n_j\leqq N_j} F_E(b,2^{-n}x). \tag{71}$$

(See (61) for the definition of $F_E(b,2^{-n}x)$.) We shall estimate these sums separately.

When we work out the sums over n, we will get a factor of the type $\sum_{n=0}^{N_j} 2^{-n\alpha_j}$, with different α_j. Each of these sums is $O\left(\frac{1}{|1-2^{-\alpha_j}|}\right)=O(1)$ if $\alpha_j>0$ and $O(2^{-\alpha_j(N_j+1)})=O(x_j^{-\alpha_j})$, if $\alpha_j<0$.

We start by estimating Σ_1. For each $\ell=1,\dots,d$, let

$$A_\ell := \{n\in\mathbb{Z}^d\colon 0\leqq n_j\leqq N_j, \min_j(2^{-n_j}x_j)=2^{-n_\ell}x_\ell\}.$$

We have

$$\begin{aligned}\Sigma_1 &\leqq N(b)\sum_{\ell=1}^{d} N(x)^{\frac{1}{2}+\varepsilon}x_\ell^{-1}\sum_{n\in A_\ell}2^{n_\ell(\frac{1}{2}-\varepsilon)}\prod_{j\neq\ell}2^{-n_j(\frac{1}{2}+\varepsilon)} \\ &\leqq N(b)N(x)^{\frac{1}{2}+\varepsilon}\sum_{\ell=1}^{d}x_\ell^{-1}2^{(\frac{1}{2}-\varepsilon)N_\ell} \ll N(b)N(x)^{\frac{1}{2}+\varepsilon}\sum_{\ell=1}^{d}x_\ell^{-1}x_\ell^{\frac{1}{2}-\varepsilon} \\ &\ll N(b)N(x)^{\frac{1}{2}+\varepsilon}x_{\min}^{-\frac{1}{2}-\varepsilon}.\end{aligned} \tag{72}$$

To give an estimate of Σ_2, we have to consider different regions, determined by the set $A(2^{-n}x)$. Let $M_j = \min\left(\log_2 \frac{x_j}{b_j}, N_j\right)$. So $\{j: b_j \geqq 1\} = \{j: M_j \leqq N_j\}$. By definition, $j \in A(2^{-n}x)$ if and only if $b_j \geqq 2^{-n_j}x_j$.

For each subset $L \subseteq \{j: b_j \geqq 1\}$ we define

$$N_L := \{n \in \mathbb{Z}^d: M_j \leqq n_j \leqq N_j, \text{ if } j \in L; 0 \leqq n_j < M_j \text{ otherwise}\}.$$

We can split up $\Sigma_2 = \sum_{L \subseteq \{j: b_j \geqq 1\}} \Sigma_{2,L}$, where

$$\Sigma_{2,L} := \sum_{n \in N_L} N(b)^{\frac{d+1}{d+2}+\varepsilon} N_-(b)^{-\varepsilon} N(2^{-n}x)^{\frac{d}{2(d+2)}+\varepsilon} \cdot \prod_{j \in A(2^{-n}x)} \left(\frac{2^{n_j}b_j}{x_j}\right)^{1+\varepsilon}. \tag{73}$$

We set $\alpha_j := \frac{d}{2(d+2)} + \varepsilon$ if $j \notin L$, and otherwise $\alpha_j := -\left(1 - \frac{d}{2(d+2)} + \varepsilon\right)$. Note that we have $A(2^{-n}x) = L$ for $n \in N_L$.

$$\begin{aligned} \Sigma_{2,L} &\ll N(b)^{\frac{d+1}{d+2}+\varepsilon} N_-(b)^{-\varepsilon} N(x)^{\frac{d}{2(d+2)}+\varepsilon} \prod_{j \in L} \left(\frac{b_j}{x_j}\right)^{1+\varepsilon} \sum_{n \in N_L} \prod_{j=1}^{d} 2^{-n_j\alpha_j} \\ &\ll N(b)^{\frac{d+1}{d+2}+\varepsilon} N_-(b)^{-\varepsilon} N(x)^{\frac{d}{2(d+2)}+\varepsilon} \prod_{j \in L} b_j^{1+\varepsilon} \prod_{j \in L} x_j^{-1-\varepsilon} \\ &\quad \cdot \prod_{j \notin L} \left(\frac{1}{|1-2^{-\alpha_j}|}\right) \prod_{j \in L} x_j^{-\alpha_j} \\ &\ll N(b)^{\frac{d+1}{d+2}+\varepsilon} N_-(b)^{-\varepsilon} N(x)^{\frac{d}{2(d+2)}+\varepsilon} \prod_{j \in L} b_j^{1+\varepsilon} \prod_{j \in L} x_j^{-\frac{d}{2(d+2)}}. \end{aligned} \tag{74}$$

The factors x_j^{ε} with $j \in L$ have been absorbed into $N(x)^{\frac{d}{2(d+2)}+\varepsilon}$.

A comparison with the estimate of Σ_1, in the x-aspect, shows that the term $N(x)^{\frac{d-1}{2d}+\varepsilon}$ is already dominated by $O\left(\left(N(x)/x_{\min}\right)^{1/2+\varepsilon}\right)$. (Note that $x_{\min} \leqq N(x)^{1/d}$.) This leaves us with a small factor if $d > 2$. We proceed under the assumption $d \geqq 2$.

$$\Sigma_{2,L} \ll N(b)^{\frac{d+1}{d+2}+\varepsilon} N_-(b)^{-\varepsilon} \left(\frac{N(x)}{x_{\min}}\right)^{1/2+\varepsilon} N(x)^{-\frac{d-2}{2d(d+2)}} \prod_{j \in L} \left(b_j^{1+\varepsilon} x_j^{-\frac{d}{2(d+2)}}\right).$$

We have assumed that $x_j \geqq b_j$ for all j. So we can use $N(x)^{-\frac{d-2}{2d(d+2)}}$ to reduce the exponent of $N(b)$ to $\frac{2d^2+d+2}{2d(d+2)}$. We have to take the sum over $L \subset \{j: b_j \geqq 1\}$. So we look at the worst case, that all $b_j \geqq 1$ have $j \in L$. This leads to the following estimate:

$$\Sigma_2 \ll N(b)^{\frac{2d^2+d+2}{2d(d+2)}+\varepsilon} N_-(b)^{-\varepsilon} \left(\frac{N(x)}{x_{\min}}\right)^{1/2+\varepsilon} N_+(b)^{\frac{d+4}{2(d+2)}+\varepsilon}. \tag{75}$$

We have to compare (68), (72) and (75). In the x-aspect, these estimates are the same. In the b-aspect, the estimate in (75) gives the main contribution

$$N_+(b)^{\frac{3d^2+5d+2}{2d(d+2)}+\varepsilon} N_-(b)^{\frac{2d^2+d+2}{2d(d+2)}-\varepsilon}.$$

To estimate Σ_3, we split up the sum in a similar way:

$$\Sigma_3 = \sum_{E \neq \emptyset} \sum_{0 \leqq n_j \leqq N_j} F_E(b, 2^{-n}x) = \sum_{E \neq \emptyset} \sum_{L \subseteq \{j:\, b_j \geqq 1\}} \sum_{n \in N_L} F_E(b, 2^{-n}x).$$

By (61) we have

$$F_E(b, 2^{-n}x) = C_E(b) N(2^{-n}x)^{\alpha_E+\varepsilon} \prod_{j \in E \setminus A(2^{-n}x)} \left(\frac{2^{-n_j}x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon}$$

$$\cdot \prod_{j \in E \cap A(2^{-n}x)} \left(\frac{2^{-n_j}x_j}{b_j}\right)^{1-\varepsilon} \prod_{j \in A(2^{-n}x) \setminus E} \left(\frac{b_j}{2^{-n_j}x_j}\right)^{1+\varepsilon},$$

where $C_E(b) = N(b)^{\frac{1}{2}+\alpha_E+\varepsilon} N_-(b)^{-\varepsilon}$, $\alpha_E = \dfrac{d-|E|}{2(d+2)}$, and $A(2^{-n}x) = \{j:\, b_j \geqq 2^{-n_j}x_j\}$.

When we shall sum this over $n \in N_L$ in the following computations, we shall use that for $j \in E \cap A(2^{-n}x)$, we have $n_j \geqq M_j = \log_2 \dfrac{x_j}{b_j}$. Otherwise we would not have $\dfrac{b_j}{2^{-n_j}x_j} \geqq 1$. Thus, we can use for these j that

$$\sum_{M_j \leqq n_j \leqq N_j} \left(\frac{x_j}{2^{n_j}b_j}\right)^{1-\varepsilon} \ll_\varepsilon \left(\frac{x_j}{b_j}\right)^{1-\varepsilon} 2^{(\varepsilon-1)M_j} = 1.$$

First we consider the most exceptional case $E = \{1, \ldots, d\}$. In this case $\alpha_E = 0$ and we obtain the following estimate:

$$\sum_{n \in N_L} F_E(b, 2^{-n}x) \ll C_E(b) \sum_{n \in N_L} N(2^{-n}x)^\varepsilon \prod_{j \notin L} \left(\frac{x_j}{2^{n_j}b_j}\right)^{2\beta(E)_j+\varepsilon} \prod_{j \in L} \left(\frac{x_j}{2^{n_j}b_j}\right)^{1-\varepsilon}$$

$$\ll C_E(b) N(x)^\varepsilon \prod_{j \notin L} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon} \prod_{j \in L} \left(\frac{x_j}{b_j}\right)^{1-\varepsilon}$$

$$\cdot \sum_{n \in N_L} \prod_{j \in L} 2^{-n_j(1-\varepsilon)} \prod_{j \notin L} 2^{-n_j(2\beta(E)_j+\varepsilon)}$$

$$\ll C_E(b) N(x)^\varepsilon \prod_{1 \leqq j \leqq d} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon}.$$

So, we have for $E = \{1, \ldots, d\}$,

$$\sum_{0 \leqq n_j \leqq N_j} F_E(b, 2^{-n}x) \ll N(b)^{\frac{1}{2}+\varepsilon} N_-(b)^{-\varepsilon} N(x)^\varepsilon \prod_{j=1}^{d} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon}. \tag{76}$$

Now, we consider $0 < |E| < d$. Let $\alpha_j = \alpha_E + 2\beta(E)_j + \varepsilon$ if $j \in E \backslash L$; $\alpha_j = \alpha_E + 1 - \varepsilon$ if $j \in E \cap L$; $\alpha_j = \alpha_E - 1 - \varepsilon$ if $j \in L \backslash E$, and $\alpha_j = \alpha_E + \varepsilon$ otherwise. We have

$$\sum_{n \in N_L} F_E(b, 2^{-n}x) \ll C_E(b) \sum_{n \in N_L} N(2^{-n}x)^{\alpha_E+\varepsilon} \prod_{j \in E \backslash L} \left(\frac{x_j}{2^{n_j} b_j}\right)^{2\beta(E)_j+\varepsilon}$$

$$\cdot \prod_{j \in E \cap L} \left(\frac{x_j}{2^{n_j} b_j}\right)^{1-\varepsilon} \prod_{j \in L \backslash E} \left(\frac{2^{n_j} b_j}{x_j}\right)^{1+\varepsilon}$$

$$\ll C_E(b) N(x)^{\alpha_E+\varepsilon} \prod_{j \in E \backslash L} \left(\left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon} \sum_{0 \leqq n_j < M_j} 2^{-n_j(\alpha_E+2\beta(E)_j+2\varepsilon)} \right)$$

$$\cdot \prod_{j \in E \cap L} \left(\left(\frac{x_j}{b_j}\right)^{1-\varepsilon} \sum_{M_j \leqq n_j \leqq N_j} 2^{-n_j(\alpha_E+1)} \right)$$

$$\cdot \prod_{j \in L \backslash E} \left(\left(\frac{b_j}{x_j}\right)^{1+\varepsilon} \sum_{M_j \leqq n_j \leqq N_j} 2^{n_j(1-\alpha_E)} \right)$$

$$\cdot \prod_{j \notin E \cup L} \sum_{0 \leqq n_j < M_j} 2^{-n_j(\alpha_E+\varepsilon)}$$

$$\ll C_E(b) N(x)^{\alpha_E+\varepsilon} \prod_{j \in E \backslash L} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon} \prod_{j \in E \cap L} \left(\frac{x_j}{b_j}\right)^{1-\varepsilon-\alpha_E-1}$$

$$\cdot \prod_{j \in L \backslash E} \left(\left(\frac{b_j}{x_j}\right)^{1+\varepsilon} x_j^{1-\alpha_E} \right)$$

$$\ll C_E(b) N(x)^{\alpha_E+\varepsilon} \prod_{j \in E \backslash L} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon} \prod_{j \in E \backslash L} (b_j^{1+\varepsilon} x_j^{-\alpha_E-\varepsilon}).$$

So, we have

$$\sum_{L \subseteq \{j:\, b_j \geqq 1\}} \sum_{n \in N_L} F_E(b, 2^{-n}x) \ll C_E(b) N(x)^{\alpha_E+\varepsilon} \prod_{j \in E} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon} \prod_{j \notin E b_j \geqq 1} b_j^{1-\alpha_E+\varepsilon}.$$

Therefore

$$\Sigma_3 \ll \sum_{E \neq \emptyset} N(b)^{\frac{1}{2}+\alpha_E+\varepsilon} N_-(b)^{-\varepsilon} N(x)^{\alpha_E+\varepsilon} \prod_{j \in E} \left(\frac{x_j}{b_j}\right)^{2\beta(E)_j+\varepsilon} \prod_{j \notin E b_j \geqq 1} b_j^{1-\alpha_E+\varepsilon}.$$

Summing up, we have proved the main result:

Theorem 4.7.1. *Let the degree d of the totally real number field F be at least 2. Let I be a non-zero ideal of the ring of integers $\mathcal{O}$ of F. Let $r, r' \in \mathcal{O}' \setminus \{0\}$. For each $\varepsilon > 0$, and for all $x \in (1, \infty)^d$ satisfying $x_j \geqq b_j = 2\pi|(rr')^{\sigma_j}|^{1/2}$ for all $j \in \{1, \ldots, d\}$, we have*

$$\sum_{c \in I \setminus \{0\}, 0 \leqq |c^{\sigma_j}| \leqq x_j} \frac{S(r, r'; c)}{|N(c)|} \ll_{\varepsilon, F} D_{r,r'} \left(\frac{N(x)}{x_{\min}}\right)^{\frac{1}{2}+\varepsilon}$$

$$+ \sum_{E \subset \{1, \ldots, d\}, E \neq \emptyset} \tilde{D}_{r,r'}(E) N(x)^{\frac{d-|E|}{2(d+2)}+\varepsilon} \prod_{j \in E} x_j^{2\beta(E)_j},$$

where $x_{\min} = \min\{x_j : 1 \leqq j \leqq d\}$. The bounds $\beta(E)_j$ of exceptional coordinates are defined in (51), *and*

$$D_{r,r'} = N_+(|rr'|)^{\frac{3d^2+5d+2}{4d(d+2)}+\varepsilon} N_-(|rr'|)^{\frac{2d^2+d+2}{4d(d+2)}-\varepsilon},$$

$$\tilde{D}_{r,r'}(E) = N(|rr'|)^{1/4+\frac{d-|E|}{4(d+2)}+\varepsilon} N_-(|rr'|)^{-\varepsilon} \prod_{j \in E} \left(\sqrt{|(rr')^{\sigma_j}|}\right)^{-2\beta(E)_j} \prod_{j \notin E, |(rr')^{\sigma_j}| \geqq 1} |(rr')^{\sigma_j}|^{\frac{1}{2}-\frac{d-|E|}{4(d+2)}},$$

$$N_+(|rr'|) = \prod_{|(rr')^{\sigma_j}| \geqq 1} |(rr')^{\sigma_j}|, \quad N_-(|rr'|) = \prod_{|(rr')^{\sigma_j}| < 1} |(rr')^{\sigma_j}|.$$

Corollary 4.7.2. *In the notations of Theorem 4.7.1, let $X \geqq 1$ be such that $X \geqq b_j = 2\pi|(rr')^{\sigma_j}|^{1/2}$, for all j. Then*

$$\sum_{c \in I \setminus \{0\}, 0 \leqq |c^{\sigma_j}| \leqq X} \frac{S(r, r'; c)}{|N(c)|} \ll D_{r,r'} X^{\frac{d-1}{2}+\varepsilon} + \sum_{E \neq \emptyset} \tilde{D}_{r,r'}(E) X^{d\frac{d-|E|}{2(d+2)}+\alpha(E)+\varepsilon}.$$

with $\alpha(E) = 2 \sum_{j \in E} \beta(E)_j$.

Remarks. *Cancellations of Kloosterman sums.* The term corresponding to $E \neq \emptyset$ gives the contribution of the eigenvalues λ for which

$$E = \{j : \lambda_j \text{ is exceptional}\} = \left\{j : 0 < \nu_j \leqq \frac{1}{4}\right\}.$$

If there are no eigenvalues of this type, the corresponding term in the theorem is absent.

We note that by inputting the best presently known bounds for exceptional eigenvalues in Theorem 4.7.1 and Corollary 4.7.2, our results imply cancellation of Kloosterman sums for any totally real number field F. This means that the sum of Kloosterman sums considered in the theorem grows strictly less rapidly than the bound in Lemma 4.1.3, obtained by using Weil's estimate.

To verify this, let us denote by $\gamma \leqq \frac{1}{4}$ an upper bound for all exceptional coordinates of cuspidal spectral parameters. If we compute the maximum contribution of the exceptional spectrum, in the notation of Corollary 7.2, we find for each $E \subset \{1, \ldots, d\}$, the exponent

$$\frac{d(d-|E|)}{2(d+2)} + \alpha(E) \leqq \frac{d(d-|E|)}{2(d+2)} + 2\gamma|E| = \frac{d^2}{2(d+2)} + \frac{|E|}{2(d+2)}(4\gamma(d+2)-d),$$

which has a maximum value that is strictly less than $\frac{d}{2}$ for any $\gamma < \frac{1}{4}$ (letting $|E| = d$ for $\gamma \geqq \frac{d}{4(d+2)}$, and $|E| = 1$ otherwise). Now, the estimate obtained by using the Weil-Salié estimates gives a growth of $X^{d/2+\varepsilon}$ in (77) (see Lemma 4.1.3), hence the asserted cancellation of Kloosterman sums holds.

The best bound for exceptional coordinates in the literature is $\gamma = \frac{1}{5}$, valid for any number field F (see [15]). Recently, Kim and Shahidi have improved this estimate to $\gamma = 1/6 - 1/51$ (private communication). An extension of the previous calculations shows that if we use this value of γ then the contribution of the exceptional spectrum is strictly smaller than the first term in Corollary 4.7.2, for any $d \geqq 3$. This implies that for any $d \geqq 3$ the main term in (77) is indeed given by $X^{(d-1)/2+\varepsilon}$. This should also be true for $d = 2$ but the known estimates for γ are not yet good enough to imply this result.

x-aspect. If we ignore the rr'-aspect, we get the bounds

$$O\left((N(x)/x_{\min})^{\frac{1}{2}+\varepsilon} + \sum_{E\neq\emptyset} N(x)^{\frac{d-|E|}{2(d+2)}+\varepsilon} \prod_{j\in E} x_j^{2\beta(E)_j}\right) \tag{77}$$

in the theorem, and

$$O\left(X^{\frac{d-1}{2}+\varepsilon} + \sum_{E\neq\emptyset} X^{d\frac{d-|E|}{2(d+2)}+\alpha(E)+\varepsilon}\right) \tag{78}$$

in the corollary.

Let $\gamma \geqq \frac{d}{4(d+2)}$ be a bound for the exceptional eigenvalues as above. We can omit anyhow from (78) the terms with $\frac{d-1}{2} \geqq d\frac{d-|E|}{2(d+2)} + 2\gamma|E|$. This means that we leave only the terms with $|E| \geqq \frac{d^2+d^2}{4\gamma(d+2)-d}$. A similar analysis of (77) does not seem useful to us.

Case $d = 1$. If $F = \mathbb{Q}$, we can do better, using the estimates (62) and (63).

We assume that there are no exceptional eigenvalues, and take $x \geqq b = 2\pi\sqrt{|rr'|}$. Note that $b \geqq 2\pi$ in this case.

$$\tilde{\Lambda}(x) = 2 \sum_{1\leqq c\leqq x,\, c\in I} \frac{S(r,r';c)}{c} = \sum_{n=0}^{\log_2 x} \Lambda(2^{-n}x).$$

This sum is split up at $N := \log_2\left(\frac{x}{b}\right)$ and $M := \log_2\left(\frac{x}{b^{1/3}}\right)$. We use, successively, (62), (63) and Lemma 4.1.3:

$$\sum_{n=0}^{N} \Lambda(2^{-n}x) \ll b^{\frac{2}{3}} \sum_{n=0}^{N} (2^{-n}x)^{\frac{1}{6}+\varepsilon} \ll b^{\frac{2}{3}} x^{\frac{1}{6}+\varepsilon},$$

$$\sum_{N<n\leqq M} \Lambda(2^{-n}x) \ll \sum_{N<n\leqq M} b^{\frac{7}{6}+\varepsilon} \ll b^{\frac{7}{6}+\varepsilon} x^{\varepsilon} \ll b x^{\frac{1}{6}+\varepsilon},$$

$$\sum_{M<n\leqq \log_2 x} \Lambda(2^{-n}x) = 2 \sum_{1\leqq c\leqq b^{1/3},\, c\in I} \frac{S(r,r';c)}{c} \ll b(b^{\frac{1}{3}})^{\frac{1}{2}+\varepsilon} \ll b x^{\frac{1}{6}+\varepsilon}.$$

Thus we obtain

$$\tilde{\Lambda}(x) \ll |rr'|^{\frac{1}{2}} x^{\frac{1}{6}+\varepsilon}. \tag{79}$$

If there are exceptional eigenvalues, these give explicit contributions, to be added to the error term in (79).

The restriction $x \geqq 2\pi\sqrt{|rr'|}$ is not essential. For smaller x, the trivial estimate in Part iii) of Lemma 4.1.2 is better anyhow.

The exponent $\frac{1}{6}+\varepsilon$ is present in Kuznetsov's estimate, [13], Theorem 3. Actually, he has $x^{1/6}(\log x)^{1/3}$. See also the discussion in [4], 4.4. This concerns the case $I = \mathbb{Z}$, for which there are no exceptional eigenvalues.

Goldfeld and Sarnak, [8], estimate sums of Kloosterman sums with use of the Linnik-Selberg series. They estimate this zeta function on vertical strips. They state that the dependence on r, r' is $O(|rr'|)$. Hejhal, see [9], App. E, p. 666, follows the approach of Goldfeld and Sarnak, and obtains similar results. Yoshida, [25], uses a better estimate of the Linnik-Selberg series, and obtains $O\left(|rr'|^{1/2}x^{1/6}(\log x)^2\right)$ for $I = N\mathbb{Z}$ with $N \leqq 17$.

Comparison. In [10], Joyner considers real quadratic number fields with class number one. He states in Lemma 4.26 a stronger result than given by our theorem, but his proof contains gaps.

A crucial point in his proof is his estimate (4.28). Here one has not only to consider the sum over $D(T_1, T_2)$, but also over sets of the form

$$D'(T_1, T_2) = \{c \in \mathcal{O}\colon T_1 < |c| < T_1 + T_0,\ 1 < |\bar{c}| < T_2\}$$

(in his notation), and subsums where one of $|c|$ and $|\bar{c}|$ is in $(0,1)$. In our approach, the estimation of similar terms turns out to be rather complicated and influences the final estimate.

The influence of the exceptional eigenvalues is included in his addendum [11]. The formulation of the correction is based on the (unfounded) assumption that it is sufficient to take into account only those eigenvalues for which both coordinates are exceptional ($E = \{1,2\}$).

We discussed the problem with Joyner and he agrees with these gaps, in particular, that his handling of the region of summation in the justification of (4.28) (hence the proof of Lemma 4.26) is incomplete. Also, the discussion in §6 of [5] indicates the problems we have with Joyner's proof of the sum formula.

5. Proof of the extended sum formula

We shall give the proof in two steps. In Subsection 5.3, we follow the method of [23] and [5] to prove the sum formula (30) for a subset of the class of test functions $\mathscr{K}^c$ (see Definition 2.5.1). The auxiliary test functions we need, form the subject of Section 5.2. Here we shall use many facts from [1], mainly Chapter 13, on Whittaker transforms. In Section 5.4, we shall extend the sum formula to the full class $\mathscr{K}^c$, by an approximation argument.

5.1. Sums over $\Gamma_{N^\kappa}\backslash\Gamma_{P^\kappa}$. Before turning to the two technical results of this subsection, we recall some notations.

In this section, we return to the general context of a general cusp κ, corresponding to parabolic subgroup $P^\kappa = N^\kappa A^\kappa M = g_\kappa P g_\kappa^{-1}$, $N^\kappa = g_\kappa N g_\kappa^{-1}$, $A^\kappa = g_\kappa A g_\kappa^{-1}$, $g_\kappa \in \mathbf{G}_\mathbf{Q}$. There is a unique decomposition $g = n_\kappa(g) g_\kappa a_\kappa(g) k_\kappa(g)$ with $n_\kappa(g) \in N^\kappa$, $a_\kappa(g) \in A$, and $k_\kappa(g) \in K$. We have the intersections $\Gamma_{P^\kappa} = \Gamma \cap P^\kappa \supset \Gamma_{N^\kappa} = \Gamma \cap N^\kappa$, and the lattice $\mathfrak{t}_\kappa$ such that $\Gamma_{N^\kappa} = \{g_\kappa n[\xi] g_\kappa^{-1} : \xi \in \mathfrak{t}_\kappa\}$. The dual lattice $\mathfrak{t}_\kappa'$ describes the characters of $\Gamma_{N^\kappa}\backslash N^\kappa$ by $\chi_r(g_\kappa n[x] g_\kappa^{-1}) = e^{2\pi i S(rx)}$. See Subsection 2.1 for further conventions.

Lemma 5.1.1. *Let $\kappa \in \mathscr{P}$, and $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$. There exists $C \geqq 0$ such that for all functions f on $\Gamma_{N^\kappa}\backslash G$ satisfying*

$$|f(g)| \leqq \prod_{j=1}^{d} \min\left(|a_\kappa(g)^{\alpha_j}|^{\alpha}, |a_\kappa(g)^{\alpha_j}|^{-\beta}\right),$$

we have for all $g \in G$:

$$\sum_{\gamma \in \Gamma_{N^\kappa}\backslash\Gamma_{P^\kappa}} |f(\gamma g)| \leqq C\left(1 + |\log a_\kappa(g)^{2\rho}|^{d-1}\right) \min\left(a_\kappa(g)^{2\alpha\rho}, a_\kappa(g)^{-2\beta\rho}\right).$$

Proof. Use $a[y]^{\alpha_j} = y_j$ and $a[y]^{2\rho} = N(y)$ to reduce the statement of this lemma to Lemma 8.1 in [5]. There we summed over $\varepsilon \in \mathcal{O}^*$, which corresponds to a lattice in a hyperplane when one takes logarithms of the coordinates. Here the $a_\kappa(\gamma g_\kappa)$ correspond to a lattice Λ_κ of the same type. □

Proposition 5.1.2. *Let $\kappa, \kappa' \in \mathscr{P}$, $r \in \mathfrak{t}_\kappa'\backslash\{0\}$, $r' \in \mathfrak{t}_{\kappa'}'\backslash\{0\}$. The Kloosterman term $K(\kappa, r; \kappa', r'; h)$ converges absolutely for each function satisfying*

(80) $$h(y) \ll \prod_{j=1}^{d} \min(|y_j|^s, |y_j|^{-t})$$

with $s, t \in \mathbb{R}$, $s + t > 0$, *and* $s > \frac{1}{2}$.

Remark. Proposition 2.5.5 shows that (80) holds for any $h = \mathrm{B}_e k$, with $k \in \mathscr{K}^e$.

Proof. We have

$$K(\kappa, r; \kappa', r'; h) \ll \sum_{c \in {}^{\kappa'}\mathscr{C}^{\kappa}} \frac{|{}^{\kappa'}\mathscr{S}^{\kappa}(c)|}{|N(c)|} \left| h\left(\frac{rr'}{c^2}\right) \right|$$
$$\ll \sum_{\gamma \in {}^{\kappa'}\mathscr{S}^{\kappa}} \prod_{j=1}^{d} \left(|c_j|^{-1} \min(|c_j|^{-2s}, |c_j|^{2t}) \right),$$

where $c_j = c^{\sigma_j}$ for $\gamma \in {}^{\kappa'}\mathscr{S}^{\kappa}(c)$. We define the positive function f on G by $f(ng_{\kappa'}ak) := \prod_{j=1}^{d} \min(|a^{\alpha_j}|^{s+1/2}, |a^{\alpha_j}|^{-t+1/2})$ for $n \in N^{\kappa'}$, $a \in A$, $k \in K$. We apply Lemma 5.1.1 and Definition 2.3.1 to estimate the sum $K(\kappa, r; \kappa', r'; h)$. The terms of this sum depend on $\gamma \in {}^{\kappa'}\Gamma^{k}$. We write $\xi = \begin{pmatrix} * & * \\ c & * \end{pmatrix} = g_{\kappa'}^{-1}\gamma g_{\kappa} = n'(\xi)a_{\xi}s_0 n''(\xi)m(\xi)$, so $a_{\xi} = a[c^{-2}]$. In this way $g_{\kappa'}a_{\xi} = g_{\kappa'}n'(\xi)^{-1}g_{\kappa'}^{-1}\gamma g_{\kappa}n''(\xi)^{-1}m(\xi)^{-1}s_0^{-1} \in N^{\kappa'}\gamma g_{\kappa}n''(\xi)^{-1}K$. This gives the following:

(81) $$K(\kappa, r; \kappa', r'; h)$$
$$\ll \sum_{\gamma \in \Gamma_{N^{\kappa'}} \backslash {}^{\kappa'}\Gamma^{\kappa} / \Gamma_{N^{\kappa}}} f(g_{\kappa'}a_{\xi})$$
$$= \sum_{\gamma \in \Gamma_{N^{\kappa'}} \backslash {}^{\kappa'}\Gamma^{\kappa} / \Gamma_{N^{\kappa}}} f\left(\gamma g_{\kappa} n''(\xi)^{-1}\right)$$
$$= \sum_{\gamma \in \Gamma_{\kappa'} \backslash {}^{\kappa'}\Gamma^{\kappa} / \Gamma_{N^{\kappa}}} \sum_{\delta \in \Gamma_{N^{\kappa'}} \backslash \Gamma_{\kappa'}} f\left(\delta\gamma g_{\kappa} n''(\xi)^{-1}\right)$$
$$\ll \sum_{\gamma \in \Gamma_{\kappa'} \backslash {}^{\kappa'}\Gamma^{\kappa} / \Gamma_{N^{\kappa}}} \min\left(a_{\kappa'}\left(\gamma g_{\kappa} n''(\xi)^{-1}\right)^{\rho+2(s-\varepsilon)\rho}, a_{\kappa'}\left(\gamma g_{\kappa} n''(\xi)^{-1}\right)^{\rho-2(t-\varepsilon)\rho}\right)$$
$$\ll \sum_{\gamma \in \Gamma_{\kappa'} \backslash {}^{\kappa'}\Gamma^{\kappa} / \Gamma_{N^{\kappa}}} a_{\kappa'}\left(\gamma g_{\kappa} n''(\xi)^{-1}\right)^{\rho+2(s-\varepsilon)\rho}$$
$$= \sum_{\gamma \in \Gamma_{\kappa'} \backslash {}^{\kappa'}\Gamma^{\kappa} / \Gamma_{N^{\kappa}}} a_{\xi}^{\rho+2(s-\varepsilon)\rho},$$

for each sufficiently small $\varepsilon > 0$. (Note that $s + t > 0$.) We turn to Eisenstein series to estimate this sum.

Take $u = s - \varepsilon > \frac{1}{2}$. The Eisenstein series $E_0(P^{\kappa'}, u, 0)$ is given by a convergent series with positive terms. Everything in the following integrals and sums is positive:

$$\begin{aligned}\int_{\Gamma_{N^\kappa}\backslash N^\kappa} E_0(P^{\kappa'}, u, 0; n)\, dn &= \sum_{\gamma\in\Gamma_{\kappa'}\backslash\Gamma}\ \int_{\Gamma_{N^\kappa}\backslash N^\kappa} a_{\kappa'}(\gamma n)^{\rho+2u\rho}\, dn \\ &= \sum_{\gamma\in\Gamma_{\kappa'}\backslash\Gamma/\Gamma_{N^\kappa}}\ \int_{N^\kappa} a_{\kappa'}(\gamma n)^{\rho+2u\rho}\, dn \\ &\geqq \sum_{\gamma\in\Gamma_{\kappa'}\backslash{}^{\kappa'}\Gamma^{\kappa}/\Gamma_{N^\kappa}}\ \int_N a_{\kappa'}(\gamma g_\kappa n g_\kappa^{-1})^{\rho+2u\rho}\, dn.\end{aligned}$$

We keep the notation $\xi = g_{\kappa'}^{-1}\gamma g_\kappa$, and use the equalities

$$\begin{aligned}\gamma g_\kappa n g_\kappa^{-1} &= g_{\kappa'} n'(\xi) g_{\kappa'}^{-1} \cdot g_{\kappa'} a_\xi s_0 n''(\xi) n g_\kappa^{-1} m(\xi), \\ a_{\kappa'}(\gamma g_\kappa n g_\kappa^{-1}) &= a_\xi \cdot a_{\kappa'}\big(s_0 n''(\xi) n g_\kappa^{-1}\big),\end{aligned}$$

to obtain

$$\infty > \sum_{\gamma\in\Gamma_{\kappa'}\backslash{}^{\kappa'}\Gamma^{\kappa}/\Gamma_{N^\kappa}} a_\xi^{\rho+2u\rho} \int_N a_{\kappa'}\big(s_0 n''(\xi) n g_\kappa^{-1}\big)^{\rho+2u\rho}\, dn.$$

As $\int_N a_{\kappa'}\big(s_0 n''(\xi) n g_\kappa^{-1}\big)^{\rho+2u\rho}\, dn$ is finite for $u > \frac{1}{2}$, and does not depend on $n''(\xi)$, we have shown that the sum in (81) converges. □

5.2. Auxiliary test functions. In this section we consider cusps κ and κ' in $\mathscr{P}$, and $r \in \mathfrak{t}'_\kappa\backslash\{0\}$, $r' \in \mathfrak{t}'_{\kappa'}\backslash\{0\}$. We have $\mathbf{e} \in \{1, -1\}^d$, given by $\mathbf{e}_j = \operatorname{sign}(r_j r'_j)$. Also, we fix $\tau \in \left(\frac{1}{2}, 1\right)$.

Notation. $\xi_r := (\xi_{r,1}, \ldots, \xi_{r,d}) := (2\pi|r_1|, \ldots, 2\pi|r_d|) \in \mathbb{R}^d_{>0}$.

Definition 5.2.1. We define $H(\tau)$ as the set of even functions h on

$$\{\nu \in \mathbb{C}\colon |\operatorname{Re}\nu| \leqq \tau\} \cup \left(\frac{1}{2} + \mathbb{Z}\right)$$

that satisfy

i) h is holomorphic on $|\operatorname{Re}\nu| \leqq \tau$,

ii) $h(\nu) \ll e^{-\frac{\pi}{2}|\operatorname{Im}\nu|}(1 + |\operatorname{Im}\nu|)^{-a}$ on $|\operatorname{Re}\nu| \leqq \tau$ for each $a \in \mathbb{R}$,

iii) $h\left(\frac{b-1}{2}\right) = 0$ for all but a finite number of $b \in 2\mathbb{Z}$.

In the next lemma, we introduce an integral transform with Whittaker functions. (See, e.g., [22], 1.7, for the definition of $W_{\varepsilon,\nu}$.)

Lemma 5.2.2. *For $h \in H(\tau)$, and $y > 0$, we put*

$$\omega_1 h(y) := \frac{1}{\pi i} \int_{\mathrm{Re}\, v=0} h(v) W_{1,v}(y) \frac{v \sin \pi v}{v^2 - \frac{1}{4}}\, dv + h\left(\frac{1}{2}\right) W_{1,1/2}(y),$$

$$\omega_{-1} h(y) := \frac{1}{\pi i} \int_{\mathrm{Re}\, v=0} h(v) W_{-1,v}(y) \left(v^2 - \frac{1}{4}\right) v \sin \pi v\, dv.$$

These integrals converge absolutely, and satisfy the estimate $\omega_{\pm 1} h(y) \ll \min(y^{1/2+\tau}, y^{1/2-\tau})$.

We have

$$h(v) = \int_0^\infty \omega_{\pm 1} h(y) W_{\pm 1,v}(y) \frac{dy}{y^2}$$

for each v *with* $|\mathrm{Re}\, v| < \tau$, *and*

$$\int_0^\infty \omega_1 h(y) \overline{\omega_1 h_1(y)} \frac{dy}{y^2} = \frac{1}{\pi i} \int_{\mathrm{Re}\, v=0} h(v) \overline{h_1(v)} \frac{v \sin \pi v}{v^2 - \frac{1}{4}}\, dv + h\left(\frac{1}{2}\right) \overline{h_1\left(\frac{1}{2}\right)},$$

$$\int_0^\infty \omega_{-1} h(y) \overline{\omega_{-1} h_1(y)} \frac{dy}{y^2} = \frac{1}{\pi i} \int_{\mathrm{Re}\, v=0} h(v) \overline{h_1(v)} \left(v^2 - \frac{1}{4}\right) v \sin \pi v\, dv$$

for $h, h_1 \in H(\tau)$.

Proof. We have $\omega_{\pm 1} h = f_h$ in the notation of Proposition 13.3.10 in [1], with $h \in F_{\pm 1,\tau}$. We quote results from [1]: Corollary 13.3.12 gives the estimate. The inversion formula is given in Proposition 13.3.13. For the scalar product formula see Propositions 13.3.5 and 13.3.10. □

Definition 5.2.3. We define $\mathscr{H}^d$ as the set of functions $h = \bigtimes_{j=1}^{d} h_j$ with all $h_j \in H(\tau)$.

If $h \in \mathscr{H}^d$, then also $\bar{h} \in \mathscr{H}^d$, where $\bar{h}\colon v \mapsto \overline{h(\bar{v})}$.

Comparison. This class of auxiliary test functions is smaller than in [5], Definition 9.1. In [5], we have derived the sum formula for as large a class of functions as the method allows. In Section 5.3, we shall use the same method, but we take it easy in the first step, and extend the class of test functions in Section 5.4. That extension step is at present not within our reach in the presence of complex places.

As in [5], Definition 9.2, we want to associate a function on G to each auxiliary test function $h \in \mathscr{H}^d$. A complication is that we cannot stick to one weight, as we want to take all irreducible components ϖ into account, see Section 2.2.

Definition 5.2.4. For each $h \in \mathscr{H}^d$ we define the function ${}^{\mathrm{e}}\mathrm{K}_{\kappa}^{r} h$ on G. For $n \in N^{\kappa}$, $y \in \mathbb{R}_{>0}^{d}$, and $\vartheta \in (\mathbb{R} \bmod 2\pi\mathbb{Z})^d$ we put

$$
{}^{\mathbf{e}}\mathsf{K}^r_\kappa h(ng_\kappa a[y]k[\vartheta]) := \chi_r(n)\prod_{j=1}^d\Bigg(\omega_{\mathbf{e}_j}h_j(4\pi|r_j|y_j)e^{2i\operatorname{sign}(r_j)\mathbf{e}_j\vartheta_j} + \left(\frac{1+\mathbf{e}_j}{2}\right)\sum_{b\geqq 4, b\in 2\mathbb{Z}}\frac{h_j\left(\frac{b-1}{2}\right)}{(b-2)!}W_{b/2,(b-1)/2}(4\pi|r_j|y_j)e^{ib\operatorname{sign}(r_j)\vartheta_j}\Bigg).
$$

We have used $\mathbf{e}$ in the definition of ${}^{\mathbf{e}}\mathsf{K}^r_\kappa$ to control which Whittaker transform is used in each of the factors.

Lemma 5.2.5. *Let* $h, h' \in \mathscr{H}^d$, $q \in (2\mathbb{Z})^d$. *We have*

$$
{}^{\mathbf{e}}\mathsf{K}^r_\kappa h(g) \ll \prod_{j=1}^d \min\left(a_\kappa(g)^{(1/2+\tau)\alpha_j}, a_\kappa(g)^{(1/2-\tau)\alpha_j}\right), \tag{82}
$$

$$
\int_{AK} a^{-2\rho}\,{}^{\mathbf{e}}\mathsf{K}^r_\kappa h(g_\kappa ak)\,\overline{{}^{\mathbf{e}}\mathsf{K}^r_\kappa h'(g_\kappa ak)}\,da\,dk \tag{83}
$$
$$
= 2^d N(\xi_r)\prod_{j=1}^d\left(\frac{1}{\pi i}\int_{\operatorname{Re}\nu=0}h_j(\nu)\overline{h_j'(\bar\nu)}\left(\nu^2-\frac14\right)^{-\mathbf{e}_j}\nu\sin\pi\nu\,d\nu + \left(\frac{1+\mathbf{e}_j}{2}\right)\sum_{b\geqq 2, b\in 2\mathbb{Z}}\frac{1}{(b-2)!}h_j\left(\frac{b-1}{2}\right)\overline{h_j'\left(\frac{b-1}{2}\right)}\right).
$$

Suppose that $\nu \in \mathbb{C}^d$ *satisfies* $|\operatorname{Re}\nu_j| < \tau$ *or* $q_j\operatorname{sign}(r_j) \geqq 2|\nu_j| + 1 \in 2\mathbb{Z}$. *Then*

$$
\int_{AK} a^{-2\rho}\,{}^{\mathbf{e}}\mathsf{K}^r_\kappa h(g_\kappa ak)\overline{\mathsf{W}^{r,\bar\nu}_{\kappa,q}(g_\kappa ak)}\,da\,dk \tag{84}
$$
$$
= \begin{cases} 2^d N(\xi_r)h(\nu) & \text{if for all } j \text{ either } q_j = 2\mathbf{e}_j\operatorname{sign}(r_j), \text{ or} \\ & \mathbf{e}_j = 1,\ q_j\operatorname{sign}(r_j) \geqq 4, \text{ and} \\ & \nu_j = \pm\frac12(q_j\operatorname{sign}(r_j)-1), \\ 0 & \text{otherwise.}\end{cases}
$$

Remark. The condition on q_j in (84) amounts to $\mathbf{e}_jq_j\operatorname{sign}(r_j) \geqq 2$, if $\mathbf{e}_j = 1$, and to $q_j = -2\operatorname{sign}(r_j)$, if $\mathbf{e}_j = -1$. The term for $b = 2$ in the sum in (83) corresponds to $h(1/2)W_{1,1/2}(y)$ in the definition of $\omega_1 h$ in Lemma 5.2.2.

Proof. For (82), we use the estimate in Lemma 5.2.2. The finitely many terms with

$$
W_{b/2,(b-1)/2}(\eta) = M_{b/2,(b-1)/2}(\eta) = \eta^{b/2}e^{-\eta/2}
$$

for $b \geqq 4$ in the factors cannot spoil the estimate.

The integral on the left in (83) converges absolutely, see (82). It can be computed factor by factor. In the computations the sums over b are present only if $\mathbf{e}_j = 1$.

$$\int_{\vartheta=0}^{2\pi}\int_{y=0}^{\infty}\frac{1}{y}\left(\omega_{e_j}h_j(2\xi_{r,j}y)e^{2ie_j\,\mathrm{sign}(r_j)\vartheta}+\sum_{b\geq 4}\frac{h_j\left(\frac{b-1}{2}\right)}{(b-2)!}W_{b/2,(b-1)/2}(2\xi_{r,j}y)e^{ib\,\mathrm{sign}(r_j)\vartheta}\right)$$

$$\cdot\left(\overline{\omega_{e_j}h_j'(2\xi_{r,j}y)}e^{-2ie_j\,\mathrm{sign}(r_j)\vartheta}+\sum_{b\geq 4}\frac{\overline{h_j'\left(\frac{b-1}{2}\right)}}{(b-2)!}W_{b/2,(b-1)/2}(2\xi_{r,j}y)e^{-ib\,\mathrm{sign}(r_j)\vartheta}\right)\frac{dy}{y}\frac{d\vartheta}{2\pi}$$

$$=\int_0^{\infty}2\xi_{r,j}\omega_{e_j}h_j(y)\overline{\omega_{e_j}h_j'(y)}\frac{dy}{y^2}+\sum_{b\geq 4}\frac{h_j\left(\frac{b-1}{2}\right)\overline{h_j'\left(\frac{b-1}{2}\right)}}{\left((b-1)!\right)^2}\int_0^{\infty}2\xi_{r,j}y^be^{-y}\frac{dy}{y^2}$$

$$=2\xi_{r,j}\left(\frac{1}{\pi i}\int_{\mathrm{Re}\,v=0}h_j(v)\overline{h_j'(v)}\left(v^2-\frac{1}{4}\right)^{-e_j}v\sin\pi v\,dv\right.$$

$$\left.+\left(\frac{1+e_j}{2}\right)h_j\left(\frac{1}{2}\right)\overline{h_j'\left(\frac{1}{2}\right)}+\sum_{b\geq 4}\frac{1}{(b-2)!}h_j\left(\frac{b-1}{2}\right)\overline{h_j'\left(\frac{b-1}{2}\right)}\right).$$

The integral in (84) can be computed as the product of local integrals as well. These converge absolutely, due to the estimate (82), and the asymptotic properties of the Whittaker function $W_{\kappa,v}$. (Note that $W_{\kappa,-v}=W_{\kappa,v}$, and, in particular, that if $v=\frac{b-1}{2}$ with $1<b\leq 2\kappa$, $b\equiv 2\kappa \bmod 2$, then $W_{\kappa,v}$ is a multiple of $M_{\kappa,v}$, which gives $W_{\kappa,\mu}(t)=O(t^{v+1/2})$ as $t\downarrow 0$, instead of $W_{\kappa,\mu}(t)=O(t^{|\mathrm{Re}\,v|+1/2-\varepsilon})$ in the general case.)

We integrate first over K and see that the integral vanishes unless the condition on the q_j is satisfied. The integral over A follows from Lemma 5.2.2 if $|q_j|=2$. Otherwise, $q_j=b\,\mathrm{sign}(r_j)$, with $b\in 2\mathbb{Z}$, $b\geq 4$, and we find the integral

$$2\xi_{r,j}\int_0^{\infty}y^{b/2-2}e^{-y/2}W_{b/2,v_j}(y)\,dy=\begin{cases}2\xi_{r,j}(b-2)! & \text{if } v_j=\pm\frac{1}{2}(b-1),\\ 0 & \text{otherwise.}\end{cases}$$

(Compare Propositions 13.3.5, 13.3.10 and 13.3.13 in [1].) □

Intertwining operators. In the Kloosterman term in the sum formula we shall need to evaluate the integral $\int_N\overline{\chi_r(n)}\,{}^{e}\mathrm{K}^{r}_{\kappa'}h(g_{\kappa'}a[t]s_0ng)\,dn$ for $t\in\mathbb{R}^d_{>0}$, $g\in G$. Remember that $s_0=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. This integral is the product of integrals

$$\int_{-\infty}^{\infty}e^{-2\pi ir_jx}({}^{e}\mathrm{K}^{r'}_{\kappa'}h)_j\left(g_{\kappa'}\begin{pmatrix}\sqrt{t}&0\\0&1/\sqrt{t}\end{pmatrix}s_0\begin{pmatrix}1&x\\0&1\end{pmatrix}g\right)\frac{dx}{\pi},\tag{85}$$

with $t>0$ and $g\in\mathrm{SL}_2(\mathbb{R})$. We have identified $g_{\kappa'}$ with its image $g_{\kappa'}^{\sigma_j}$ in $\mathrm{SL}_2(\mathbb{R})$.

We shall use results from [1]. We first express $({}^{\tilde e}\mathrm{K}^{\tilde r}_{\tilde\kappa}h)_j(g)$, with $\tilde\kappa\in\mathscr{P}$, $\tilde r\in \mathfrak{t}'_{\tilde\kappa}\setminus\{0\}$, $\tilde e\in\{1,-1\}^d$, in terms of a function $\Phi\xi$ as defined in Proposition 13.5.5 of [1].

The even function $({}^{\mathrm{e}}\mathrm{K}_{\tilde\kappa}^{\tilde r}h)_j$ on $\mathrm{SL}_2(\mathbb{R})$ can be considered as a function on the universal covering group G_0 of $\mathrm{SL}_2(\mathbb{R})$ that is invariant under translations by elements of the center of G_0. But [1] works with functions on the universal covering group of $\mathrm{PSL}_2(\mathbb{R})$, which has two components. We pay now for that generality by having a harder job in digging out the desired results.

Three number references in the sequel refer to [1]. There, the component of 1 in the covering group is called G_0. It is the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, and of $\mathrm{PSL}_2(\mathbb{R})$. The other component is $G_0 j$, where j lies above $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In G_0 are elements $n(x)$ above $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $a(y)$ above $\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$ and $k(\vartheta)$ above $\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$; see (2.2.1). Another convention is $j_1 = 1$, $j_{-1} = j$.

In these notations, we have:

$$\begin{aligned}
&({}^{\mathrm{e}}\mathrm{K}_{\tilde\kappa}^{\tilde r}h)_j(g_{\tilde\kappa}n(x)a(y)k(\vartheta)) \\
&\quad = e^{2\pi i\tilde r_j x}\Bigg(\frac{1}{\pi i}\int_{\mathrm{Re}\,\nu=0} h_\nu(\nu)\,W_{\tilde{\mathrm{e}}_j,\nu}(4\pi|\tilde r_j|y)\,\nu\sin\pi\nu\left(\nu^2-\frac14\right)^{-\tilde{\mathrm{e}}_j} d\nu\, e^{2i\tilde{\mathrm{e}}_j\,\mathrm{sign}(\tilde r_j)\vartheta} \\
&\qquad + \frac{1+\mathrm{e}_j}{2}\sum_{b\geqq 2,\, b\in 2\mathbb{Z}}\frac{1}{(b-2)!}h_j\left(\frac{b-1}{2}\right)W_{b/2,(b-1)/2}(4\pi|\tilde r_j|y)\,e^{ib\tilde{\mathrm{e}}_j\,\mathrm{sign}(\tilde r_j)\vartheta}\Bigg).
\end{aligned}$$

We construct an element $\xi \in H_\tau^0$ as in Definition 13.5.1. Note that $\sigma_{\text{there}} = \tau_{\text{here}}$, and $\tau_{\text{there}} = 0$ for even weights.

$$\xi(s) := \frac12\Gamma\left(\frac12+s-\tilde{\mathrm{e}}_j\right)h_j(s)\left(\varphi^{1}_{2\tilde{\mathrm{e}}_j\,\mathrm{sign}(\tilde r_j)}(s)+\mathrm{sign}(\tilde r_j)\varphi^{-1}_{2\tilde{\mathrm{e}}_j\,\mathrm{sign}(\tilde r_j)}(s)\right),$$

$$\xi_b := \begin{cases} \sqrt{\dfrac{b-1}{2}}\,h_j\left(\dfrac{b-1}{2}\right)\varphi_{b\,\mathrm{sign}(\tilde r_j)}[b] & \text{if } \tilde{\mathrm{e}}_j = 1, \\ 0 & \text{if } \tilde{\mathrm{e}}_j = -1. \end{cases}$$

(Use (3.5.6) to check condition (13.5.2).)

Proposition 13.5.5 defines a function Φ on $G_0 \cup G_0 j$ in the following way:

$$\begin{aligned}
&\Phi\xi\left(j_{\pm\,\mathrm{sign}(\tilde r_j)}a(2\xi_{\tilde r,j})n(x)a(y)k(\vartheta)\right) \\
&\quad = \Phi\xi\left(n(\pm 4\pi\tilde r_j x)a(4\pi|\tilde r_j|y)j_{\pm\,\mathrm{sign}(\tilde r_j)}k(\vartheta)\right) \\
&\quad = \frac{1}{4\pi i}\int_{\mathrm{Re}\,s=0}\frac{-2s\sin 2\pi s}{\pi}\cdot\frac12\Gamma\left(\frac12+s-\tilde{\mathrm{e}}_j\right)h_j(s)e^{\pm 2\pi i\tilde r_j x}(1\pm 1) \\
&\qquad \cdot\Gamma\left(\frac12-s\mp\tilde{\mathrm{e}}_j\right)W_{\pm\tilde{\mathrm{e}}_j,s}(4\pi|\tilde r_j|y)e^{2i\tilde{\mathrm{e}}_j\,\mathrm{sign}(\tilde r_j)\vartheta}\,ds \\
&\qquad + \left(\frac{1+\tilde{\mathrm{e}}_j}{2}\right)\sum_{b\geqq 2,\, b\in 2\mathbb{Z}}\sqrt{\frac{b-1}{2}}\,\frac{h_j\left(\dfrac{b-1}{2}\right)}{\sqrt{b-1}\,(b-2)!} \\
&\qquad \cdot\frac{1\pm 1}{2}\sqrt{2}\,e^{2\pi i\tilde r_j x}W_{b/2,(b-1)/2}(4\pi|\tilde r_j|y)e^{ib\,\mathrm{sign}(\tilde r_j)\vartheta}.
\end{aligned}$$

We have used Definition 5.5.1, Definition 5.4.2, and Definition 5.4.4. We conclude that $\Phi\xi$ vanishes on $j_{-\mathrm{sign}(\tilde r_j)}G_0$, and satisfies

$$\Phi\xi\big(j_{\mathrm{sign}(\tilde r_j)}a(2\xi_{\tilde r,j})g\big) = ({}^{\mathbf{e}}\mathrm{K}^{\tilde r}_{\tilde\kappa}h)_j(g_{\tilde\kappa}g) \tag{86}$$

for $g \in G_0$.

Now we take $r \in \mathfrak{t}'_\kappa\setminus\{0\}$, $r' \in \mathfrak{t}'_{\kappa'}\setminus\{0\}$, and put $\mathbf{e}_j = \mathrm{sign}(r_jr'_j)$. We work with even functions, so we can replace s_0^{-1} in the argument by $s_0^{-\mathrm{sign}(r')}$. We see that the integral (85) is equal to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-2\pi i r_j x}\Phi\xi\big(j_{\mathrm{sign}(r'_j)}a(2\xi_{r',j})a(t)s_0^{-\mathrm{sign}(r'_j)}n(x)g\big)\,dx$$

$$= \frac{1}{4\pi^2|r_j|}\cdot\int_{-\infty}^{\infty} e^{-\frac{1}{2}ix}\Phi\xi\big(s_0^{-1}j_{\mathbf{e}_j}a(4\xi_{r,j}\xi_{r',j}t)^{-1}n(x)j_{\mathrm{sign}(r_j)}a(2\xi_{r,j})g\big)\,dx$$

$$= \frac{1}{4\pi^2|r_j|^{1/2}}V^{\mathbf{e}_j}(4\xi_{r,j}\xi_{r',j}t)\Phi\xi\big(j_{\mathrm{sign}(r_j)}a(2\xi_{r,j})g\big).$$

See Definition 6.2.2 in [1], for the meaning of $V^{\mathbf{e}_j}$. In Corollary 14.3.3, and (14.3.12), (14.3.5), (14.2.11), (14.3.13), and (23), we see that (85) is equal to $\Phi\eta\big(j_{\mathrm{sign}(r_j)}a(2\xi_{r,j})g\big)$, where $\eta \in H^0_\tau$ is given by

$$\eta(s) = \frac{1}{2}\Gamma\Big(\frac{1}{2}+s-\mathbf{e}_j\Big)h_j(s)\mathscr{B}_{\mathbf{e}_j,j}\Big(\xi_{r,j}\xi_{r',j}\frac{t}{4\pi^2},s\Big)\sqrt{t|r'_j/r_j|}$$
$$\cdot\big(\varphi^1_{2\mathbf{e}_j\,\mathrm{sign}(r'_j)}(s) + \mathbf{e}_j\,\mathrm{sign}(r'_j)\varphi^{-1}_{2\mathbf{e}_j\,\mathrm{sign}(r'_j)}\big),$$

$$\eta_b = \begin{cases} \sqrt{\frac{b-1}{2}}h_j\Big(\frac{b-1}{2}\Big)\sqrt{t|r'_j/r_j|} & \\ \quad\cdot\mathscr{B}_{\mathbf{e}_j,j}\Big(\xi_{r,j}\xi_{r',j}\frac{t}{4\pi^2},\frac{b-1}{2}\Big)\varphi_{b\,\mathrm{sign}(r'_j)}[b] & \text{if } \mathbf{e}_j = 1,\\ 0 & \text{if } \mathbf{e}_j = -1.\end{cases}$$

We look for $h_t = \times_j h_{t,j}$ such that $\Phi\eta\big(j_{\mathrm{sign}(r_j)}a(2\xi_{r,j})g\big) = ({}^{\mathbf{p}}\mathrm{K}^r_\kappa h_t)_j(g_\kappa g)$, where $\mathbf{p} = (1,1,\ldots,1)$. According to (86), we have

$$({}^{\mathbf{p}}\mathrm{K}^r_\kappa h_t)_j(g_\kappa g) = \Phi\tilde\eta\big(j_{\mathrm{sign}(r_j)}a(2\xi_{r,j})g\big),$$

with

$$\tilde\eta(s) = \frac{1}{2}\Gamma\Big(-\frac{1}{2}+s\Big)h_{t,j}(s)\big(\varphi^1_{2\,\mathrm{sign}(r_j)}(s) + \mathrm{sign}(r_j)\varphi^{-1}_{2\,\mathrm{sign}(r_j)}(s)\big),$$

$$\tilde\eta_b = \sqrt{\frac{b-1}{2}}h_{t,j}\Big(\frac{b-1}{2}\Big)\varphi_{b\,\mathrm{sign}(r_j)}[b].$$

We take

$$h_{t,j}(s) = \Gamma\left(\frac{1}{2}+s-\mathbf{e}_j\right)\Gamma\left(-\frac{1}{2}+s\right)^{-1}\sqrt{t\left|\frac{r_j'}{r_j}\right|}\mathscr{B}_{\mathbf{e}_j,j}(|r_jr_j'|t,s)h_j(s),$$

and obtain $\tilde{\eta} = \eta$. (We use that $\mathscr{B}_{-1,j}\left(\cdot,\frac{b-1}{2}\right) = 0$ for $b \geqq 2$, and $\eta_b = 0$ if $\mathbf{e}_j = -1$.) We check that $h_{t,j}$ has the right properties to conclude that $h_t \in \mathscr{H}^d$. This gives the following result:

Proposition 5.2.6. *Let* $\mathbf{e} \in \{1,-1\}^d$ *be given by* $\mathbf{e}_j = \mathrm{sign}(r_jr_j')$. *For* $h \in \mathscr{H}^d$ *and* $t \in \mathbb{R}^d_{>0}$, *we define* h_t *by*

$$h_t(v) := N(\xi_{r'})^{1/2}N(\xi_r)^{-1/2}N(t)^{1/2}\mathscr{B}_{\mathbf{e}}(T,v)h(v)\prod_{j,\mathbf{e}_j=-1}\left(v_j^2-\frac{1}{4}\right),$$

with $T \in \mathbb{R}^d_{>0}$, $T_j = |r_jr_j'|t_j$, *and* $\xi_r = (2\pi|r_1|,\ldots,2\pi|r_d|)$. *Then* $h_t \in \mathscr{H}^d$, *and*

$$\int_N \overline{\chi_r(n)}\,{}^{\mathbf{e}}\mathsf{K}_{\kappa'}^{r'}h(g_{\kappa'}a[t]s_0ng_\kappa^{-1}g)\,dn = {}^{\mathbf{p}}\mathsf{K}_\kappa^r h_t(g).$$

5.3. Sum formula for a restricted class of test functions. We consider $\kappa,\kappa' \in \mathscr{P}$, $r \in \mathfrak{t}_\kappa'\backslash\{0\}$, $r' \in \mathfrak{t}_{\kappa'}'\backslash\{0\}$, and $\mathbf{e},\mathbf{p} \in \{1,-1\}^d$ given by $\mathbf{e}_j = \mathrm{sign}(r_jr_j')$, $\mathbf{p}_j = 1$. We take $h, h' \in \mathscr{H}^d$.

Poincaré series. For $\tilde{\kappa} \in \mathscr{P}$, $\tilde{r} \in \mathfrak{t}_{\tilde{\kappa}}'\backslash\{0\}$, $\tilde{\mathbf{e}} \in \{1,-1\}^d$, and $\tilde{h} \in \mathscr{H}^d$, we define a Poincaré series ${}^{\tilde{\mathbf{e}}}P_{\tilde{\kappa}}^{\tilde{r}}\tilde{h}(g) := \sum_{\gamma\in\Gamma_{N\tilde{\kappa}}\backslash\Gamma} {}^{\tilde{\mathbf{e}}}\mathsf{K}_{\tilde{\kappa}}^{\tilde{r}}\tilde{h}(\gamma g)$. The absolute convergence is proved in the same way as in [5], Section 10. The idea is to write ${}^{\tilde{\mathbf{e}}}P_{\tilde{\kappa}}^{\tilde{r}}\tilde{h}(g) = \sum_{\gamma\in\Gamma_{P\tilde{\kappa}}\backslash\Gamma}\ \sum_{\delta\in\Gamma_{N\tilde{\kappa}}\backslash\Gamma_{P\tilde{\kappa}}} {}^{\tilde{\mathbf{e}}}\mathsf{K}_{\tilde{\kappa}}^{\tilde{r}}(\delta\gamma g)$, and to use (82) and Lemma 5.1.1 to estimate $\sum_{\delta\in\Gamma_{N\tilde{\kappa}}\backslash\Gamma_{P\tilde{\kappa}}} |{}^{\tilde{\mathbf{e}}}\mathsf{K}_{\tilde{\kappa}}^{\tilde{r}}\tilde{h}(\delta\gamma g)|$ by means of a function

$$F(g) = \min\left(a(g_{\tilde{\kappa}}^{-1}g)^{(1+2\tau-\varepsilon)\rho}, a(g_{\tilde{\kappa}}^{-1}g)^{(1+2\tau+\varepsilon)\rho}\right).$$

The resulting sum over $\gamma \in \Gamma_{P\tilde{\kappa}}\backslash\Gamma$ is compared with an absolutely convergent Eisenstein series. The function ${}^{\tilde{\mathbf{e}}}P_{\tilde{\kappa}}^{\tilde{r}}$ is bounded and Γ-invariant on the left.

The convergence of these Poincaré series would be difficult if we had defined ${}^{\mathbf{e}}\mathsf{K}_\kappa^r$ in a less complicated way in Definition 5.2.4. If we use weight zero in the term with $\omega_{\pm1}h_j$, then the term with $b = 2$ would cause convergence problems.

Throughout the rest of this section, we use two different methods to compute the scalar product $\langle{}^{\mathbf{p}}P_\kappa^r h, {}^{\mathbf{e}}P_{\kappa'}^{r'}h'\rangle$. The resulting equality is the sum formula.

Fourier coefficients. To find the spectral decomposition of ${}^{\tilde{\mathbf{e}}}P_{\tilde{\kappa}}^{\tilde{r}}\tilde{h}$, we need to compute $\langle{}^{\tilde{\mathbf{e}}}P_{\tilde{\kappa}}^{\tilde{r}}\tilde{h}, f\rangle$, where f is a square integrable automorphic form in the basis $\mathscr{M}_q$, or an Eisenstein series with purely imaginary values of the spectral parameter. The weights q should be those that occur in ${}^{\tilde{\mathbf{e}}}P_{\tilde{\kappa}}^{\tilde{r}}\tilde{h}$. That is, the weights that satisfy $q_j = 2\tilde{\mathbf{e}}_j\,\mathrm{sign}(\tilde{r}_j)$, or

$\mathrm{sign}(\tilde r - j)q_j \geqq 4$ and $\tilde{\mathrm{e}}_j = 1$. We denote the spectral parameter of f by ν. Then $|\mathrm{Re}\,\nu_j| < \frac{1}{2}$ or $\nu_j \in \left\{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{|q_j|-1}{2}\right\}$. Note that $\bar\nu_j = \pm\nu_j$.

The following integral converges absolutely:

$$\langle {}^{\tilde{\mathrm{e}}}P^{\tilde r}_{\tilde\kappa}\tilde h, f\rangle = \int_{\Gamma_{N^\kappa}\backslash G} {}^{\tilde{\mathrm{e}}}\mathrm{K}^{\tilde r}_{\tilde\kappa}\tilde h(g)\overline{f(g)}\,dg$$

$$= \mathrm{vol}(\Delta_{\tilde\kappa}\backslash N^{\tilde\kappa})\overline{a_{\tilde\kappa}(\tilde r, f)} \int_{AK} {}^{\mathrm{e}}\mathrm{K}^{\tilde r}_{\tilde\kappa}\tilde h(g_{\tilde\kappa}ak)\overline{\mathrm{W}^{\tilde r,\bar\nu}_{\tilde\kappa,q}(g_{\tilde\kappa}ak)}a^{-2\rho}\,da\,dk.$$

We apply (84) to this relation. From Proposition 2.2.3 it follows that if $|\mathrm{Re}\,\nu_j| \geqq \frac{1}{2}$, then $a_{\tilde\kappa}(\tilde r, f) \neq 0$ implies that $q_j\,\mathrm{sign}(r_j) \geqq 2|\nu_j| + 1$. So if the Fourier coefficient is non-zero, we find that $\langle {}^{\mathrm{e}}P^{\tilde r}_{\tilde\kappa}\tilde h, f\rangle$ is equal to

$$2^d\,\mathrm{vol}(\Delta_{\tilde\kappa}\backslash N^{\tilde\kappa})\overline{a_{\tilde\kappa}(\tilde r, f)}N(\xi_{\tilde r})\tilde h(\nu)$$

if for all j, either $q_j = 2\tilde{\mathrm{e}}_j\,\mathrm{sign}(\tilde r_j)$, or $\tilde{\mathrm{e}}_j = 1$ and $q_j\,\mathrm{sign}(\tilde r_j) = 2\nu_j + 1 \geqq 4$, and equal to 0 otherwise.

In particular, if f is an Eisenstein series with purely imaginary spectral parameter $i\nu + i\mu$, then the scalar product is zero if $|q_j| \geqq 4$ for some j.

Spectral decomposition. Define the weight $q(r)$ by $q(r)_j = 2\,\mathrm{sign}(r_j)$. For the spectral decomposition of ${}^{\mathrm{p}}P^r_\kappa h$ we need the Eisenstein integral for weight $q(r)$, and the orthonormal elements $\psi_{\varpi, q(r,\varpi)}$ with $q(r,\varpi)$ determined in the following way:

ϖ_j	$q(r,\varpi)_j$
$H(\nu_j)$ with $\frac{1}{4} - \nu_j^2 \geqq 0$	$2\,\mathrm{sign}(r_j)$
$D_b^{\pm}$ with $b \geqq 2$, $b \in 2\mathbb{Z}$, and $\pm r_j > 0$	$b\,\mathrm{sign}(r_j)$

The other ϖ do not contribute to the expansion of ${}^{\mathrm{p}}P^r_\kappa h$. For ${}^{\mathrm{e}}P^{r'}_{\kappa'}h'$, we need the Eisenstein integral in weight $q(r)$, and a subset of the ϖ given above. (In fact, if $\mathrm{e}_j = -1$, then we need only $\varpi_j \cong H(\nu_j)$.)

We obtain the following equality:

$$\langle {}^{\mathrm{p}}P^r_\kappa h, {}^{\mathrm{e}}P^{r'}_{\kappa'}h'\rangle = 4^d\,\mathrm{vol}(\Gamma_{N^\kappa}\backslash N^\kappa)\,\mathrm{vol}(\Gamma_{N^{\kappa'}}\backslash N^{\kappa'})N(\xi_r)N(\xi_{r'})$$

$$\cdot\Bigg(\sum_{\varpi}\|\psi_{\varpi,q(r,\varpi)}\|^{-2}\overline{a_\kappa(r,\psi_{\varpi,q(r,\varpi)})}a_{\kappa'}(r',\psi_{\varpi,q(r,\varpi)})h(\nu_\varpi)\overline{h'(\nu_\varpi)}$$

$$+\sum_{\lambda\in\mathscr{P}} c_\lambda \sum_{\mu}\int_{-\infty}^{\infty}\overline{a_\kappa(r, E_{q(r')}(P^\lambda, iy, i\mu))}$$

$$\cdot\, a_{\kappa'}(r', E_{q(r')}(P^\lambda, iy, i\mu))h(iy+i\mu)\overline{h'(iy+i\mu)}\,dy\Bigg).$$

A computation based on (15) and (16) shows that for all ϖ that occur in the sum

$$\begin{aligned}
&\|\psi_{\varpi,q(r,\varpi)}\|^{-2}\overline{a_\kappa(r,\psi_{\varpi,q(r,\varpi)})}a_{\kappa'}(r',\psi_{\varpi,q(r,\varpi)})\\
&\quad\cdot \operatorname{vol}(\Gamma_{N^\kappa}\backslash N^\kappa)\operatorname{vol}(\Gamma_{N^{\kappa'}}\backslash N^{\kappa'})N(\xi_r)^{1/2}N(\xi_{r'})^{1/2}\\
&\quad= \overline{c^r_\kappa(\varpi)}c^{r'}_{\kappa'}(\varpi)\prod_{j=1}^d \Gamma\left(\frac{1}{2}-\nu_{\varpi,j}+\frac{1}{2}q(r)_j\operatorname{sign}(r_j)\right)^{-1}\\
&\quad\cdot \prod_{j=1}^d \Gamma\left(\frac{1}{2}+\nu_{\varpi,j}+\frac{1}{2}q(r)_j\operatorname{sign}(r'_j)\right)^{-1}.
\end{aligned}$$

Checking the cases $|\mathrm{Re}\,\nu_{\varpi,j}| \leqq \tau$ and $\nu_{\varpi,j} = \dfrac{b-1}{2} > \tau$ separately, we see that this quantity is equal to $\overline{c^r_\kappa(\varpi)}c^{r'}_{\kappa'}(\varpi)\varphi^{\mathrm{e}}(\nu_\varpi)$, where $\varphi^{\mathrm{e}} := \bigtimes_{j=1}^{d} \varphi^{\mathrm{e}}_j$ is given by

$$(87)\qquad \varphi^{\mathrm{e}}_j(\nu) := \begin{cases} \dfrac{1}{\pi}\cos\pi\nu\left(\dfrac{1}{4}-\nu 2\right)^{-1} & \text{if } |\mathrm{Re}\,\nu| \leqq \tau, \mathrm{e}_j = 1,\\ -\dfrac{1}{\pi}\cos\pi\nu & \text{if } |\mathrm{Re}\,\nu| \leqq \tau, \mathrm{e}_j = -1,\\ \dfrac{1}{(b-1)!} & \text{if } \nu = \dfrac{b-1}{2}, b\in 2\mathbb{Z}, b \geqq 2, \mathrm{e}_j = 1,\\ 0 & \text{if } \nu \in \dfrac{1}{2}+\mathbb{Z}, \mathrm{e}_j = -1. \end{cases}$$

We treat the Fourier coefficients of the Eisenstein series in a similar way, and obtain the following expression for $\langle {}^{\mathrm{p}}P^r_\kappa h, {}^{\mathrm{e}}P^{r'}_{\kappa'}h'\rangle$:

$$\begin{aligned}
&4^d N(\xi_r)^{1/2}N(\xi_{r'})^{1/2}\Bigg(\sum_\varpi h(\nu_\varpi)\overline{h'(\overline{\nu_\varpi})}\varphi^{\mathrm{e}}(\nu_\varpi)\overline{c^r_\kappa(\varpi)}c^{r'}_{\kappa'}(\varpi)\\
&\quad+\sum_{\lambda\in\mathscr{P}} c_\lambda \sum_\mu \int_{-\infty}^{\infty} h(iy+i\mu)\overline{h'(-iy-i\mu)}\\
&\quad\cdot\varphi^{\mathrm{e}}(iy+i\mu)\overline{D^{\kappa,r}_\lambda(iy,i\mu)}D^{\kappa',r'}_\lambda(iy,i\mu)\,dy\Bigg)\\
&= 4^d N(\xi_r)^{1/2}N(\xi_{r'})^{1/2}\int_Y h(\nu)\overline{h'}(\nu)\varphi^{\mathrm{e}}(\nu)\,d\sigma^{\kappa,\kappa'}_{r,r'}(\nu),
\end{aligned}$$

see Definition 2.2.4. In particular, $h\overline{h'}\varphi^{\mathrm{e}}$ is integrable for the measure $d\sigma^{\kappa,\kappa'}_{r,r'}$.

For other ϖ than indicated above, at least one of the Fourier coefficients is zero. So we may as well let ϖ run over a total orthogonal system of irreducible components of $L^2_d(\Gamma\backslash G)$, except $\varpi = \mathbf{1}$ (constant functions).

Geometric computation of the scalar product. The absolute convergence and square integrability of the Poincaré series gives the following absolutely convergent expression:

$$\langle {}^{\mathrm{p}}P^r_\kappa h, {}^{\mathrm{e}}P^{r'}_{\kappa'} h'\rangle = \int_{\Gamma_{N^\kappa}\backslash G} {}^{\mathrm{p}}\mathrm{K}^r_\kappa h(g)\, \overline{{}^{\mathrm{e}}P^{r'}_{\kappa'} h'(g)}\, dg$$
$$= \int_K\int_A \int_{\Gamma_{N^\kappa}\backslash N^\kappa} \chi_r(n)\, {}^{\mathrm{p}}\mathrm{K}^r_\kappa h(g_\kappa ak)\, \overline{{}^{\mathrm{e}}P^{r'}_{\kappa'}(ng_\kappa ak)}\, dn a^{-2\rho}\, da\, dk$$
$$= \sum_{\gamma\in\Gamma_{N^{\kappa'}}\backslash\Gamma} I(\gamma),$$

where

$$I(\gamma) := \int_K\int_A a^{-2\rho}\, {}^{\mathrm{p}}\mathrm{K}^r_\kappa h(g_\kappa ak) \int_{\Gamma_{N^\kappa}\backslash N^\kappa} \chi_r(n)\, \overline{{}^{\mathrm{e}}\mathrm{K}^{r'}_{\kappa'} h'(\gamma n g_\kappa ak)}\, dn\, da\, dk.$$

We split this up as $I_1 + I_2$, $I_1 := \sum_{\gamma\in\Gamma_{N^{\kappa'}}\backslash(\Gamma\cap g_{\kappa'}Pg_\kappa^{-1})} I(\gamma)$, and $I_2 := \sum_{\gamma\in\Gamma_{N^{\kappa'}}\backslash{}^{\kappa'}\Gamma^\kappa} I(\gamma)$.

Delta term. We first consider I_1. As $\mathscr{P}$ consists of inequivalent cusps for Γ, this term is non-empty only if $\kappa = \kappa'$. We write $\gamma = n_\gamma a_\gamma$, with $n_\gamma = n_\kappa(\gamma g_\kappa)$, $a_\gamma = g_\kappa a_\kappa(\gamma g_\kappa) g_\kappa^{-1}$. Then $\gamma n g_\kappa ak = n_\gamma \cdot a_\gamma n a_\gamma^{-1} \cdot a_\gamma g_\kappa ak$. Hence

$$I(\gamma) = \overline{\chi_{r'}(n_\gamma)} \int_K\int_A a^{-2\rho}\, {}^{\mathrm{p}}\mathrm{K}^r_\kappa h(g_\kappa ak)\, \overline{{}^{\mathrm{e}}\mathrm{K}^{r'}_\kappa h'(a_\gamma g_\kappa ak)} \int_{\Gamma_{N^\kappa}\backslash N^\kappa} \chi_r(n)\overline{\chi_{r'}(a_\gamma n a_\gamma^{-1})}\, dn\, da\, dk.$$

The integral over $\Gamma_{N^\kappa}\backslash N^\kappa$ is non-zero if and only if $\chi_{r'}(a_\gamma n a_\gamma^{-1}) = \chi_r(n)$ for all $n\in N^\kappa$. This is precisely the condition in Definition 2.6.1. Under this condition, ${}^{\mathrm{e}}\mathrm{K}^{r'}_\kappa h'(a_\gamma g_\kappa ak) = {}^{\mathrm{e}}\mathrm{K}^{r'}_\kappa h'(g_\kappa a_\kappa(\gamma g_\kappa)ak) = {}^{\mathrm{e}}\mathrm{K}^r_\kappa h'(g_\kappa ak)$, $N(\xi'_r) = N(\xi_r)$, and $\mathrm{e} = \mathrm{p}$. We have used that $a_\kappa(\gamma g_\kappa) = a[\eta]$ with $N(\eta) = 1$ and $\eta_j r'_j = r_j$. For such γ we obtain from (83) the following equality:

$$I(\gamma) = \overline{\chi_{r'}(n_\kappa(\gamma g_\kappa))}\, \mathrm{vol}(\Gamma_{N^\kappa}\backslash N^\kappa) \int_K\int_A a^{-2\rho}\, {}^{\mathrm{p}}\mathrm{K}^r_\kappa h(g_\kappa ak)\, \overline{{}^{\mathrm{p}}\mathrm{K}^r_\kappa h'(g_\kappa ak)}\, da\, dk$$
$$= \overline{\chi_{r'}(n_\kappa(\gamma g_\kappa))}\, 2^d\, \mathrm{vol}(\Gamma_{N^\kappa}\backslash N^\kappa) N(\xi_r)$$
$$\cdot \prod_{j=1}^d \left(\frac{1}{\pi i}\int_{\mathrm{Re}\, \nu=0} h_j(\nu)\overline{h'_j(\nu)}\, \frac{\nu \sin \pi\nu}{\nu^2 - \frac{1}{4}}\, d\nu + \sum_{b\geqq 2,\, b\in 2\mathbb{Z}} \frac{1}{(b-2)!}\, h_j\!\left(\frac{b-1}{2}\right) \overline{h'_j\!\left(\frac{b-1}{2}\right)} \right).$$

Thus we find that $I_1 = 4^d N(\xi_r)^{1/2} N(\xi_{r'})^{1/2} \Delta^{\kappa,\kappa'}_{r,r'}(h\overline{h'}\varphi^{\mathrm{e}})$.

Contribution of the big cell. We have $I_2 = \sum_{\gamma\in{}^{\kappa'}\mathscr{S}^\kappa} \sum_{\delta\in\Gamma_{N^\kappa}} I(\gamma\delta)$, see Definition 2.3.1 and Proposition 2.3.2. Let $c\in{}^{\kappa'}\mathscr{C}^\kappa$, $\gamma\in{}^{\kappa'}\mathscr{S}^\kappa(c)$.

$$\sum_{\delta\in\Gamma_{N^\kappa}} I(\gamma\delta) = \int_K\int_A a^{-2\rho}\, {}^{\mathrm{p}}\mathrm{K}^r_\kappa h(g_\kappa ak) \int_{N^\kappa} \chi_r(n)\, \overline{{}^{\mathrm{e}}\mathrm{K}^{r'}_{\kappa'} h'(\gamma n g_\kappa ak)}\, dn\, da\, dk.$$

Write $\xi = g_{\kappa'}^{-1}\gamma g_\kappa = \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix}$. Then $\gamma = g_{\kappa'} n'(\xi) m(\xi) a_\xi s_0 n''(\xi) g_\kappa^{-1}$. For $g\in G$ we find

$$\int_{N^\kappa} \overline{\chi_r(n)}\, {}^{\mathbf{e}}\mathrm{K}^{r'}_{\kappa'} h'(\gamma n g_\kappa g)\, dn = \int_N \overline{\chi_r(n)}\, {}^{\mathbf{e}}\mathrm{K}^{r'}_{\kappa'} h'\big(g_{\kappa'} n'(\xi) a_\xi s_0 n''(\xi) n g\big)\, dn$$
$$= \chi_{r'}\big(n'(\xi)\big)\chi_r\big(n''(\xi)\big)\, {}^{\mathbf{p}}\mathrm{K}^r_\kappa h'_{1/c^2}(g_\kappa g),$$

see Proposition 5.2.6. Let $T_j = |r_j r'_j|(c^{\sigma_j})^{-2}$. We apply Equation (83) with $\mathbf{e} = \mathbf{p}$:

$$\sum_{\delta \in \Gamma_{N^\kappa}} I(\gamma\delta) = \overline{\chi_{r'}\big(n'(\xi)\big)\chi_r\big(n''(\xi)\big)} \int_K \int_A a^{-2\rho}\, {}^{\mathbf{p}}\mathrm{K}^r_\kappa h(g_\kappa a k)\, \overline{{}^{\mathbf{p}}\mathrm{K}^r_\kappa h'_{1/c^2}(g_\kappa a k)}\, da\, dk$$
$$= \overline{\chi_{r'}\big(n'(\xi)\big)\chi_r\big(n''(\xi)\big)} 2^d N(\xi_r) \cdot N(\xi_{r'})^{1/2} N(\xi_r)^{-1/2} |N(c)|^{-1}$$
$$\times \prod_{j=1}^d \Bigg(\frac{1}{\pi i} \int_{\mathrm{Re}\, \nu = 0} h_j(\nu) \mathscr{B}_{\mathbf{e}_j, j}(T_j, \nu) \overline{h'_j(\nu)} \frac{\nu \sin \pi\nu}{\left(\nu^2 - \frac{1}{4}\right)^{(\mathbf{e}_j+1)/2}}\, d\nu$$
$$+ \left(\frac{1+\mathbf{e}_j}{2}\right) \sum_{b \geqq 2, b \in 2\mathbb{Z}} \frac{1}{(b-2)!} h_j\left(\frac{b-1}{2}\right) \mathscr{B}_{\mathbf{e}_j, j}\left(T_j, \frac{b-1}{2}\right) \overline{h'_j\left(\frac{b-1}{2}\right)} \Bigg)$$
$$= \overline{\chi_{r'}\big(n'(\xi)\big)\chi_r\big(n''(\xi)\big)} 4^d N(\xi_r)^{1/2} N(\xi_{r'})^{1/2} |N(c)|^{-1}$$
$$\cdot \int_{Y^{\mathbf{e}}} h(\nu) \overline{h'}(\nu) \varphi^{\mathbf{e}}(\nu) \mathscr{B}_{\mathbf{e}}(T, \nu)\, d\eta^{\mathbf{e}}(\nu)$$
$$= \overline{\chi_{r'}\big(n'(\xi)\big)\chi_r\big(n''(\xi)\big)} 4^d N(\xi_r)^{1/2} N(\xi_{r'})^{1/2} |N(c)|^{-1} \mathsf{B}_{\mathbf{e}}(h\overline{h'}\varphi^{\mathbf{e}})(T).$$

Thus, we have obtained the following expressions for I_2:

$$4^d \sqrt{N(\xi_r) N(\xi_{r'})} \sum_{c \in {}^{\kappa'}\mathscr{C}^\kappa} |N(c)|^{-1} \mathsf{B}_{\mathbf{e}}(h\overline{h'}\varphi^{\mathbf{e}}) \left(\left| \frac{rr'}{c^2} \right| \right) S(\kappa, -r; \kappa', -r'; c)$$
$$= 4^d N(\xi_r)^{1/2} N(\xi_{r'})^{1/2} K^{\kappa,\kappa'}_{-r,-r'}\big(\mathsf{B}_{\mathbf{e}}(h\overline{h'}\varphi^{\mathbf{e}})\big).$$

Conclusion. Dividing the equality we have obtained by $4^d N(\xi_r)^{1/2} N(\xi_{r'})^{1/2}$, we obtain the following restricted version of the sum formula:

Proposition 5.3.1. *Let $\kappa, \kappa' \in \mathscr{P}$, $r \in \mathfrak{t}'_\kappa \setminus \{0\}$, $r' \in \mathfrak{t}'_{\kappa'} \setminus \{0\}$. Define $\mathbf{e} \in \{1, -1\}^d$ by $\mathbf{e}_j = \mathrm{sign}(r_j r'_j)$. For all $h, h' \in \mathscr{H}^d$ the function k_0: $\nu \mapsto h(\nu)\overline{h'(\overline{\nu})}\varphi^{\mathbf{e}}(\nu)$ is integrable for the measures $d\sigma^{\kappa,\kappa'}_{r,r'}$ and $d\eta^{\mathbf{p}}$, the Kloosterman term $K^{\kappa,\kappa'}_{-r,-r'}(\mathsf{B}_{\mathbf{e}} k_0)$ converges absolutely, and*

$$\int_{Y^{\mathbf{e}}} k_0(\nu)\, d\sigma^{\kappa,\kappa'}_{r,r'}(\nu) = \Delta^{\kappa,\kappa'}_{r,r'}(k_0) + K^{\kappa,\kappa'}_{-r,-r'}(\mathsf{B}_{\mathbf{e}} k_0).$$

See (87) for the function $\varphi^{\mathbf{e}}$.

5.4. Extension of the sum formula. We take $\kappa, \kappa', r, r', \mathbf{e}, k$ as in Theorem 2.7.1. To prove the sum formula for the test function $k \in \mathscr{K}^{\mathbf{e}}$, we approximate k by functions $h\overline{h'}\varphi^{\mathbf{e}}$ with $h, h' \in \mathscr{H}^d$, and use Proposition 5.3.1.

There exists a function $\tilde{k}$ such that $k(\nu) = \tilde{k}(\nu)\varphi^{\mathfrak{e}}(\nu)$. To determine $\tilde{k} = \bigtimes_{j=1}^{d} \tilde{k}_j$ completely, we impose the conditions that $\tilde{k}_j$ is holomorphic on $|\mathrm{Re}\,\nu| \leqq \tau$, and $\tilde{k}_j\left(\frac{b-1}{2}\right) = 0$ for all $b \in 2\mathbb{Z}$ if $\mathfrak{e}_j = -1$. Note that k_j and $\varphi_j^{\mathfrak{e}}$ both have a zero at $\pm\frac{1}{2}$ if $\mathfrak{e}_j = -1$. If $\mathfrak{e}_j = 1$, we have $\tilde{k}_j\left(\frac{b-1}{2}\right) = (b-1)!\ k_j\left(\frac{b-1}{2}\right)$ for $b \in 2\mathbb{Z}$. On the strip $|\mathrm{Re}\,\nu| \leqq \tau$ we have $\tilde{k}_j(\nu) \ll (1 + |\mathrm{Im}\,\nu|)^{-a} e^{-\pi|\mathrm{Im}\,\nu|}$ for some $a > 2$.

Approximation. For $0 \leqq t < \tau^2$, we define $z_t = \bigtimes_{j=1}^{d} z_{t,j}$ by

$$z_{t,j}(\nu) := e^{t(\nu^2 - 1/4)} \quad \text{if } |\mathrm{Re}\,\nu| \leqq \tau,$$
$$z_{t,j}\left(\frac{b-1}{2}\right) := 1 \quad \text{if } b \in 2\mathbb{Z}, \left|\frac{b-1}{2}\right| \leqq t^{-1/2},$$
$$z_{t,j}\left(\frac{b-1}{2}\right) := 0 \quad \text{if } b \in 2\mathbb{Z}, \left|\frac{b-1}{2}\right| > t^{-1/2}.$$

This defines $z_t \in \mathscr{H}^d$ if $t > 0$. This function is bounded on its domain, uniformly in $t \in [0, \tau^2)$, and $\lim_{t \downarrow 0} z_t(\nu) = z_0(\nu) = 1$. Moreover, $\overline{z_t(\bar{\nu})} = z_t(\nu)$, and $z_t \tilde{k} \in \mathscr{H}^d$ if $t > 0$. Application of Proposition 5.3.1 shows that for all $t > 0$

$$\int_{Y^{\mathfrak{e}}} z_t^2(\nu) k(\nu)\, d\sigma_{r,r'}^{\kappa,\kappa'}(\nu) = \Delta_{r,r'}^{\kappa,\kappa'}(z_t^2 k) + K_{-r,-r'}^{\kappa,\kappa'}(\mathsf{B}_{\mathfrak{e}}(z_t^2 k)),$$

with absolutely convergent sums and integrals. We shall prove the sum formula for the test function k itself by taking the limit as $t \downarrow 0$.

Delta term. Suppose that $\alpha(\kappa, r; \kappa', r') \neq 0$. The test function k itself is integrable for the positive measure $d\eta^{\mathfrak{p}}$. The uniform boundedness of z_t implies that

$$\lim_{t \downarrow 0} \Delta_{r,r'}^{\kappa,\kappa'}(z_t^2 k) = \Delta_{r,r'}^{\kappa,\kappa'}(k).$$

Kloosterman term. We proceed as in the proof of Proposition 2.5.5. Equation (26) implies that $\beta_{\mathfrak{e}_j}(z_t^2 k)_j \ll_{y_0} y^{\tau}$ for $0 < y \leqq y_0$, uniformly in t. The estimate $\beta_{\mathfrak{e}_j}(z_t^2 k)(y) \ll 1$ for $y > 0$ is also valid uniformly in t, use (25) and (24). We apply Proposition 5.1.2 to $\mathsf{B}_{\mathfrak{e}}(z_t^2 k)$, and note that the implicit constant can be chosen uniform in t. Hence $K_{-r,-r'}^{\kappa,\kappa'}(\mathsf{B}_{\mathfrak{e}} k)$ converges absolutely, and

$$\lim_{t \downarrow 0} K_{-r,-r'}^{\kappa,\kappa'}(\mathsf{B}_{\mathfrak{e}}(z_t^2 k)) = K_{-r,-r'}^{\kappa,\kappa'}(\mathsf{B}_{\mathfrak{e}} k).$$

Spectral term. We first consider the case $\kappa = \kappa'$, $r = r'$, and $k = m = \bigtimes_{j=1}^{d} m_j \in \mathscr{K}^{\mathfrak{e}}$ with

$$m_j(\nu) = (p^2 - \nu^2)^{-a} \qquad \text{if } |\mathrm{Re}\,\nu| \leqq \tau, \mathbf{e}_j = 1,$$

$$m_j(\nu) = (p^2 - \nu^2)^{-a-1}\left(\frac{1}{4} - \nu^2\right) \qquad \text{if } |\mathrm{Re}\,\nu| \leqq \tau, \mathbf{e}_j = -1,$$

$$m_j(\nu) > 0 \qquad \text{for } b \geqq 2, b \in 2\mathbb{Z}, \text{ if } \mathbf{e}_j = 1,$$

with $\tau < p < \frac{3}{2}$, for a given $a > 2$.

On the support of the positive measure $d\sigma_{r,r}^{\kappa,\kappa}$, the family of positive functions $t \mapsto z_t^2 m$ is increasing in $-t$. The limit $\lim_{t \downarrow 0} \int_{Y^{\mathfrak{c}}} z_t(\nu)^2 m(\nu)\, d\sigma_{r,r}^{\kappa,\kappa}(\nu)$ exists, as it is the sum of the limits of the corresponding delta term and Kloosterman term. Fatou's lemma implies that m is integrable for the measure $d\sigma_{r,r}^{\kappa,\kappa}$.

For any $k \in \mathscr{K}$ there exists an m as above such that $|z_t^2 k| \leqq Cm$ on $Y^{\mathfrak{p}}$ for all $t \in [0, \tau^2)$, for some $C \geqq 0$. Lebesgue's theorem on bounded convergence shows that k is integrable for $d\sigma_{r,r}^{\kappa,\kappa}$, and

$$\lim_{t \downarrow 0} \int_{Y^{\mathfrak{p}}} z_t(\nu)^2 k(\nu)\, d\sigma_{r,r}^{\kappa,\kappa}(\nu) = \int_{Y^{\mathfrak{p}}} k(\nu)\, d\sigma_{r,r}^{\kappa,\kappa}(\nu).$$

In the general case, we use Lemma 3.1.1 to see that k is integrable for $d\sigma_{r,r'}^{\kappa,\kappa'}$. Take $f = (z_t^2 - 1)k$ to see that

$$\lim_{t \downarrow 0} \int_{Y^{\mathfrak{c}}} z_t(\nu)^2 k(\nu)\, d\sigma_{r,r'}^{\kappa,\kappa'}(\nu) = \int_{Y^{\mathfrak{c}}} k(\nu)\, d\sigma_{r,r'}^{\kappa,\kappa'}(\nu).$$

End of proof. We have shown that for each term in the sum formula for $z_t^2 k$, the limit as $t \downarrow 0$ is equal to the corresponding term for k. As for all $t \in (0, \tau^2)$ the sum formula holds for $z_t^2 k$, it holds also for k.

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