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New Lattice Point Asymptotics for Products of Upper Half-planes

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Sadly, Fritz Grunewald died when this work was in the process of revision. We dedicate this paper to his memory.

Let Γ be an irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})^d$ ($d \in \mathbb{N}$) and z a point in the d-fold direct product of the upper half-plane. We study the discrete set of componentwise distances $\mathbf{D}(\Gamma, z) \subset \mathbb{R}^d$ defined in (2). We prove asymptotic results on the number of $\gamma \in \Gamma$ such that $\mathrm{dist}(z, \gamma z)$ is contained in strips expanding in some directions and also in expanding hypercubes. The results improve the existing error terms, [6], and generalize the best known error term for $d = 1$, due to Selberg.

1 Introduction

Let $\mathfrak{H} = \{ x + iy \in \mathbb{C} : y > 0 \}$ be the upper half-plane equipped with the hyperbolic metric $d : \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ and its invariant measure induced by $\mathrm{d}x\,\mathrm{d}y/y^2$. The group of orientation preserving isometries of this metric space is $\mathrm{PSL}_2(\mathbb{R})$. Let now d be a natural number and consider the semisimple Lie group $\mathrm{PSL}_2(\mathbb{R})^d$ as acting on its corresponding symmetric

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space $\mathfrak{H}^d$. We write $z=(z_1,\ldots,z_d)$ for the coordinates $z_1,\ldots,z_d \in \mathfrak{H}$ of a point $z \in \mathfrak{H}^d$. Let us consider the vector-valued distance function

$$\operatorname{dist}(z,u) := (\operatorname{dist}(z_1,u_1),\ldots,\operatorname{dist}(z_d,u_d)) \in \mathbb{R}^d \tag{1}$$

for points $z=(z_1,\ldots,z_d)$, $u=(u_1,\ldots,u_d) \in \mathfrak{H}^d$. The canonical invariant distance of z, u is then the euclidean norm of $\operatorname{dist}(z,u)$. But other choices of norms (like the maximum norm) also induce $\mathrm{PSL}_2(\mathbb{R})^d$-invariant metrics on $\mathfrak{H}^d$.

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})^d$ be an irreducible lattice. A lattice in $\mathrm{PSL}_2(\mathbb{R})^d$ is a discrete subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^d$ of finite covolume, that is, the volume of the quotient $\Gamma\backslash\mathfrak{H}^d$ in the canonical measure is finite. The lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R})^d$ is called irreducible if all projections of Γ to non-trivial subproducts of $\mathrm{PSL}_2(\mathbb{R})^d$ are dense. A main example is the *Hilbert modular group* $\mathrm{PSL}_2(\mathcal{O})$ for the ring of integers $\mathcal{O}$ of a totally real number field F of degree d over $\mathbb{Q}$, embedded in the product $\mathrm{PSL}_2(\mathbb{R})^d$ by the d embeddings of F into $\mathbb{R}$. The embedded group $\mathrm{PSL}_2(\mathcal{O})$ and all its subgroups of finite index are irreducible lattices in $\mathrm{PSL}_2(\mathbb{R})^d$. They are not cocompact, which means that the quotient $\Gamma\backslash\mathfrak{H}^d$ is not compact. In case $d \geq 2$, every irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})^d$ that is not cocompact contains a subgroup of finite index, which is $\mathrm{PSL}_2(\mathbb{R})^d$-conjugate to a subgroup of finite index in one of the $\mathrm{PSL}_2(\mathcal{O})$. Irreducible cocompact lattices in $\mathrm{PSL}_2(\mathbb{R})^d$ are constructed from quaternion algebras over totally real number fields F. In case $d \geq 2$, these are up to conjugacy, the only examples by Margulis' arithmeticity theorem. See Section 2 for more details.

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})^d$ be an irreducible lattice and let $z \in \mathfrak{H}^d$ be fixed. Consider the set of vector-valued distances

$$\mathbf{D}(\Gamma,z) := \{\operatorname{dist}(z,\gamma z) \in \mathbb{R}^d \;:\; \gamma \in \Gamma\}. \tag{2}$$

This clearly is an infinite discrete subset of $\mathbb{R}^d$. But what more can be said? In this paper, we shall prove results that describe the distribution of the points of $\mathbf{D}(\Gamma,z)$ in various regions like strips or expanding polyhedra in $\mathbb{R}^d$. (We will count the point $\operatorname{dist}(z,\gamma z)$ with multiplicity $\#\Gamma_z$, where Γ_z is the subgroup of Γ fixing z.)

To give a precise formulation of our main results, we need to discuss some aspects of the spectral theory of $L^2(\Gamma\backslash\mathfrak{H}^d)$. This Hilbert space has a subspace $L^{2,\mathrm{discr}}(\Gamma\backslash\mathfrak{H}^d)$ with an orthonormal basis $\{\psi_\ell\}$ of joint eigenvectors of the Laplace operators $\Delta_j = -y_j^2\partial_{x_j}^2 - y_j^2\partial_{y_j}^2$ $(j=1,\ldots d)$ in the factors. The corresponding multi-eigenvalues λ_ℓ have finite multiplicities and form a discrete set in $[0,\infty)^d$. Among the eigenfunctions is the constant function $\psi_0(z) = \left(\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)\right)^{-1/2}$ for which the eigenvalues of all Δ_j are all equal to 0. For $\ell \geq 1$, one knows that $\lambda_{\ell,j} > 0$ for all $j = 1, \ldots, d$. If Γ is cocompact, then $L^{2,\mathrm{discr}}(\Gamma\backslash\mathfrak{H}^d)$ is all of $L^2(\Gamma\backslash\mathfrak{H}^d)$. Otherwise, the elements of the orthogonal complement of $L^{2,\mathrm{discr}}(\Gamma\backslash\mathfrak{H}^d)$ can be described as sums of integrals of Eisenstein series.

We call a multi-eigenvalue λ_ℓ *exceptional* if $0 < \lambda_{\ell,j} < 1/4$ for some coordinate $j \in \{1, \ldots, d\}$. If $d \geq 2$, there may be infinitely many exceptional eigenvalues, since there is no bound on the other coordinates. If $0 < \lambda_{\ell,j} < 1/4$ for all j, we call λ_ℓ *totally exceptional*. There are at most finitely many totally exceptional eigenvalues.

For a further discussion, we use the parametrization $\lambda = (1/4) - \tau^2$ by the *spectral parameter* τ. In $L^2(\Gamma\backslash\mathfrak{H}^d)$, all eigenvalues of local Laplace operators are in $[0, \infty)$, so we can choose $\tau \in i[0, \infty) \cup \left[0, 1/2\right]$. Thus, we have $\tau_{0,j} = 1/2$ for all j, and $\mathrm{Re}\,\tau_{\ell,j} < 1/2$ for all j.

For a congruence subgroup Γ of a Hilbert modular group, it has been shown by Kim and Shahidi, [15], that $\mathrm{Re}\,\tau_{\ell,j} \leq 1/9$ for all $\ell \geq 1$ and all j. For this situation, a conjecture called after Selberg says that $\mathrm{Re}\,\tau_{\ell,j} = 0$ for all $\ell \geq 1$ and all j. Below we will discuss other results concerning $\mathrm{Re}\,\tau_{\ell,j}$, $\ell \geq 1$. For the formulation of our results, we summarize the information concerning exceptional eigenvalues in the quantity

$$\hat{\tau} \;=\; \hat{\tau}(\Gamma) \;:=\; \sup_{\ell \geq 1,\, 1 \leq j \leq d} \mathrm{Re}\,\tau_{\ell,j}. \tag{3}$$

This, by definition, is an element of $\left[0, 1/2\right]$. Since there may be infinitely many λ_ℓ, the value 1/2 might occur in (3), although here we omit $\ell = 0$. If $\hat{\tau} < 1/2$, one says that $\Gamma\backslash\mathfrak{H}^d$ has a *strong spectral gap*. In our later arguments, we use that $\hat{\tau} < 1/2$, a fact proved in [14] by Kelmer and Sarnak for cocompact Γ. If Γ is not cocompact and $d \geq 2$, then Γ contains a subgroup of finite index that is conjugate to a congruence subgroup of a Hilbert modular group $\mathrm{SL}_2(\mathcal{O})$, for which the results of Kim and Shahidi, [15], imply that $\hat{\tau} \leq 1/9$. Here we use the important fact, proved in [28] that every subgroup of finite index in $\mathrm{SL}_2(\mathcal{O})$ is a congruence subgroup.

Let now $E \subset \{1, \ldots, d\}$ be a non-empty subset and let $I := \{\, I_j \;:\; j \in E \,\}$ be a set of bounded intervals $I_j := [A_j, B_j) \subset [0, \infty)$. Define for $T > 0$

$$S(E, I; T) := \{\, (x_1, \ldots x_d) \in \mathbb{R}^d \;:\; x_j \in I_j \text{ for } j \in E,\; 0 \leq x_j \leq T \text{ for} j \notin E \,\}. \tag{4}$$

We think of $S(E, I; T)$ as a strip of increasing height T in $\mathbb{R}^d$. Given $z \in \mathfrak{H}^d$, we introduce the counting quantity

$$N_E(z; T) := \#\{\, \gamma \in \Gamma \;:\; \mathrm{dist}(z, \gamma z) \in S(E, I; T) \,\}. \tag{5}$$

We show

Theorem 1.1. Let Γ be an irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})^d$, with $d \geq 2$. Let $E \subset \{1, \dots, d\}$ be a subset with $e := \#E \geq 1$. Define $Q := \{1, \dots, d\} \setminus E$ and assume $q := \#Q \geq 1$. Let finite intervals $[A_j, B_j) \subset [0, \infty)$ be given for $j \in E$, and let $n(z)$ be as defined in (19). The quantity $N_E(z; T)$ in (5) satisfies as $T \to \infty$:

$$N_E(z;T) \;=\; \frac{\pi^d 2^e}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)}\; e^{qT} \prod_{j\in E}\bigl(\cosh B_j - \cosh A_j\bigr) + \begin{cases} \mathrm{O}_{\Gamma,E}\Bigl(n(z)\,\exp\Bigl(\dfrac{d+1}{d+2}\,qT\Bigr)\Bigr) & \text{if } \hat\tau \leq \dfrac{q}{2(d+2)}\,, \\ \mathrm{O}_{\Gamma,E}\Bigl(n(z)\,\exp\Bigl(\dfrac{1+2\hat\tau+e}{2+e}\,qT\Bigr)\Bigr) & \text{if } \dfrac{q}{2(d+2)} \leq \hat\tau < \dfrac{1}{2}. \end{cases}$$

The quantity $n(z)$ in the error term equals 1 far away from the cusps and approaches ∞ as z approaches a cusp. Indeed, for z deep in the cusp sector associated to the cusp κ we have $n(z) = \Pi_j \,\mathrm{Im}\, g_{\kappa,j}^{-1} z_j$, where $g_\kappa \in \mathrm{PSL}_2(\mathbb{R})^d$ sends ∞ to κ.

This theorem is a specialization of Theorem 5.5, where we allow somewhat more general conditions on the components of the vector-valued distances. The E in $\mathrm{O}_{\Gamma,E}$ implies an implicit dependence on the intervals $[A_j, B_j)$ with $j \in E$. In Theorem 1.1, we have $d \geq 2$ and $1 \leq q = \#Q \leq d-1$. As explained above, $\hat\tau < 1/2$ holds and it depends on the relative sizes of d and q which of the error terms is applicable.

We may reformulate the theorem in terms of the volume

$$\mathrm{vol}(E, I; T) \;:=\; \mathrm{vol}\bigl\{z' \in \mathfrak{H}^d \;:\; \mathrm{dist}(z, z') \in S(e, I; T)\bigr\} \tag{6}$$

with respect to the invariant measure on $\mathfrak{H}^d$. Normalizing this measure by taking $\mathrm{d}x_j\,\mathrm{d}y_j/y_j^2$ in the factors we have

$$\mathrm{vol}(E, I; T) \;=\; \pi^d 2^{-q} e^{qT}\bigl(1 + \mathrm{O}(e^{-T})\bigr) \prod_{j\in E}\bigl(\cosh B_j - \cosh A_j\bigr).$$

Hence, the estimate in the theorem is equivalent to

$$N(z;T) \;=\; 2^d \frac{\mathrm{vol}(E, I; T)}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)} + \begin{cases} \mathrm{O}_{\Gamma,E}\Bigl(n(z)\,\mathrm{vol}(E, I; T)^{(d+1)/(d+2)}\Bigr) & \text{if } \hat\tau \leq \frac{q}{2(d+2)}\,, \\ \mathrm{O}_{\Gamma,E}\Bigl(n(z)\,\mathrm{vol}(E, I; T)^{(1+2\hat\tau+e)/(2+e)}\Bigr) & \text{if } \frac{q}{2(d+2)} \leq \hat\tau < \frac{1}{2}. \end{cases} \tag{7}$$

We shall describe now another result pertaining to the more standard lattice point problems. We consider the asymptotic distribution of the orbit points γz ($\gamma \in \Gamma$) for a given point $z \in X$ and a discontinuous group of motions Γ acting on a symmetric space X. In the case when $X = \mathfrak{H}$ is the upper half-plane, many authors have contributed to this problem, for instance [10], [11], [12], [22], and [13]. The best result concerning error terms is due to Selberg (see the Bombay and Göttingen lectures in [27]). It gives

$$\begin{aligned} \#\{\gamma \in \Gamma \ : \ \mathrm{dist}(\gamma z, z) \leq T\} \ &= \ \frac{\pi}{\mathrm{vol}(\Gamma\backslash\mathfrak{H})}\, e^{T} \\ &+ \sum_{\ell} \pi^{1/2}\, |\psi_\ell(z)|^2\, \frac{\Gamma(\tau_\ell)}{\Gamma(\tau_\ell + 3/2)}\, e^{(1/2+\tau_\ell)T} + \mathrm{O}\big(e^{\frac{2}{3}T}\big) \quad (T \to \infty). \end{aligned} \tag{8}$$

The functions ψ_j form an finite orthonormal system (possibly empty) of eigenfunctions with eigenvalue $1/4 - \tau_\ell^2$ of the hyperbolic Laplace operator acting on $L^2(\Gamma\backslash\mathfrak{H})$ with $0 < \tau_\ell < 1/4$. This result holds for all cofinite discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$, cocompact or not.

Let Γ now be an irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})^d$, with $d \in \mathbb{N}$. For $z \in \mathfrak{H}^d$, we define

$$N(z; T) := \#\,\big\{\gamma \in \Gamma \ : \ \max\big(\mathrm{dist}(z, \gamma z)\big) \leq T\big\}, \tag{9}$$

where $\max(x)$ is the maximum of the absolute values of the coordinates of the vector $x \in \mathbb{R}^d$. We show

Theorem 1.2. Let Γ be an irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})^d$, with $d \in \mathbb{N}$ and let $z \in \mathfrak{H}^d$ be given. With $\hat{\tau} = \hat{\tau}(\Gamma)$ as in (3), and with the quantity $n(z)$ as defined in (19), the counting function $N(z; T)$ has the following asymptotic behavior as $T \to \infty$:

- If $0 \leq \hat{\tau}(\Gamma) \leq \frac{d}{2(d+2)}$ *(large spectral gap)*, then

$$N(z; T) \ = \ \frac{\pi^d}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)}\, e^{dT} + \mathrm{O}_\Gamma\bigg(n(z)\exp\bigg(\frac{d+1}{d+2}dT\bigg)\bigg). \qquad \square$$

- If $\frac{d}{2(d+2)} \leq \hat{\tau}(\Gamma) \leq \frac{1}{2}$ *(small spectral gap)*, then

$$\begin{aligned} N(z; T) = \frac{\pi^d}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)} e^{dT} + \sum_{\ell \geq 1\,,\ \forall_j\ \tau_{\ell,j} \in (0,1/2)} \big|\psi_\ell(z)\big|^2 \prod_{j=1}^{d} \bigg(\frac{\sqrt{\pi}\,\Gamma(\tau_{\ell,j})}{\Gamma(3/2 + \tau_{\ell,j})} e^{(1/2+\tau_{\ell,j})T}\bigg) \\ + \mathrm{O}_\Gamma\bigg(n(z)\exp\bigg(\frac{2d + 2(d-1)\hat{\tau}}{3}T\bigg)\bigg). \end{aligned}$$

Theorem 1.2 is a special case of Theorem 5.4, where we allow a more general counting quantity than $N(z;T)$. The function $n(z)$ in (19) is positive on $\Gamma\backslash\mathfrak{H}^d$ and grows when z approaches a cusp.

A reformulation in terms of volumes uses

$$\mathrm{vol}(T) = \mathrm{vol}\{z' \in \mathfrak{H}^d : d_j(z,z') \le T \text{ for all } j\} = \pi^d 2^{-d} e^{dT}\left(1 + \mathrm{O}(e^{-T})\right),$$

and leads to

$$\begin{aligned} N(z;T) &= 2^d \frac{\mathrm{vol}(T)}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)} \\ &\quad + \mathrm{O}_\Gamma\left(n(z)\mathrm{vol}(T)^{(d+1)/(d+2)}\right) \quad (\text{if } \hat{\tau} \text{ small}), \\ &\quad + \Bigg(\sum_{\ell\ge 1,\ \forall_j\ \tau_{\ell,j}\in(0,1/2)} \left|\psi_\ell(z)\right|^2 \left(2\mathrm{vol}(T)/\pi\right)^{\sum_{j=1}^d (\frac{1}{2}+\tau_{\ell,j})/d} \\ &\qquad \cdot \prod_{j=1}^d \frac{\sqrt{\pi}\,\Gamma(\tau_{\ell,j})}{\Gamma(3/2+\tau_{\ell,j})} \\ &\qquad + \mathrm{O}_\Gamma\left(n(z)\mathrm{vol}(T)^{\frac{1}{3}(2+2\hat{\tau}-2\hat{\tau}/d)}\right)\Bigg) \quad (\text{if } \hat{\tau} \text{ large}). \end{aligned} \tag{10}$$

We note that we do not try to put one distance function on $\mathfrak{H}^d$, but work with the vector of the distances in the factors. Partly, this is because in this way it is easier to apply the spectral theory. Partly, it reflects the fact that there is not one distance function on $\mathfrak{H}^d$ that is preserved by the action of $\mathrm{PSL}_2(\mathbb{R})^d$, but infinitely many.

For a small spectral gap, totally exceptional eigenfunctions appear explicitly in the asymptotic estimate. Of course, some of these exceptional contributions may happen to be absorbed by the error term. For large spectral gaps, further improvement of our knowledge of $\hat{\tau}(\Gamma)$ does not improve the quality of the error term in our asymptotic formula. This holds in particular for the congruence case, in which we know that $\hat{\tau} \le 1/9$. The main term is always larger than the error term, even if we would have $\hat{\tau}(\Gamma) = 1/2$ (no spectral gap).

The case $d = 1$ in Theorem 1.2 concerns lattice point counting for groups acting on the upper half-plane. The best known error term $\mathrm{O}(e^{(2/3)T})$ coincides with the error term in Theorem 1.2 for $d = 1$. The papers [7], [1], [17], [18], and [2] treat lattice point counting for other symmetric spaces of rank one. The situation in this paper, with rank d, falls within the scope of [5] and [6], in which Gorodnik and Nevo consider counting of lattice points over quite general families of sets in quotients of more general

Lie groups. Their error terms for $\Gamma\backslash\mathfrak{H}^d$ are weaker than those in Theorem 1.2. They get $\mathrm{O}\big[\exp\big((5/6)T\big)\big]$ in the case $d=1$ and Γ not cocompact, and $\mathrm{O}\big\{\exp\big[\big((4d+1)/(4d+2)\big)dT\big]\big\}$ for the Hilbert modular case and $d\geq 2$, which should be compared with Selberg's bound $\mathrm{O}\big((2/3)T\big)$ for $d=1$, and with $\mathrm{O}\big\{\exp\big[\big((d+1)/(d+2)\big)dT\big]\big\}$ in Theorem 1.2 for general d. We emphasize that the class of counting problems considered by Gorodnik and Nevo is much larger than ours. They consider quite general families $t\mapsto G_t$ of regions in much more general groups than $\mathrm{PSL}_2(\mathbb{R})^d$. They use an ergodic method that can be applied in all these cases, without using more spectral information than the size of the spectral gap.

In the proofs, we apply the spectral theory of automorphic forms. We give the main proof in Section 5. The approach is sketched in the introduction to Section 4 and in Sections 5.1 and 5.2. The idea is that the sums $N(z;T)$ and $N_E(z;T)$ are replaced by smooth approximations. This smoothness ensures that the new quantities have a spectral decomposition that converges pointwise. In this spectral expansion, we single out the terms corresponding to the constant functions and to totally exceptional eigenfunctions, if these are present. These give the main terms in the asymptotic expansion. The remaining part of the spectral decomposition is estimated using the estimate of the spectral measure in Theorem 4.2. We use an approach similar to one in [13] that makes explicit the dependence on the point $z\in\mathfrak{H}^d$.

For the handling of the spectral decomposition, the Selberg transform discussed in Section 4.1 is essential. In Section 6, we prove the estimates and other facts that we need. In the proof of the main theorems, we also use some estimates of the counting function (Lemmas 3.1 and 3.2) obtained without the use of spectral theory. A main role in the proofs is played by an estimate of the spectral function given in Theorem 4.2, proved in Section 7.2.

We end the introduction with an explanation of the level of generality that we have chosen. The error terms in the estimates depend on the group Γ and the point $z\in\mathfrak{H}^d$. The dependence on the group Γ is hard to make explicit in a simple way. In the proof of Lemmas 3.1 and 3.2, we use geometrical properties of $\Gamma\backslash\mathfrak{H}^d$. Furthermore, in the introduction of Section 5.4 we use a property of the spectrum of $L^2(\Gamma\backslash\mathfrak{H}^d)$, and in the proof of Lemma 5.1 the size of the square integrable automorphic forms. The dependence on the point z is given by the quantity $n(z)$ in (19), which arises in a geometrical way in the proof of Lemma 3.2. It is used in the proof in Section 7.2 of the spectral estimate Theorem 4.2. It also enters in the proof of Lemma 5.1, in a rough estimate of the growth of residues of Eisenstein series. The methods in this paper would allow to count $\mathrm{dist}(w,\gamma z)$ with $w,z\in\mathfrak{H}^d$, instead of $\mathrm{dist}(z,\gamma z)$, but then changes would be needed at all these places. In the situation that $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1, an irreducible lattice in $\mathrm{SL}_2(\mathbb{R})^{d_1}$ and Γ_2 in $\mathrm{SL}_2(\mathbb{R})^{d_2}$, with $d_1, d_2 \geq 1$ and $d_1 + d_2 = d$, we may derive from the results for Γ_1 and Γ_2 similar, but more complicated, results for the reducible lattice $\Gamma = \Gamma_1 \times \Gamma_2$. If $\Gamma_1 \times \Gamma_2$ has finite index in Γ, it is harder to get information on counting quantities for Γ from that for $\Gamma_1 \times \Gamma_2$. Thus, to avoid many complications, in this paper we have restricted ourselves to irreducible lattices and to the case when $z = w$.

2 Lie Groups and Discrete Subgroups

Let G be the Lie group $\mathrm{PSL}_2(\mathbb{R})^d$ for some integer $d \geq 1$. The group G acts on the product $\mathfrak{H}^d$ of upper half-planes by fractional linear transformations in each factor. We will use the letter j to index these factors. G leaves invariant the vector-valued distance function

$$\mathrm{dist}(z, w) = \left(\mathrm{dist}_j(z_j, w_j)\right)_{j\in\{1,\dots,d\}}, \tag{11}$$

where dist_j is the hyperbolic distance in the jth factor. By $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we denote the class in G represented by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

We consider an irreducible lattice $\Gamma \subset G$, as described in Definition 5.20 and Corollary 5.21 of [24]. So, for each of the genuine subproducts H of $\mathrm{PSL}_2(\mathbb{R})^d$ the projection $\Gamma \to H$ has dense image. In particular, $\Gamma\backslash\mathfrak{H}^d$ has finite volume, and the projection to each of the factors is injective on Γ. (See also Corollary 5.23 in *loc. cit.*)

Hilbert modular groups $\mathrm{PSL}_2(\mathcal{O})$ and their subgroups of finite index, mentioned in the introduction, are examples. Cases for which $\Gamma\backslash G$ is compact can be derived from quaternion algebras $\mathcal{H}$ over a totally real number field F for which there is a non-empty set S of infinite places j for which the tensor product $F_j \otimes_F \mathcal{H}$, where F_j is the completion of F, is a division algebra. Suppose that $\#S = d > 0$. Let $\mathcal{H}_{\mathcal{O}}$ be an order in $\mathcal{H}$. Then the elements of reduced norm 1 in $\mathcal{H}_{\mathcal{O}}$ have as their image in $\prod_{j\notin S} \mathrm{PSL}_2(F_j)$ a cocompact discrete subgroup satisfying the assumptions above.

In the case $d = 1$, most of the subgroups of finite index in $\mathrm{PSL}_2(\mathbb{Z})$ are not the image of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Moreover, there are many irreducible discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ that are not commensurable to $\mathrm{PSL}_2(\mathbb{Z})$. For $d \geq 2$, Margulis has shown that all irreducible discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})^d$ are arithmetic, that is, commensurable to a Hilbert modular group or to a unit group of a quaternion algebra. (See Theorem (1.11) in Chapter IX of [20], or the discussion in Section 7 of [26].) Serre, [28], has shown that all subgroups of finite index in $\mathrm{SL}_2(\mathcal{O})$ are congruence subgroups.

So all non-cocompact irreducible lattices contain a conjugate of a congruence subgroup as a subgroup of finite index.

3 A Priori Estimates

From here on, we follow the usual practice of not working directly with the hyperbolic distance dist on $\mathfrak{H}$, but with

$$\begin{aligned} u(z, z') &= \frac{|z-z'|^2}{4yy'} = \left(\sinh \tfrac{1}{2}\mathrm{dist}(z, z')\right)^2, \\ \mathrm{dist}(z, z') &= 2\log\left(\sqrt{u(z, z')} + \sqrt{u(z, z')+1}\right). \end{aligned} \tag{12}$$

For $U, V \in [0, \infty)^d$ such that $U_j < V_j$ for all j, we consider the counting quantity

$$\mathbf{N}(U, V; z) = \#\left\{\gamma \in \Gamma \ :\ U_j \le u\big((\gamma z)_j, z_j\big) < V_j \text{ for all } j\right\}. \tag{13}$$

To relate this to the quantities $N(z; T)$ and $N_E(z; T)$ used in the introduction, we will use that $\mathrm{dist} \downarrow 0$ corresponds to $u \downarrow 0$ in such a way that

$$u = \frac{\mathrm{dist}^2}{4} + \mathrm{O}(\mathrm{dist}^4)\,, \qquad \mathrm{dist} = 2\sqrt{u} + \mathrm{O}(u^{3/2}), \tag{14}$$

and that $u \to \infty$ corresponds to $\mathrm{dist} \to \infty$ in such a way that

$$u = \frac{1}{4}e^{\mathrm{dist}} + \mathrm{O}(1)\,, \qquad \mathrm{dist} = \log u + \log 4 + \mathrm{O}(u^{-1}). \tag{15}$$

We will need a starting point for the estimation of $\mathbf{N}(U, V; z)$. We give two estimates for the counting function. The first is based on a simple volume argument. The second is important for the dependence of our results on the geometry of $\Gamma\backslash\mathfrak{H}^d$. As $z \in \mathfrak{H}^d$ approaches a cusp there are more and more $\gamma \in \Gamma$ for which γz is near z.

Lemma 3.1. For $z \in \mathfrak{H}^d$ and $U, V \in [0,\infty)^d$ such that $U_j < V_j$ for all j, we have

$$\mathbf{N}(U, V; z) \ll_{\Gamma,z} \prod_j (V_j - U_j + 1).$$

□

Proof. For $w \in \mathfrak{H}^d$ and $\delta > 0$, we put

$$B(w,\delta) = \{v \in \mathfrak{H}^d : \forall_j \; u(w_j, v_j) < \delta\}.$$

Let $z \in \mathfrak{H}^d$ be given. The subgroup Γ_z of Γ fixing z is finite. See, for example, Remark 2.14 in Chapter I of [4]. By the discontinuity of the action, there is $\delta > 0$ such that $B(z,\delta) \cap B(\gamma z, \delta) = \emptyset$ for all $\gamma \in \Gamma \smallsetminus \Gamma_z$. The Γ-invariance of u implies that $B(\gamma_1 z, \delta) \cap B(\gamma_2 z, \delta) = \emptyset$ for all $\gamma_1, \gamma_2 \in \Gamma$ for which $\gamma_1 z \neq \gamma_2 z$.

For $P, Q \in [0,\infty)^d$, denote by $A(P, Q)$ the multi-annulus

$$\{v \in \mathfrak{H}^d : \forall_j \; P_j \le u(v_j, z_j) < Q_j\}.$$

The pairwise disjoint sets $B(\gamma z, \delta)$ with $\gamma z \in \Gamma z \cap A(U, V)$ are contained in a slightly larger multi-annulus $A\big(U(\delta), V(\delta)\big)$ with $U(\delta)_j = U_j - \mathrm{O}(1)$ and $V(\delta)_j = V_j + \mathrm{O}(1)$. See (12). Thus, we have

$$\#\big(\Gamma z \cap A(U,V)\big) \le \mathrm{vol}\big(A(U(\delta), V(\delta))\big)/\mathrm{vol}(B(z,\delta)) \ll_\delta \prod_j (V_j - U_j + 1).$$

(The volume computation is easiest in a distance coordinate $u = u(z, i)$ and an angular coordinate ϕ. Then $d\mu$ on $\mathfrak{H}$ is given by $4\,du\,d\phi$. See (1.17) in [13].) Since $\mathbf{N}(U, V; z) = \#\Gamma_z \cdot \#\big(\Gamma z \cap A(U,V)\big)$, this proves the lemma. ■

Next we aim at an estimate of $\mathbf{N}(0, V; z)$ when all V_j are small. In this estimate, the dependence on z will be explicit. In order to do this, we take into account some facts concerning the geometry of the action of Γ on $\mathfrak{H}^d$. (The approach is motivated by that in the proof of Corollary 2.12 in [13].)

The assumptions on Γ imply that we can find a fundamental domain $\mathfrak{F}$ that is compact in the case of cocompact Γ, and is contained in a union of Siegel domains otherwise:

$$\mathfrak{F} \subset \bigcup_{\kappa} g_\kappa \mathfrak{S}(X_\kappa, Y_\kappa, V_\kappa),$$
$$\mathfrak{S}(X_\kappa, Y_\kappa, V_\kappa) = \Big\{ z \,:\, x_j \in [-X_\kappa, X_\kappa] \text{ for all } j\,,\ y_1 y_2 \cdots y_d \geq Y_\kappa\,, \quad V_\kappa^{-1} \leq \frac{y_j}{y_{j+1}} \leq V_\kappa \text{ for } 1 \leq j \leq d-1 \Big\}$$

where κ runs through a finite set of representatives $\kappa = g_\kappa \infty$ of the Γ-classes of cusps, with $g_\kappa \in G$, $X_\kappa, Y_\kappa > 0$ and $V_\kappa > 1$. (See [4], Chapter I, Section 2.) Enlarging Y_κ decreases $\mathfrak{S}(X_\kappa, Y_\kappa, V_\kappa)$. There exists $A > 0$ such that the $g_\kappa \mathfrak{S}(X_\kappa, A, V_\kappa)$ are disjoint. For each $B \geq A$, there is a compact set C_B such that

$$\mathfrak{F} \subset C_B \cup \bigcup_{\kappa} g_\kappa \mathfrak{S}(X_\kappa, B, V_\kappa). \tag{16}$$

We fix a fundamental domain and a disjoint decomposition of it induced by (16), and define Γ-invariant functions $\mathfrak{y}_1, \ldots, \mathfrak{y}_d$ on $\mathfrak{H}$ determined by the requirement that for $z \in \mathfrak{F}$:

$$\mathfrak{y}_j(z) = \begin{cases} 1 & \text{if } \Gamma \text{ is cocompact, or if } z \in C_A, \\ \mathrm{Im}(g_{\kappa,j}^{-1} z_j) & \text{if } z \in g_\kappa \mathfrak{S}(X_\kappa, A, V_\kappa). \end{cases} \tag{17}$$

The product of the $\mathfrak{y}_j(z)$ measures how far up in a cusp sector the point z is situated. Note that the $\mathfrak{y}_j$ may be discontinuous, but are bounded away from 0.

For $T \in (0, \infty)^d$, we put

$$n_j(T_j, z) = \max(1, \mathfrak{y}_j(z)/T_j)\,, \qquad n(T, z) = \prod_j n_j(T_j, z). \tag{18}$$

We take

$$n(z) = n(\mathbf{1}, z) = \prod_j \max\big(1, \mathfrak{y}_j(z)\big)\,, \tag{19}$$

with $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d$. The quantity $n(z)$ occurs in the error terms in the final estimates in Theorems 5.4 and 5.5, describing the dependence on $z \in \mathfrak{H}^d$. It is a Γ-invariant function, bounded away from zero, and tending to infinity deep in cuspidal sectors.

Lemma 3.2. For all sufficiently small $\delta_1 > 0, \dots, \delta_d > 0$, we have for $0 \in \mathbb{R}^d$ and $\delta = (\delta_j)_j$:

$$\mathbf{N}(z; 0, \delta) \ll_\Gamma n\big(\delta^{-1/2}, z\big),$$

where $\delta^{-1/2} = (\delta_j^{-1/2})_j$. □

Proof. It suffices to consider z in a fundamental domain $\mathfrak{F}$ chosen as indicated above. As long as z stays in a compact region, the value of $\mathbf{N}(z; 0, \delta)$ is at most the maximal order of subgroups Γ_z fixing points of $\mathfrak{F}$, provided we take the $\delta_j > 0$ sufficiently small. These groups are finite, and there are only finitely many points z in a given compact region for which Γ_z is non-trivial. See Remark 2.14, Chapter I, [4]. This proves the lemma for cocompact Γ.

For other Γ, we fix $B > 2A$. If $z \in C_B$, with C_B as in (16), we have $\mathbf{N}(z; 0, \delta) = \mathrm{O}_\Gamma(1)$ for all sufficiently small δ. Suppose now that $z \in g_\kappa \mathfrak{S}(X_\kappa, B, V_\kappa)$. If the δ_j are sufficiently small, then all $\gamma \in \Gamma$ such that $u\big((\gamma z)_j, z_j\big) \le \delta_j$ lie in $\Gamma \cap P_\kappa$, where P_κ is the parabolic subgroup fixing κ.

For the remaining computations, we can assume that $\kappa = \infty$ and $g_\kappa = 1$. Denote by $N = \left\{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in G\right\}$ the unipotent radical of $P_\infty = \left\{\begin{bmatrix} t & x \\ 0 & 1/t \end{bmatrix} \in G\right\}$. Elements in $N_\Gamma = N \cap \Gamma \subset \Gamma_\infty = P_\infty \cap \Gamma$ have the form $\begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}$ with ω running through a lattice $\Lambda \subset \mathbb{R}^d$. The quotient Γ_∞ / N_Γ is represented by elements of the form $\begin{bmatrix} \varepsilon & \alpha/\varepsilon \\ 0 & 1/\varepsilon \end{bmatrix}$, where α mod Λ is determined by ε, and where ε runs through a discrete subgroup of $(\mathbb{R}^*)^d$ such that $\varepsilon^2 \Lambda = \Lambda$, and such that the $\big(\log|\varepsilon_1|, \dots, \log|\varepsilon_d|\big)$ run through a lattice in the hyperplane $\sum_j x_j = 0$ in $\mathbb{R}^d$.

We need a bound for the number of $\gamma \in \Gamma_\infty$ with

$$u\big(\varepsilon_j^2(z_j + \alpha_j), z_j\big) = \frac{(\varepsilon_j^2 - 1)^2 Y_j^2 + \big((\varepsilon_j^2 - 1)x_j + \alpha_j\big)^2}{4\varepsilon_j^2 Y_j^2} \le \delta_j,$$

for $\delta_j \in (0, 1)$ for all j. Hence, the following quantities have to be non-negative:

$$4\delta_j \varepsilon_j^2 Y_j^2 - (\varepsilon_j^2 - 1)^2 Y_j^2. \tag{20}$$

This implies

$$\log\Big(1 + 2\delta_j - 2\sqrt{\delta_j + \delta_j^2}\Big) \le 2 \log|\varepsilon_j| \le \log\Big(1 + 2\delta_j + 2\sqrt{\delta_j + \delta_j^2}\Big).$$

Since $\log|\varepsilon|$ runs through a lattice in a hyperplane in $\mathbb{R}^d$, this leaves O(1) possibilities for the choice of ε. Taking the maximum of the quantity in (20), we find for all j:

$$\left|(\varepsilon_j^2 - 1)x_j + \alpha_j\right| \;\leq\; 2\sqrt{\delta_j + \delta_j^2}\; y_j.$$

Since α runs through a coset modulo the lattice Λ, this gives at most

$$\mathrm{O}\Bigl(\prod_j (1 + \sqrt{\delta_j}\; y_j)\Bigr) \ll \prod_j n_j(\delta_j^{-1/2}, z)$$

possibilities for the choice of ω.

For $z \in g_\kappa \mathfrak{S}(X_\kappa, B, V_\kappa)$, replace y_j by $\mathrm{Im}\, g_{\kappa,j}^{-1} z_j$. Together with the bound O(1) for $z \in C_B$, we get the statement in the lemma. ■

4 The Selberg Transform and Spectral Estimates

If $k_1, \ldots, k_d$ are bounded functions on $[0, \infty)$ with compact support, then the sum

$$K(z, w) \;=\; \sum_{\gamma \in \Gamma} \prod_j k_j\bigl(u((\gamma z)_j, w_j)\bigr) \tag{21}$$

converges absolutely, and defines a function on $(\Gamma\backslash\mathfrak{H}^d) \times (\Gamma\backslash\mathfrak{H}^d)$. If we take each k_j equal to the characteristic function of the interval $[U_j, V_j)$, then

$$K(z, z) \;=\; \mathbf{N}(U, V; z).$$

It will turn out preferable to use smooth k_j, so we will take for the k_j approximations of those characteristic functions. In this case, $K(z, z)$ is only an approximation of $\mathbf{N}(U, V; z)$, but its spectral expansion as an element of $L^2(\Gamma\backslash\mathfrak{H}^d)$ converges pointwise, and we can write

$$K(z, z) \;=\; K_{\mathrm{expl}}(z, z) + K'(z, z)$$

for each $z \in \mathfrak{H}^d$, where $K_{\mathrm{expl}}(z, z)$ is the contribution to the spectral expansion of a finite number of ψ_ℓ (among them ψ_0), and where $K'(z, z)$ is the remainder. The main idea is that $K_{\mathrm{expl}}(z, z)$ will yield the explicit terms in the asymptotic expansion of $\mathbf{N}(U, V; z)$, and that estimates of the difference $K(z, z) - \mathbf{N}(U, V; z)$ and of $K'(z, z)$ will contribute to the error term.

To carry this out, we have to see how the spectral expansion depends on the functions k_j. That leads us to a study of the Selberg transform (Section 4.1). We also have to know what is the size of the contributions of various parts of the spectrum (Section 4.3). In this section, we state the results that we need, and refer for most proofs to Section 6 and Section 7.

4.1 The Selberg transform

We can do most of the work on $\mathfrak{H}^d$ factor by factor. So we work first on $\mathfrak{H}$.

Functions k on $[0,\infty)$ yield kernel operators on functions f on $\mathfrak{H}$:

$$L_k f(z) \;=\; \int_{\mathfrak{H}} k(u(z,w))\; f(w)\,\mathrm{d}\mu(w), \tag{22}$$

where $\mathrm{d}\mu(w) = \mathrm{d}\,\mathrm{Re}\,w\;\mathrm{d}\,\mathrm{Im}\,w/(\mathrm{Im}\,w)^2$ is the invariant measure associated to the Riemannian metric on $\mathfrak{H}$. We take $k \in C_c^\infty[0,\infty)$. (This implies in particular that all derivatives are well defined and continuous at $u = 0$.) We assume that the function f is continuous. That suffices for the convergence in (22).

The *Selberg transform* associates to the function $k \in C_c^\infty[0,\infty)$ an even holomorphic function h on $\mathbb{C}$, given by the following three steps:

$$\begin{aligned} q(p) &= \int_p^\infty k(u)\,\frac{\mathrm{d}u}{\sqrt{u-p}}, && \text{for } p \geq 0,\\ g(r) &= 2q\left((\sinh(r/2))^2\right), && \text{for } r \in \mathbb{R},\\ h(\tau) &= \int_{-\infty}^\infty e^{r\tau} g(r)\,\mathrm{d}r\,, && \text{for } \tau \in \mathbb{C}. \end{aligned} \tag{23}$$

See, for example, [13], page 33, but note that Iwaniec uses ir as the variable in h. (See also [25].) The relation can be described in one step:

$$h(\tau) \;=\; \int_{\mathfrak{H}} k(u(z,i))\; y^{\frac{1}{2}-\tau}\;\mathrm{d}\mu(z). \tag{24}$$

See (1.62′) and the proof of Theorem 1.16 in [13]. In fact, h is the *spherical transform* of k. See, for example, [16], Chapter V, Section 4. We have in particular

$$h\big(\tfrac{1}{2}\big) \;=\; 4\pi \int_0^\infty k(u)\,\mathrm{d}u. \tag{25}$$

The Selberg transform has the important property that if $\Delta f = \big((1/4) - \tau^2\big) f$, then

$$L_k f = h(\tau)\, f \tag{26}$$

(Theorem 1.16 in [13]).

Next we consider $h_1, \ldots, h_d \in C_c^\infty[0,\infty)$, and form the kernel function

$$k(z,w) = \prod_j k_j(u(z_j, w_j)) \tag{27}$$

on $\mathfrak{H}^d \times \mathfrak{H}^d$. Thus, we have the operator

$$L_k f(z) = \int_{\mathfrak{H}^d} k(z,w)\, f(w)\, \mathrm{d}\mu(w), \tag{28}$$

with $\mathrm{d}\mu = \prod_j \mathrm{d}\mu_j$ the product of the invariant measures. This converges absolutely if f is continuous on $\mathfrak{H}^d$. If moreover we have $\Delta_j f = \Big((1/4) - \tau_j^2\Big) f$ for the local Laplace operators $\Delta_j = -y_j^2(\partial^2/\partial x_j^2) - y_j^2(\partial^2/\partial y_j^2)$, then

$$L_k f = \left(\prod_j h_j(\tau_j)\right) f, \tag{29}$$

where h_j is the Selberg transform of h_j.

By Lemma 3.1, the sum

$$K(z,w) := \sum_{\gamma\in\Gamma} k(\gamma z, w) \tag{30}$$

converges absolutely, and defines a function in $C^\infty\big((\Gamma\backslash\mathfrak{H}^d) \times (\Gamma\backslash\mathfrak{H}^d)\big)$ that satisfies

$$K(z,w) = \mathrm{O}_z(1). \tag{31}$$

The boundedness of $K(z,w)$ is uniform for z varying in compact sets. If f is square integrable on $\Gamma\backslash\mathfrak{H}^d$ for the invariant measure $\mathrm{d}\mu$, then

$$\mathcal{K}_k f(z) = \int_{\Gamma\backslash\mathfrak{H}^d} K(z,w)\, f(w)\, \mathrm{d}\mu(w) \tag{32}$$

converges absolutely, and defines an operator

$$\mathcal{K}_k : L^2(\Gamma\backslash\mathfrak{H}^d) \longrightarrow C^\infty(\Gamma\backslash\mathfrak{H}^d),$$

where $f \mapsto \mathcal{K}_k f(z)$ is continuous on $L^2(\Gamma\backslash\mathfrak{H}^d)$ for each $z \in \mathfrak{H}^d$.

4.2 Spectral decomposition

A consequence of the irreducibility assumption for the lattice Γ is that the spectral theory $L^2(\Gamma\backslash\mathfrak{H}^d)$ is well known. The Hilbert space $L^2(\Gamma\backslash\mathfrak{H}^d) = L^2(\Gamma\backslash\mathfrak{H}^d, \mathrm{d}\mu)$ has a spectral decomposition in terms of automorphic forms. In the cocompact case, each element can be written in L^2-sense as

$$\sum_{\ell \geq 0} a_\ell \, \psi_\ell, \tag{33}$$

where the ψ_ℓ form a complete orthonormal system in $L^2(\Gamma\backslash\mathfrak{H}^d)$ of simultaneous eigenfunctions of the Δ_j:

$$\Delta_j \psi_\ell \;=\; \left(\tfrac{1}{4} - \tau_{\ell,j}^2\right)\psi_\ell, \tag{34}$$

with $\tau_{\ell,j} \in i[0,\infty) \cup (0, 1/2]$. Among these eigenfunctions we choose $\psi_0 = 1/\sqrt{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)}$, a constant function; hence $\tau_{0,j} = 1/2$ for all j. For each $\ell \geq 1$, we know that $\tau_{\ell,j} \in i[0,\infty) \cup (0, 1/2)$. The a_ℓ form a sequence in the Hilbert space ℓ^2.

If Γ has cusps, there is a subspace $L^{2,\mathrm{discr}}(\Gamma\backslash\mathfrak{H}^d)$ with the same structure as in the cocompact case. It always contains the constant function ψ_0. If $d = 1$, there may be finitely many $\ell \geq 1$ for which ψ_ℓ is a residue of an Eisenstein series and at most countably many ψ_ℓ that are cusp forms. The orthogonal complement $L^{2,\mathrm{cont}}(\Gamma\backslash\mathfrak{H}^d)$ is a sum of direct integrals. Elements of this space can be written in L^2-sense in the form

$$\sum_{\kappa} 2c_\kappa \sum_{\mu \in \mathcal{L}_\kappa} \int_0^\infty b_{\kappa,\mu}(t)\, E(\kappa; it, i\mu)\, \mathrm{d}t. \tag{35}$$

Here κ runs over representatives of the finitely many cuspidal Γ-orbits, the c_κ are positive constants, $\mathcal{L}_\kappa$ is a lattice in the hyperplane $\sum_j x_j = 0$ in $\mathbb{R}^d$, and $E(\kappa; s, i\mu)$ is an Eisenstein series, satisfying

$$\Delta_j E(\kappa; s, i\mu) = \left(\tfrac{1}{4} - (s + i\mu_j)^2\right) E(\kappa; s, i\mu)$$

for each j. For $f \in L^2(\Gamma\backslash\mathfrak{H}^d)$, we have $a_\ell = (f, \psi_\ell)$. If f is bounded and sufficiently smooth, then $b_{\mu,\kappa}$ is given by integration against $\overline{E(\kappa; it, i\mu)}$.

The quantity $\hat{\tau}(\Gamma) = \sup_{\ell\geq 1,\, 1\leq j\leq d} \operatorname{Re}\tau_{\ell,j}$ in (3) is related to the quantity $p(\Gamma\backslash G) \in [2,\infty)$ in [14], with $G = \mathrm{PSL}_2(\mathbb{R})^d$, by

$$p(\Gamma\backslash G) \;\geq\; \frac{1}{\frac{1}{2}-\hat{\tau}} \quad \text{or equivalently} \quad \tau \;\leq\; \frac{1}{2} - \frac{1}{p(\Gamma\backslash G)}. \tag{36}$$

So $\hat{\tau} = 1/2$ would imply $p(\Gamma\backslash G) = \infty$ (no strong spectral gap), and $p(\Gamma\backslash G) = 2$ implies $\hat{\tau} = 0$ (no exceptional eigenvalues at all). We have to be careful to use inequalities in (36). Kelmer and Sarnak take all irreducible representations of $G = \mathrm{PSL}_2(\mathbb{R})^d$ in $L^{2,\mathrm{discr}}(\Gamma\backslash G)$ into account. Such a representation is visible in $L^{2,\mathrm{discr}}(\Gamma\backslash\mathfrak{H}^d)$ only if all d components of the representation have a non-trivial PSO(2)-invariant vector. We recall that in the congruence case (including all non-cocompact Γ if $d \geq 2$) we have $\hat{\tau}(\Gamma) \leq 1/9$. For all cocompact Γ, we have $\hat{\tau}(\Gamma) < 1/2$.

We return to the kernel function K in (30). By (29) and the invariance of the kernel $k(z, w)$, we have for fixed $z \in \mathfrak{H}^d$:

$$\int_{\Gamma\backslash\mathfrak{H}^d} K(z,w)\,\overline{\psi_\ell(w)}\,\mathrm{d}\mu(w) \;=\; \int_{\mathfrak{H}^d} k(z,w)\,\overline{\psi_\ell(w)}\,\mathrm{d}\mu(w) \;=\; h(\tau_\ell)\overline{\psi_\ell(z)},$$

with

$$h(\tau) \;=\; \prod_j h_j(\tau_j). \tag{37}$$

Therefore, the scalar product of $K(z,\cdot)$ with ψ_ℓ makes sense. If Γ is not cocompact, we find in a similar way that the coefficients $b_{\kappa,\mu}(t)$ in (35) are given by $\prod_j h_j(it + i\mu_j)\,\overline{E(\kappa; it, i\mu)}$. Thus, we obtain the spectral expansion of $K(z,\cdot) \in L^2(\Gamma\backslash\mathfrak{H}^d)$:

$$\begin{aligned} K(z,\cdot) \;=\; &\sum_\ell h(\tau_\ell)\,\overline{\psi_\ell(z)}\,\psi_\ell \\ &+ \sum_\kappa 2c_\kappa \sum_{\mu\in\mathcal{L}_\kappa} \int_0^\infty h(it + i\mu)\,\overline{E(\kappa; it, i\mu; z)}\,E(\kappa; it, i\mu)\,\mathrm{d}t. \end{aligned} \tag{38}$$

In the cocompact case, we understand the sum over κ to be absent.

This spectral expansion converges in the Hilbert space $L^2(\Gamma\backslash\mathfrak{H}^d)$. To use it to investigate the counting function $\mathbf{N}(U, V; z)$ in the way indicated in the introduction of this section, we need it to make sense pointwise.

Theorem 4.1. Let $f \in C^{2d}(\Gamma\backslash\mathfrak{H}^d)$ be bounded, and suppose that the derivatives $\Delta_1^{a_1}\Delta_2^{a_2}\cdots\Delta_d^{a_d} f$ are bounded for all choices of $a_j \in \{0, 1, 2\}$. Then the spectral expansion of f converges absolutely and uniformly on compacta.

In particular, if the k_j in (27) are in $C_c^\infty[0, \infty)$ for all j, then the expansion

$$K(z, w) = \sum_\ell h(\tau_\ell)\,\overline{\psi_\ell(z)}\,\psi_\ell(w) + \sum_\kappa 2c_\kappa \sum_{\mu\in\mathcal{L}_\kappa} \int_0^\infty h(it + i\mu)\,\overline{E(\kappa; it, i\mu; z)}\,E(\kappa; it, i\mu; w)\,\mathrm{d}t \tag{39}$$

converges absolutely for each choice $z, w \in \mathfrak{H}^d$. □

This result is more or less standard. We will sketch a proof in Section 7.1.

4.3 Spectral measure

As indicated in the introduction of this section, we will need to know how the various parts of the spectral set

$$\left(i\mathbb{R} \cup (0, \frac{1}{2}]\right)^d$$

contribute to the spectral expansion of $K(z, z)$. We write $i\mathbb{R}$ instead of $i[0, \infty)$, since in the term in the spectral expansion (39) corresponding to the continuous spectrum there are quantities $i\left(t + \mu_j\right)$, which in some cases are in $i(-\infty, 0)$.

For $X \in [1, \infty)^d$, we put

$$Y(X) = \prod_j \left((0, \frac{1}{2}] \cup i(-X_j, X_j)\right), \tag{40}$$

and define

$$S(X; z, w) = \sum_{\ell,\ t_\ell\in Y(X)} \overline{\psi_\ell(z)}\psi_\ell(w) + \sum_\kappa 2c_\kappa \sum_{\mu\in\mathcal{L}_\kappa} \int_{t\geq 0,\ (t+\mu_j)_j\in Y(X)} \overline{E(\kappa; it, i\mu; z)}\,E(\kappa; it, i\mu; w)\,\mathrm{d}t.$$

This is a smooth function on $(\Gamma\backslash\mathfrak{H}^d) \times (\Gamma\backslash\mathfrak{H}^d)$. (The sum over ℓ is finite. The region of integration is finite for all (κ, μ), and empty for almost all (κ, μ).)

The following estimate of the spectral function $S(X; z, z)$ will play an important role in Section 5 in the proof of our main results:

Theorem 4.2. For $X \in [1, \infty)^d$ and $z \in \mathfrak{H}^d$:

$$S(X; z, z) \ll_\Gamma X_1^2 X_d^2 \cdots X_d^2 \, n(X, z). \qquad \square$$

The quantity $n(X, z)$ has been defined in (19). It makes explicit the dependence of the spectral measure on the point $z \in \mathfrak{H}$. This constitutes a difference with [2], where we used a result of Hörmander to estimate the spectral measure uniformly for z in compact sets, obtaining an asymptotic formula for the lattice point counting function on symmetric spaces of rank one that was uniform for z varying in compact sets only.

We prove Theorem 4.2 in Section 7.2. The proof uses the a priori estimate of the lattice point counting function in Lemma 3.2.

5 Proof of the Lattice Points Theorems

This section is the heart of this paper, where we carry out the plan sketched in the introduction of Section 4. We consider the asymptotic behavior of the quantity $\mathbf{N}(U, V; z)$ defined in (13), with $U, V \in [0, \infty)^d$, $U_j < V_j$ for all j. In Theorem 5.4, which is slightly more general than Theorem 1.2 in the introduction, all V_j tend to ∞, whereas in Theorem 5.5, generalization of Theorem 1.1, some intervals $[U_j, V_j)$ stay fixed. For most of the section, we handle the proofs simultaneously.

5.1 The main parameters

We partition the set $\{1, \dots, d\}$ into two disjoint subsets Q and E, with the requirement that Q is non-empty.

For each $j \in Q$, we let V_j be a large parameter, tending to ∞, and choose $U_j = \alpha_j V_j$ for a fixed $\alpha_j \in [0, 1)$. (This choice may depend on the place $j \in Q$.) The simplest is to take all V_j with $j \in Q$ equal to each other. We wish also to include the case that $V_j = T^{a_j}$ with positive exponent a_j, where T tends to infinity. However, we do not let the V_j run apart too much, by fixing a parameter $\hat{q}$ satisfying

$$\hat{q} \geq \#Q, \quad \text{and requiring} \quad \min_{j \in Q} V_j^{\hat{q}} = \prod_{j \in Q} V_j. \tag{41}$$

Table 1 Overview of the parameters in Section 5

Main parameters		
Q		a non-empty subset of $\{1, \dots, d\}$
E		the complement $\{1, \dots, d\} \setminus Q$
V_j	$j \in Q$	$V_j \to \infty$,
	$j \in E$	$V_j > 0$ fixed
α_j	$j \in Q$	$0 \le \alpha_j < 1$
U_j	$j \in Q$	$U_j = \alpha_j V_j$
	$j \in E$	$U_j \in [0, V_j)$ fixed
$\hat{q}$		$\hat{q} \ge \#Q$ fixed , $\hat{q} \log V_{\min} = \sum_{j \in Q} \log V_j$
$V_{\min}$		$V_{\min} = \min_{j \in Q} V_j \ge 2$
Auxiliary parameters		
ϑ		$\vartheta \in (0, 1)$
Y_E		$Y_E \in (0, 1]$, $Y_E \downarrow 0$
Y_j	$j \in Q$	$Y_j = V_j^{\vartheta} \le \frac{1}{2}(V_j - U_j)$, $Y_j \ge 1$, $Y_j \to \infty$
	$j \in E$	$Y_j = Y_E \downarrow 0$
c		$0 < c < \frac{1}{2}$, if $\hat{\tau} > 0$ then $c < \hat{\tau}$

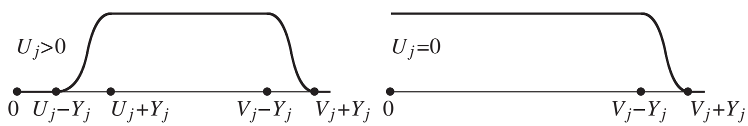
Thus, if all V_j with $j \in Q$ are equal to each other, then $\hat{q} = \#Q$. For each $j \in E$, we keep the non-empty interval $[U_j, V_j)$ fixed.

These are the parameters used in Theorems 5.4 and 5.5. They constitute the "main parameters" in Table 1.

5.2 Test functions and auxiliary parameters

In the introduction of Section 4, we have sketched our plan to prove the main results in Sections 5.7 and 5.8. We take the approximations $h_j \in C_c^\infty(0, \infty)$ of the characteristic functions $[U_j, V_j)$ in the following way:

$$
\begin{aligned}
0 \;\le\; k \;\le\; 1\,, \qquad k_j^{(l)} \;&=\; \mathrm{O}_l(Y^l) \text{ for } l \in \mathbb{N}, \\
k_j \;=\; 1 \text{ on } &\begin{cases} [U_j + Y_j, V_j - Y_j] & \text{if } U_j > 0, \\ [0, V_j - Y_j] & \text{if } U_j = 0, \end{cases} \\
k_j \;=\; 0 \text{ on } &\begin{cases} [0, U_j - Y_j] \cup [V_j + Y_j, \infty) & \text{if } U_j > 0, \\ [V_j + Y_j, \infty) & \text{if } U_j = 0. \end{cases}
\end{aligned}
\tag{42}
$$



The parameters Y_j control how quickly k_j changes from 0 to 1 and back. We require

$$Y_j \le \frac{V_j - U_j}{2}. \tag{43}$$

If one fixes a smooth function $\omega \in C^\infty(\mathbb{R})$ that increases from 0 to 1 on an interval contained in $(-1, 1)$, then, in the case $U_j > 0$, we take $h_j(u) = \omega\big((u - U_j)u/Y_j\big)$ on $[U_j - Y_j, U_j + Y_j]$ satisfies on this interval the condition on the derivatives, and goes from 0 to 1. On $[V_j - Y_j, V_j + Y_j]$, we proceed similarly. This gives a choice such that $k_j = 0$ outside $[U_j - Y_j, V_j + Y_j]$. We take $k_j = 1$ on $[U_j + Y_j, V_j - Y_j]$. If $U_j = 0$, we proceed similarly.

The Y_j are new parameters. They play a role in the proof, not in the theorems. At the end of the proof we try to choose them optimally. To avoid having to keep track of too many auxiliary parameters, we assume from the start that $Y_j = V_j^\vartheta$ for $j \in Q$, with $\vartheta \in (0, 1)$ a single auxiliary parameter. So the Y_j are large parameters for $j \in Q$. (The conditions on $Y_j = V_j^\vartheta$ in (43) imply lower bounds for the V_j, depending on ϑ.)

We let the Y_j with $j \in E$ tend to zero. It seems that we do not loose much if we take all these parameters equal to a quantity Y_E tending to 0, for which we require that $2Y_E \le V_j - U_j$ for all $j \in E$, and $2Y_E \le U_j$ for all $j \in E$ with $U_j > 0$.

The estimates of Selberg transforms in Section 6 depend on a small positive parameter $c \in \big(0, 1/2\big)$. In Lemma 6.3 c), there is also a small positive parameter δ. We choose δ such that $\delta < U_j$ for all $j \in E$ with $U_j > 0$. The dependence on δ of the implicit constants in the estimates is absorbed in the dependence of the choice of the intervals $[U_j, V_j]$ with $j \in E$. The positive constant c will turn up in exponents in some estimates. We take $c < \hat{\tau}$ if $\hat{\tau} > 0$. (We recall that $\hat{\tau}$ measures the spectral gap, which is maximal if $\hat{\tau} = 0$.)

If $\alpha_j > 0$ for some $j \in Q$, we assume that $U_j = \alpha_j V_j > \delta$. This implies an additional assumption on the lower bound of V_j.

5.3 Terms in the asymptotic formula

With the test functions in (42), we form the Γ-invariant kernel

$$K(z,w) \;=\; \sum_{\gamma\in\Gamma} k\big(u(z,w)\big), \qquad k(u) \;=\; \prod_j k_j(u_j), \tag{44}$$

as indicated in the introduction of Section 4. The diagonal value $K(z,z)$ gives an approximation of $\mathbf{N}(U,V;z)$. By $h(\tau)=\prod_j h_j(\tau_j)$, we denote the product of the Selberg transforms of the k_j.

From the absolutely convergent spectral decomposition in (39), we single out

$$K_{\mathrm{expl}}(z,z) \;:=\; h(\tau_0)\,\frac{1}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)} + \sum_{\ell\geq 1\,,\ \forall_j\ 0<\tau_{\ell,j}<\frac12} h(\tau_\ell)\,\big|\psi_\ell(z)\big|^2. \tag{45}$$

Here we have made the choice to take not only the contribution of the constant functions, but also all of the terms corresponding to totally exceptional eigenvalues. This term $K_{\mathrm{expl}}(z,z)$ will lead to the explicit term in our asymptotic expansions.

As the explicit term in the final results, we use

$$\begin{aligned}\mathcal{E}(U,V;z) \;=\; &\sum_{\ell\geq 0\,,\ \forall_j\ \tau_{\ell,j}\in(0,\frac12]} \big|\psi_\ell(z)\big|^2 \prod_{j\in E}\eta(U_j,V_j;\tau_{\ell,j})\\ &\cdot\prod_{j\in Q}\frac{\sqrt{\pi}\,2^{1+2\tau_{\ell,j}}\,\Gamma(\tau_{\ell,j})}{\Gamma\big(\frac32+\tau_{\ell,j}\big)}\big(V_j^{\frac12+\tau_{\ell,j}}-U_j^{\frac12+\tau_{\ell,j}}\big),\end{aligned} \tag{46}$$

where

$$\eta(a,b;\tau) \;=\; \int_{z\in\mathfrak{H}\,,\ a\leq u(z,i)<b} y^{\frac12+\tau}\,\mathrm{d}\mu(z) \tag{47}$$

is the Selberg transform of the characteristic function of $[a,b)$. So we will need to estimate the difference between $\mathcal{E}(U,V;z)$ and $K_{\mathrm{expl}}(z,z)$. We note that there might not exist totally exceptional eigenvalues for the group Γ. In that case, $\mathcal{E}(U,V;z)$ is equal to the term for $\ell=0$:

$$\frac{1}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)}\,(4\pi)^d\prod_j\big(V_j-U_j\big). \tag{48}$$

(See (92) in Lemma 6.1.)

Table 2 Overview of the error term estimates

$\mathbf{N}(U, V; z) = \mathcal{E}(U, V; z) + \mathrm{O}\big(\mathrm{Err}_1 + \mathrm{Err}_2 + \mathrm{Err}_3\big)$			
Err_1	estimate of	$\sum_{n \in \mathbb{N}^d} K_n(z, z)$	in Section 5.5
Err_2	estimate of	$K_{\mathrm{expl}}(z, z) - \mathcal{E}(U, V; z)$	in Section 5.4
Err_3	estimate of	$\mathbf{N}(U, V; z) - K(z, z)$	in Section 5.6

The remaining part of the spectral decomposition is split up according to subsets $Z(n)$ of the space of spectral parameters. For $n \in \mathbb{N}^d$, we put

$$
\begin{aligned}
Z(n) \;=\; \Big\{ \tau \in \big(i[0, \infty) \cup (0, \tfrac{1}{2})\big)^d \;:\; & \\
& \tau_j \in i[-n_j, 1 - n_j] \cup i[n_j - 1, n_j) \text{ if } n_j > 1, \\
& \tau_j \in (0, \tfrac{1}{2}) \cup i(-1, 1) \text{ if } n_j = 1 \Big\}.
\end{aligned} \tag{49}
$$

For $n \in \mathbb{N}^d$, $n \neq \mathbf{1} = (1, 1, \ldots, 1)$, we define

$$
\begin{aligned}
K_n(z, z) \;=\; & \sum_{\ell \geq 1\,,\ \tau_\ell \in Z(n)} h(\tau_\ell) \left|\psi_\ell(z)\right|^2 \\
& + \sum_\kappa 2c_\kappa \sum_{\mu \in \mathcal{L}_\kappa} \int_{t \geq 0\,,\ i(t+\mu) \in Z(n)} h(it + i\mu) \left|E(\kappa; it, i\mu; z)\right|^2 \mathrm{d}t.
\end{aligned} \tag{50}
$$

For $n = \mathbf{1}$, we modify the term from the discrete spectrum by requiring not only $\tau_\ell \in Z(\mathbf{1})$, but also $\tau_{\ell,j} \in i[0, \infty)$ for some j. (The totally exceptional terms go into $\mathcal{E}(U, V; z)$.) With these definitions, we have

$$
K(z, z) - K_{\mathrm{expl}}(z, z) \;=\; \sum_{n \in \mathbb{N}^d} K_n(z, z). \tag{51}
$$

It will be hard work to estimate this sum.

Finally, we also will have to estimate the difference between $K(z, z)$ and the counting quantity $\mathbf{N}(U, V; z)$. Table 2 gives an overview of the estimates to be carried out.

5.4 The explicit term

The explicit term $\mathcal{E}(U, V; z)$ in (46) is a finite sum. Each of its terms contains as a factor, for $j \in E$, the Selberg transform $\eta(U_j, V_j; \tau_{\ell,j})$ of the characteristic function of $[U_j, V_j)$, and, for $j \in Q$, the approximation of this Selberg transform given in (94) in Lemma 6.1. That approximation is uniform on intervals $[c, 1/2]$ for each $c > 0$. Here we want to apply it with τ equal to the coordinates $\tau_{\ell,j}$ of the totally exceptional eigenvalues. These coordinates form a finite subset of $(0, 1/2)$. We take $c \in (0, 1/2)$ smaller than the minimum of these finitely many $\tau_{\ell,j}$, and then apply (94) uniformly. Thus, this parameter c depends on the group Γ, and will lead to an implicit dependence of the error terms on Γ.

Lemma 5.1. The explicit term satisfies for all U_j, V_j with $j \in \Omega$, for all $\vartheta \in (0,1)$ and all sufficiently small values of the steepness parameter Y_E, under the conditions in Table 1, for each $z \in \mathfrak{H}$, the estimate

$$\mathcal{E}(U, V; z) - K_{\mathrm{expl}}(z, z) \ll_{\Gamma,E,\alpha} n(z)\left(V_{\min}^{\vartheta-1} + Y_E\right) \prod_{j\in\Omega} V_j. \tag{52}$$

The factor $n(z)$ has been defined in (19). The implicit constant depends on the group Γ, the U_j, V_j with $j \in E$, and the quotients $\alpha_j = U_j/V_j$ with $j \in \Omega$. □

Proof. For each of the finitely many $\tau = \tau_\ell$ occurring in $\mathcal{E}(U, V; z)$ and $K_{\mathrm{expl}}(z, z)$, we have by Lemma 6.3 a):

$$\begin{aligned}&\prod_j \eta(U_j, V_j; \tau_j) - \prod_j h_j(\tau_j)\\ &\quad\ll \sum_j \left|\eta(U_j, V_j; \tau_j) - h_j(\tau_j)\right| \sum_{l\neq j}\Big(\left|\eta(U_j, V_j; \tau_j)\right| + \left|h_j(\tau_j)\right|\Big).\end{aligned} \tag{53}$$

We apply Lemmas 6.1 and 6.2 with $c \in \big(0, 1/2\big)$ chosen so that $c < \tau_{\ell,j}$ for all j for all ℓ occurring in the explicit term. We find, uniformly for $c \le \tau_j \le 1/2$:

$$\begin{aligned} j \in \Omega:\ \eta(U_j, V_j; \tau_j) - h_j(\tau_j) &\ll_{c,\alpha} Y_j V_j^{\tau_j-\frac12} = V_j^{\tau_j+\vartheta-\frac12} && \text{Lemma 6.2 c)},\\ \eta(U_j, V_j; \tau_j) &\ll_c V_j^{\tau_j+\frac12} && (94),\\ j \in E:\ \eta(U_j, V_j; \tau_j) - h_j(\tau_j) &\ll_E Y_j = Y_E && \text{Lemma 6.2 c)},\\ \eta(U_j, V_j; \tau_j) &\ll_E 1. \end{aligned}$$

This implies that $h_j(\tau_j) \ll_c V_j^{\tau_j+(1/2)} + V_j^{\tau_j-(1/2)} \ll V_j^{\tau_j+(1/2)}$ for $j \in \Omega$ and $h_j(\tau_j) \ll_E 1 + Y_E \ll 1$ for $j \in E$. Note that we leave implicit the influence of the fixed quantities U_j and V_j with $j \in E$, but keep explicit the parameter $Y_j \le (V_j - U_j)/2$. The difference in (53) is estimated by the following quantity, uniformly in the $\tau = \tau_\ell$ under consideration:

$$\begin{aligned}&\ll_{\Gamma,E,\alpha} \sum_{j\in\Omega} V_j^{\tau_j+\vartheta-\frac12} \cdot \prod_{l\in\Omega\setminus\{j\}} V_l^{\tau_l+\frac12} \cdot \mathrm{O}(1) + \sum_{j\in E} Y_E \cdot \prod_{l\in\Omega} V_l^{\tau_l+\frac12} \cdot \mathrm{O}(1)\\ &\ll \left(\sum_{j\in\Omega} V_j^{\vartheta-1} + Y_E\right) \prod_{l\in\Omega} V_l^{\tau_l+\frac12} \ll \left(V_{\min}^{\vartheta-1} + Y_E\right) \prod_{j\in\Omega} V_j,\end{aligned}$$

where we have used $\tau_j \le 1/2$ in the last step.

We still have to estimate the finitely many $\psi_\ell(z)$. If ψ_ℓ is a cusp form or if $\ell = 0$, then $|\psi_\ell(z)| = \mathrm{O}_\Gamma(1)$. If ψ_ℓ, with $\ell \ge 1$, arises from a residue of an Eisenstein series, it satisfies $\psi_\ell(g_\kappa z) = \mathrm{O}(N(y)^{(1/2)-\rho_\ell})$ as $N(y) = \prod_j y_j \to \infty$ for all cusps κ, for some $\rho_\ell \in (0, 1/2)$. Hence, $|\psi_\ell(z)| \ll n(z)^{1/2}$. Since the explicit term and $K_e(z, z)$ run over finitely many ℓ, this estimate can be used uniformly, thus giving the lemma. Note that here arises another implicit dependence of the error terms on the group Γ. ■

5.5 Sum over the spectrum

We turn to the estimation of $\sum_{n\in\mathbb{N}^d} K_n(z, z)$, as defined in (50), with the given modification for $n = \mathbf{1}$. We will use that

$$K_n(z, z) \ \le \ M(n)\, S_n, \tag{54}$$

where, with $Z(n)$ as defined in (49)

$$M(n) \ = \ \sup_{\tau\in Z(n)} |h(\tau)|, \tag{55}$$

$$S_n \ = \sum_{\ell\ge 1\,,\ \tau_\ell\in Z(n)} \left|\psi_\ell(z)\right|^2 + \sum_\kappa 2c_\kappa \sum_{\mu\in\mathcal{L}_\kappa} \int_{t\ge 0\,,\ \left(i(t+\mu_j)\right)_j\in Z(n)} \left|E(\kappa; it, i\mu; z)\right|^2 \mathrm{d}t. \tag{56}$$

Lemma 5.2. For each sufficiently small $c \in \left(0, 1/2\right)$, the quantity $M(n)$ has for each $n \in \mathbb{N}^d$ and each $l \in \mathbb{N}^d$ an estimate

$$M(n) \ \ll_{E,l,c,\alpha} \prod_j f_{l_j,j}(n_j),$$

where for all $l \in \mathbb{N}$

$$f_{l,j}(1) \ = \ \begin{cases} V_j^{\hat\tau+\frac12} & \text{if } j \in Q \text{ and } \hat\tau > 0,\\ V_j^{c+\frac12} & \text{if } j \in Q \text{ and } \hat\tau = 0,\\ 1 & \text{if } j \in E,\end{cases} \tag{57}$$

$$f_{l,j}(n) \ = \ \begin{cases} n^{-l-\frac12}\, Y_j^{1-l} V_j^{l-\frac12} & \text{if } j \in Q \text{ and } n \ge 2,\\ n^{-l-\frac12}\, Y_E^{1-l} & \text{if } j \in E \text{ and } n \ge 2.\end{cases}$$

This estimate holds under the same conditions as the previous lemma. The implicit constant depends also on the choice of the l_j and of c.

If $E = \emptyset$ and $\hat{\tau} > 0$, we have the slightly better estimate

$$M(\mathbf{1}) \ll_c V_{\min}^{c-\hat{\tau}} \prod_j \mathfrak{f}_{l_j,j}(1). \tag{58}$$

□

We recall that $\hat{\tau} \in [0, 1/2]$ is the supremum of the real parts $\mathrm{Re}\, \tau_{\ell,j}$, $\ell \geq 1$, of the spectral parameters. It measures the spectral gap.

Proof. If $\hat{\tau} > 0$, we take $c \in (0, \hat{\tau})$. We use the estimates in Section 6 of the Selberg transforms h_j of the k_j with this value.

If $n = 1$, we have to consider $\tau_j \in (0, \hat{\tau}) \cup i[-1, 1]$. For $\tau \in [c, 1/2)$, which can occur only if $\hat{\tau} > 0$, we use Lemma 6.2 c) and (94) in Lemma 6.1 to get

$$h_j(\tau_j) \ll_c V_j^{\tau_j + \frac{1}{2}} + Y_j\, m,$$

with

$$m = \max(V_j^{\tau_j - \frac{1}{2}}, U_j^{-\frac{1}{2}}) \text{ if } U_j > 0, \qquad m = \max(V_j^{-\frac{1}{2}}, V_j^{\tau_j - \frac{1}{2}}) \text{ if } U_j = 0.$$

If $j \in Q$, we get a bound by $\mathrm{O}_\alpha(V_j^{\tau_j + (1/2)}) = \mathrm{O}_\alpha(V_j^{\hat{\tau}+(1/2)})$. For $j \in E$, the dependence on U_j and V_j is left implicit, so we can use the bound 1. If $|\tau_j| \leq c$, Lemma 6.3 b) gives the bound $\mathrm{O}_c(V_j^{c+(1/2)})$ for $j \in Q$, and $\mathrm{O}_{c,E}$ if $j \in E$. For $\tau_j \in i\mathbb{R}$, $c \leq |\tau_j| \leq 1$, we use Lemma 6.3 c) with $l = 1$. For $j \in E$, we take care to choose the δ in Lemma 6.3 such that $U_j \geq \delta$ if $U_j > 0$. We use that $k' \ll Y^{-1}$ to find $\mathrm{O}(V_j^{1/2})$ if $j \in Q$ and $\mathrm{O}(1)$ if $j \in E$.

If $n \geq 2$, we use Lemma 6.3 c) and the condition $k_j^{(l)} = \mathrm{O}(Y^{-l})$ to obtain the bounds by $\mathfrak{f}_{l,j}(n)$.

In the case of $M(\mathbf{1})$, we have the additional information that $\tau_j \in i[-1, 1]$ for at least one j. If $E = \emptyset$, this leads to the estimate in (58). ■

The problem with S_n in (56) is that we do not have a direct estimate for it. All we have is Theorem 4.2, which gives

$$\sum_{m \in \mathbb{N}^d,\, \forall_j\, m_j \leq n_j} S_m \ll_\Gamma \mathfrak{n}(n, z) \prod_j n_j^2 = \prod_j \max(n_j^2, \mathfrak{y}_j(z)\, n_j). \tag{59}$$

(See (18) for $n(n, z)$.) So we need to carry out a d-dimensional partial summation.

Lemma 5.3. Let $z \in \mathfrak{H}$. For $c \in \left(0, 1/2\right)$ as in the previous lemma, we have if $\hat{\tau} > 0$

$$\sum_{n\in\mathbb{N}^d} K_n(z,z) \ll_{\Gamma,E,c,\alpha} n(z) \cdot \begin{cases} Y_E^{-\frac{1}{2}\#E} \prod_{j\in Q} V_j^{1-\frac{1}{2}\vartheta} & \text{if } \vartheta \le 1-2\hat{\tau}, \\ Y_E^{-\frac{1}{2}\#E} \prod_{j\in Q} V_j^{\hat{\tau}+\frac{1}{2}} & \text{if } \vartheta \ge 1-2\hat{\tau} \\ & \quad \text{and } E \neq \emptyset, \\ V_{\min}^{\frac{1}{2}-\hat{\tau}-\frac{\vartheta}{2}} \prod_{j=1}^d V_j^{\hat{\tau}+\frac{1}{2}} & \text{if } 1-2\hat{\tau} \le \vartheta \le 1-2c \\ & \quad \text{and } E = \emptyset, \\ V_{\min}^{c-\hat{\tau}} \prod_{j=1}^d V_j^{\hat{\tau}+\frac{1}{2}} & \text{if } \vartheta \ge 1-2c \\ & \quad \text{and } E = \emptyset, \end{cases} \tag{60}$$

and if $\hat{\tau} = 0$

$$\sum_{n\in\mathbb{N}^d} K_n(z,z) \ll_{\Gamma,E,c,\alpha} n(z) \cdot \begin{cases} Y_E^{-\frac{1}{2}\#E} \prod_{j\in Q} V_j^{1-\frac{1}{2}\vartheta} & \text{if } \vartheta \le 1-2c, \\ Y_E^{-\frac{1}{2}\#E} \prod_{j\in Q} V_j^{c+\frac{1}{2}} & \text{if } \vartheta \ge 1-2c. \end{cases} \tag{61}$$

These estimates hold under the conditions in Lemma 5.1. The implicit constants depend also on the value of c. □

Proof. We use that $\sum_n K_n(z,z) \le \sum_n M(n)\, S_n$. Lemma 5.2 gives an estimate of $M(n)$.

$$\sum_{n\in\mathbb{N}^d,\, n\neq\mathbf{1}} K_n(z,z) \ll_{E,L,c,\alpha} \sum_{n\in\mathbb{N}^d,\, n\neq\mathbf{1}} S_n\, P(n), \tag{62}$$

where $P(n)$ is a bound given by Lemma 5.2, for some choice of the l_j. We choose the l_j from a finite set L of values, which influences the implicit constant in the estimate, but will not depend on other choices.

We have $S_n = \sum_{H\subset\{1,\dots,d\}} (-1)^{\#H} S(Y(n-\mathbf{1}_H); z, z)$, where $\mathbf{1}_H \in \mathbb{N}_{\ge 0}^d$ has coordinate 1 if $j \in H$ and coordinate 0 otherwise. We understand that $S(Y(m); z, z)$ is zero if one of the coordinates of m vanishes. Thus, we obtain

$$\sum_{n\in\mathbb{N}^d,\, n\neq\mathbf{1}} S_n\, P(n) = \sum_{m\in\mathbb{N}^d} S\big(Y(m); z, z\big) \sum_{H\subset\{1,\dots,d\}} (-1)^{\#H}\, P(m+\mathbf{1}_H). \tag{63}$$

We'll handle the exceptional estimate for $M(\mathbf{1})$ in Lemma 5.2 later, and first consider the sum T_F over those $m \in \mathbb{N}^d$ that satisfy $m_j \geq 2$ if and only if $j \in F$, where F runs over the non-empty subsets of $\{1, \ldots, q\}$.

$$T_F = \sum_{m \in \mathbb{N}^d,\, m_j \geq 2 \Leftrightarrow j \in F} S\big(Y(m); z, z\big) \prod_j \big(f_{l,j}(m_j) - f_{l,j}(m_j+1) \big). \tag{64}$$

In the $f_{l,j}(m_j)$, we take $l = 3$ if $m_j > A_j$ and $l = 1$ if $1 \leq m_j \leq A_j$, with

$$A_j = V_j / Y_j = V_j^{1-\vartheta} \quad \text{if } j \in Q, \qquad A_j = Y_j^{-1} = Y_E^{-1} \quad \text{if } j \in E. \tag{65}$$

The A_j are large quantities with the choices made before. For $m_j = 1$, the choice of l in (57) does not matter. In this way, the set of values of l is finite.

In Lemma 5.2, we see that the differences $f_{l,j}(m_j) - f_{l,j}(m_j+1)$ are non-negative, except possibly for $m_j = [A_j]$ if $j \in F$, and for $m_j = 1$ if $j \notin F$. In those cases, we use $\big| f_{l,j}(m_j) - f_{l,j}(m_j+1) \big| \leq f_{l,j}(m_j) + f_{l,j}(m_j+1)$. Thus, we obtain (64):

$$T_F \ll \sum_{m \in \mathbb{N}^d,\, m_j \geq 2 \Leftrightarrow j \in F} S\big(Y(m); z, z\big) \prod_j d_j(m_j),$$

$$\text{for } j \in Q: \quad d_j(m_j) = \begin{cases} V_j^{\max(\hat{\tau}, c) + \frac{1}{2}} & \text{if } m_j = 1, \\ m_j^{-\frac{5}{2}} V_j^{\frac{1}{2}} & \text{if } 2 \leq m_j \leq A_j - 1, \\ Y_j^{\frac{3}{2}} V_j^{-1} & \text{if } A_j - 1 < m_j \leq A_j, \\ m_j^{-\frac{9}{2}} Y_j^{-2} V_j^{\frac{5}{2}} & \text{if } m_j > A_j\,; \end{cases} \tag{66}$$

$$\text{for } j \notin E: \quad d_k(m_j) = \begin{cases} 1 & \text{if } m_j = 1, \\ m_j^{-\frac{5}{2}} & \text{if } 2 \leq m_j \leq A_j - 1, \\ Y_E^{\frac{3}{2}} & \text{if } A_j - 1 < m_j \leq A_j, \\ m_j^{-\frac{9}{2}} Y_E^{-2} & \text{if } m_j > A_j. \end{cases}$$

Now we have an expression with non-negative quantities, and obtain with Theorem 4.2:

$$\begin{aligned} T_F &\ll_\Gamma \sum_{m \in \mathbb{N}^d,\, m_j \geq 2 \Leftrightarrow j \in F} \prod_j \max\big(m_j^2, \mathfrak{y}_j(z)\, m_j\big)\, d_j(m_j) \\ &= \prod_{j \in F} \Bigg(\sum_{m_j \geq 2} \max\big(m_j^2, \mathfrak{y}_j(z)\, m_j\big)\, d_j(m_j) \Bigg) \cdot \prod_{j \notin F} \mathfrak{y}_j(z)\, d_j(1). \end{aligned} \tag{67}$$

If $\mathfrak{y}_j(z) \geq A_j$, then the sum in the factor for $j \in Q \cap F$ in the estimate is split according to $2 \leq m_j \leq A_j - 1$, $m_j \approx A_j$, $A_j < m_j \leq \mathfrak{y}_j(z)$, and $m_j > \mathfrak{y}_j(z)$. This leads to an estimate by

$$\mathfrak{y}_j(z) V_j^{\frac{1}{2}} + \mathfrak{y}_j(z) Y_j^{\frac{1}{2}} + \mathfrak{y}_j(z) A_j^{-\frac{5}{2}} Y_j^2 V_j^{\frac{5}{2}} + \mathfrak{y}_j(z)^{-\frac{3}{2}} Y_j^{-2} V_j^{\frac{5}{2}} \ll \mathfrak{y}_j(z) V_j^{\frac{1}{2}} + V_j^{1-\frac{\vartheta}{2}}.$$

In the last step, we have used that $\mathfrak{y}_j(z)^{-(3/2)} \leq V_j^{-(3/2)(1-\vartheta)}$.

If $\mathfrak{y}_j(z) < A_j$, we split the sum according to $2 \leq m_j \leq \mathfrak{y}_j(z)$, $\mathfrak{y}_j(z) < m_j \leq A_j - 1$, $A_j - a < m_j \leq A_j$, and $m_j > A_j$, to obtain the estimate

$$\mathfrak{y}_j(z) V_j^{\frac{1}{2}} + A_j^{\frac{1}{2}} V_j^{\frac{1}{2}} + A_j^2 Y_j^{\frac{3}{2}} V_j + A_j^{-\frac{3}{2}} Y_j^{-2} V_j^{\frac{5}{2}} \ll \mathfrak{y}_j(z) V_j^{\frac{1}{2}} + V_j^{1-\frac{\vartheta}{2}}.$$

Thus, both cases lead to the same estimate.

For $j \in E \cap F$, we find similarly the estimate

$$\mathfrak{y}_j(z) + Y_E^{-\frac{1}{2}},$$

both for $\mathfrak{y}_j(z)$ larger and smaller than A_j. Taking into account the factors $d_j(1)$ for $j \notin F$, we arrive at

$$\begin{aligned} T_F &\ll_\Gamma \prod_{j \in F \cap Q} \left(\mathfrak{y}_j(z) V_j^{\frac{1}{2}} + V_j^{1-\frac{\vartheta}{2}} \right) \prod_{j \in F \cap E} \left(\mathfrak{y}_j(z) + Y_E^{-\frac{1}{2}} \right) \\ &\quad \cdot \prod_{j \in Q \smallsetminus F} \mathfrak{y}_j(z) V_j^{\max(\hat{\tau}, c) + \frac{1}{2}} \prod_{j \in E \smallsetminus F} \mathfrak{y}_j(z) \\ &\ll \prod_{j \in F \cap Q} n_j\big(V_j^{\frac{1-\vartheta}{2}}, z\big)\, V_j^{\frac{1}{2} - \frac{\vartheta}{2}} \prod_{j \in F \cap E} n_j\big(Y_E^{-\frac{1}{2}}, z\big)\, Y_E^{-\frac{1}{2}} \\ &\quad \cdot \Big(\prod_{j \in Q \smallsetminus F} V_j \Big)^{\max(\hat{\tau}, c) + \frac{1}{2}} \prod_{j \notin F} \mathfrak{y}_j(z). \end{aligned} \tag{68}$$

Since we have already $n(z)$ in the error term in Lemma 5.1, it seems sensible to replace $n_j(*, z)$ by $n_j(1, z)$ in these estimates:

$$T_F \ll_\Gamma n(z)\, Y_E^{-\#(F \cap E)/2} \prod_{j \in Q \cap F} V_j^{\frac{1-\vartheta}{2}} \prod_{j \in Q \smallsetminus F} V_j^{\max(\hat{\tau}, c) + \frac{1}{2}}. \tag{69}$$

The next step is to determine which non-empty $F \subset \{1, \ldots, d\}$ has the maximal contribution. The factors for $j \in E$ are maximal if $j \in F$. So we consider $F \supset E$. The factors for $j \in Q$ are maximal for $j \notin F$ if $\vartheta \geq 1 - 2\max(c, \hat{\tau})$, and maximal for $j \in F$

otherwise. If $E = \emptyset$, we have to put one place in F anyhow, which gives the maximal contribution if $V_j = V_{\min}$. We find the following maximal value:

$$\begin{cases} n(z)\, Y_E^{-\frac{1}{2}\#E} \prod_{j\in Q} V_j^{\max(c,\hat\tau)+\frac12} & \text{if } \vartheta \ge 1-2\max(c,\hat\tau) \text{ and } E \neq \emptyset, \\ n(z)\, V_{\min}^{\frac12-\max(c,\hat\tau)-\frac12\vartheta} \prod_{j\in Q} V_j^{\max(c,\hat\tau)+\frac12} & \text{if } \vartheta \ge 1-2\max(c,\hat\tau) \text{ and } \#Q = d, \\ n(z)\, Y_E^{-\frac12\#E} \prod_{j\in Q\cap F} V_j^{1-\frac12\vartheta} & \text{if } \vartheta \le 1-2\max(c,\hat\tau). \end{cases} \tag{70}$$

In the latter case, the maximum is attained for $F = \{1, \dots, d\}$, and in the former case for $F = E$, if $E \neq \emptyset$. If $E = \emptyset$, one place has to be in F, and j such that $V_j = V_{\min}$ gives the maximal value.

Finally, we have to consider the term with $m = \mathbf{1}$ on the right in (63). It is estimated by

$$S\big(Y(\mathbf{1}); z, z\big)\, M(\mathbf{1}) \ll n(z) \cdot \begin{cases} \prod_{j\in Q} V_j^{\max(c,\hat\tau)+\frac12} & \text{if } E \neq \emptyset, \\ V_{\min}^{c-\max(c,\hat\tau)} \prod_{j\in Q} V_j^{\max(c,\hat\tau)+\frac12} & \text{if } E = \emptyset. \end{cases} \tag{71}$$

If $E \neq \emptyset$ or if $\vartheta \le 1 - 2\max(c, \hat\tau)$, this is absorbed in the term that we have already obtained. In the case $E = \emptyset$ and $\vartheta \ge 1 - 2\max(c, \hat\tau)$, we have to compare the factors $V_{\min}^{(1/2)-\max(c,\hat\tau)-(1/2)\vartheta}$ and $V_{\min}^{c-\max(c,\hat\tau)}$. If $\hat\tau = 0$, we have $\max(c, \hat\tau) = c$, in which case the latter factor, $V_{\min}^0 = 1$, is the largest. In the remaining case, there is another transition point at $\vartheta = 1 - 2c$.

This leads to the statements in the lemma. We resist the temptation to simplify the lemma by choosing $c < (1 - \vartheta)/2$. That would cause a dependence of the implicit constant in the estimates on the auxiliary parameter ϑ. ■

5.6 Difference between sums with sharp and smooth bounds

We have obtained $K(z, z) = \mathcal{E}(U, V; z) + \mathrm{Err}_1 + \mathrm{Err}_2$ for the sum $K(z, z)$ in (44), the explicit term $\mathcal{E}(U, V; z)$ in (46), with error terms Err_1 estimated in Lemma 5.1 and Err_2 in Lemma 5.3. The sum $K(z, z)$ depends on the choice of the local test functions k_j as indicated in (42). In particular, the estimate is valid for the sum $K^+(z, z)$ based on test functions with $k_j^+ = 1$ on $[U_j, V_j]$ for all j, and also for the sum $K^-(z, z)$ built with $\mathrm{Supp}(k_j^-) \subset [U_j, V_j]$. Since the characteristic function χ of $\prod_j [U_j, V_j)$ satisfies $\prod_j k_j^- \le \chi \le \prod_j k_j^+$, we have $K^-(z, z) \le \mathbf{N}(U, V; z) \le K^+(z, z)$. Thus, we have also

$$\mathbf{N}(U, V; z) \;=\; \mathcal{E}(U, V; z) + \mathrm{Err}_1 + \mathrm{Err}_2, \tag{72}$$

with error terms satisfying the estimates in Lemmas 5.1 and 5.3.

5.7 Asymptotic estimate, case $E = \emptyset$

First we choose the auxiliary parameters in the case $E = \emptyset$. This leads to the asymptotic result in Theorem 5.4, of which Theorem 1.2 is a special case.

The auxiliary parameter $\vartheta \in (0, 1)$ has to be adapted to the V_j to get the minimal value of the bound

$$\mathbf{N}(U, V; z) - \mathcal{E}(U, V; z) \ll_{\Gamma,c,\alpha} n(z) V_{\min}^{\vartheta-1} \prod_j V_j + n(z) \cdot \begin{cases} \prod_j V_j^{1-\frac{1}{2}\vartheta} & \text{if } \vartheta \leq 1 - 2\max(c, \hat{\tau}), \\ V_{\min}^{\max\left(\frac{1-\vartheta}{2}, c\right) - \hat{\tau}} \prod_j V_j^{\frac{1}{2}+\max(c,\hat{\tau})} & \text{if } \vartheta \geq 1 - 2\max(c, \hat{\tau}). \end{cases} \tag{73}$$

The parameter $c \in (0, 1/2)$ is allowed to depend on Γ, but not on the V_j. We have assumed that $0 < c < \hat{\tau}$ if $\hat{\tau} > 0$. The V_j influence the estimate by their product $\prod_j V_j$, which tends to ∞. We have prescribed that the minimal V_j is coupled to the product by $V_{\min}^{\hat{q}} = \prod_j V_j$, with $\hat{q} \geq d$. (See (41).) Expressing the logarithm of the quantity to consider in terms of $\log V_{\min}$, we arrive at the following quantity to minimize:

$$\begin{cases} \max\left(\vartheta - 1 + \hat{q},\ \hat{q} - \frac{1}{2}\hat{q}\vartheta\right) & \text{if } \vartheta \leq 1 - 2\max(c, \hat{\tau}), \\ \max\left(\vartheta - 1 + \hat{q},\ \frac{\hat{q}}{2} + (\hat{q}-1)\hat{\tau} + \max\left(\frac{1-\vartheta}{2}, c\right)\right) & \text{if } \vartheta \geq 1 - 2\hat{\tau},\ \hat{\tau} > 0, \\ \max\left(\vartheta - 1 + \hat{q},\ \hat{q}\left(\frac{1}{2} + c\right) + \max\left(\frac{1-\vartheta}{2}, c\right)\right) & \text{if } \vartheta \geq 1 - 2c,\ \hat{\tau} = 0. \end{cases} \tag{74}$$

We choose $0 < c < 1/18$ in addition to the requirement that $c < \hat{\tau}$ if $\hat{\tau} > 0$.

If $\hat{\tau} = 0$, the value ϑ_1 for which $\vartheta_1 - 1 + \hat{q} = \hat{q} - (1/2)\hat{q}\vartheta_1$ is $\vartheta_1 = 2/(\hat{q}+2)$. Since $\hat{q} \geq d \geq 1$, we have $\vartheta_1 \leq 2/3 < 1 - 2c$. Hence, this is the optimal choice.

For $\hat{\tau} > 0$, we have $\vartheta_1 = 2/(\hat{q}+2)$ and $\vartheta_2 = 1 - (1/3)\hat{q} + (2/3)(\hat{q}-1)\hat{\tau}$, for the intersections of the graph of $\vartheta \mapsto \vartheta - 1 + \hat{q}$ with, respectively, $\vartheta \mapsto \hat{q} - (1/2)\hat{q}\vartheta$ and $\vartheta \mapsto (1/2)\hat{q} + (\hat{q}-1)\hat{\tau} + ((1-\vartheta)/2)$. If $\hat{\tau} \leq \hat{q}/(2(\hat{q}+2))$, then $\vartheta_1 \leq 1 - 2\hat{\tau}$ gives the optimal choice. Otherwise, $\vartheta_2 \geq 1 - 2\hat{\tau}$ is optimal, since it is between $1 - 2\hat{\tau}$ and $1 - 2c$.

This leads to the following optimal bound of the quantity in (74):

$$\hat{q}\frac{\hat{q}+1}{\hat{q}+2} \text{ if } 0 \leq \hat{\tau} \leq \frac{\hat{q}}{2(\hat{q}+2)}, \qquad \hat{q}\frac{2(\hat{\tau}+1)}{3} - \frac{2\hat{\tau}}{3} \text{ if } \hat{\tau} \geq \frac{\hat{q}}{2(\hat{q}+2)}. \tag{75}$$

Now we have chosen c depending only on quantities determined by Γ, and we have arrived at the following estimate:

$$\mathbf{N}(U, V; z) - \mathcal{E}(U, V; z) \ll_\Gamma n(z) \cdot \begin{cases} \prod_j V_j^{\frac{\hat{q}+1}{\hat{q}+2}} & \text{if } \hat{\tau} \le \frac{\hat{q}}{2(\hat{q}+2)}, \\ \prod_j V_j^{\frac{2}{3}(\hat{\tau}+1)-\frac{2\hat{\tau}}{3\hat{q}}} & \text{if } \hat{\tau} \ge \frac{\hat{q}}{2(\hat{q}+2)}. \end{cases} \tag{76}$$

If there are totally exceptional eigenvalues, the explicit sum $\mathcal{E}(U, V; z)$ in (46) contains the corresponding terms, which are of size $O\big(n(z)\prod_j V_j^{(1/2)+\hat{\tau}}\big)$. Since $\tau_{\ell,j} \le \hat{\tau}$ for all exceptional coordinates, these terms are swallowed by the error term obtained for the case $\hat{\tau} \le \hat{q}/(2(\hat{q}+2))$.

We have thus obtained the following asymptotic result for the counting function:

Theorem 5.4. Let Γ be an irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})^d$ with $d \in \mathbb{N}$. Let $\hat{\tau}$ be the quantity in (3), measuring the spectral gap. For each $j \in \{1, \ldots, d\}$, we fix $\alpha_j \in [0, 1)$, and denote by V_j a large quantity going to ∞. The V_j are subject to the condition $\min_j V_j^{\hat{q}} = \prod_j V_j$ for a fixed number $\hat{q} \ge d$. Choose $U_j = \alpha_j V_j$ for each $j = 1, \ldots, d$.

Let $z \in \mathfrak{H}^d$. The number $\mathbf{N}(U, V; z)$ of $\gamma \in \Gamma$ such that $U_j \le u\big((\gamma z)_j, z_j\big) \le V_j$ for all j, with $u(\cdot, \cdot)$ as in (12), satisfies

$$\mathbf{N}(U, V; z) = \frac{(4\pi)^d}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)} \prod_{j=1}^{d} (V_j - U_j) + O_{\Gamma,\alpha}\left(n(z) \prod_{j=1}^{d} V_j^{(\hat{q}+1)/(\hat{q}+2)} \right) \tag{77}$$

if $\hat{\tau} \le \frac{\hat{q}}{2(\hat{q}+2)}$, and

$$\begin{aligned} \mathbf{N}(U, V; z) = {} & \frac{(4\pi)^d}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)} \prod_{j=1}^{d} (V_j - U_j) + \sum_{\ell \ge 1,\ \forall_j\ \tau_{\ell,j} \in (0, \frac{1}{2})} \left|\psi_\ell(z)\right|^2 \\ & \cdot \prod_{j=1}^{d} \frac{\sqrt{\pi}\, 2^{1+2\tau_{\ell,j}}\, \Gamma(\tau_{\ell,j})}{\Gamma\big(\frac{3}{2} + \tau_{\ell,j}\big)} \big(V_j^{\frac{1}{2}+\tau_{\ell,j}} - U_j^{\frac{1}{2}+\tau_{\ell,j}}\big) \\ & + O_{\Gamma,\alpha}\left(n(z) \prod_{j=1}^{d} V_j^{\frac{2}{3}(\hat{\tau}+1)-2\hat{\tau}/3\hat{q}} \right) \end{aligned} \tag{78}$$

if $\hat{\tau} \ge \hat{q}/(2(\hat{q}+2))$. The factor $n(z)$ is as in (19). □

By $O_{\Gamma,\alpha}$, we indicate that the implicit constant depends on the group and on the choice of the constants α_j.

Note that even if there is no spectral gap ($\hat{\tau} = 1/2$) the error term is still smaller than the main term.

In the special case when all V_j are equal to the same quantity V, we have $\hat{q} = d$. The relation (12) implies that the condition $\mathrm{dist}\big((\gamma z)_j, z_j\big) \leq T$ is equivalent to $u\big((\gamma z)_j, z_j\big) \leq (1/4)e^T\big(1 + \mathrm{O}(e^{-T})\big)$. Thus, we obtain Theorem 1.2.

5.8 Asymptotic estimate, case $E \neq \emptyset$

We turn to the case when both parts of the partition $\{1, \dots, d\} = \mathcal{Q} \sqcup E$ are non-empty. This implies $d \geq 2$; in this situation it is known that $\hat{\tau} < 1/2$, as discussed in the introduction.

We have obtained the following estimate for the error terms:

$$\ll_{\Gamma, c, E, \alpha} \; n(z)\big(V_{\min}^{\vartheta-1} + Y_E\big) \prod_{j \in \mathcal{Q}} V_j + n(z) Y_E^{-\frac{1}{2}\#E} \prod_{j \in \mathcal{Q}} V_j^{\max(1-\frac{1}{2}\vartheta, \frac{1}{2}+\hat{\tau}, \frac{1}{2}+c)}. \tag{79}$$

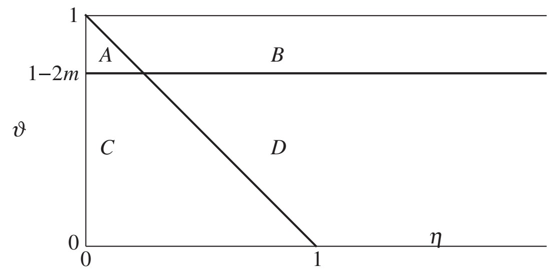
We try to choose ϑ and Y_E optimally, depending on the V_j, $j \in \mathcal{Q}$. The parameter $c \in \big(0, 1/2\big)$ satisfies $c < \hat{\tau}$ if $\hat{\tau} > 0$, and can be further adapted to the situation, but is not allowed to depend on the V_j with $j \in \mathcal{Q}$.

We take $x = \log V_{\min}$ as the large variable. Then $\prod_{j \in \mathcal{Q}} V_j = e^{\hat{q}x}$, with $\hat{q} \geq q$ fixed. We assume that $Y_E = V_{\min}^{-\eta}$ with some $\eta > 0$ is a sensible choice. To simplify the formulas, we work for the moment with the notations $e = \#E$, $m = \max(c, \hat{\tau})$. So $0 < m \leq 1/2$. We try to choose $\vartheta \in [0, 1]$ and $\eta \geq 0$ such that the following quantity is minimal:

$$M(\eta, \vartheta) \;=\; \max\left(\vartheta - 1 + \hat{q},\, \hat{q} - \eta,\, \frac{e}{2}\eta + \hat{q}\left(-\frac{\vartheta}{2}\right),\, \frac{e}{2}\eta + \hat{q}\left(\frac{1}{2} + m\right)\right). \tag{80}$$

We allow for the moment ϑ and η to assume boundary values. If we end up with an optimal choice on the boundary, we will see how to handle the problem of satisfying the conditions in Section 5.2.

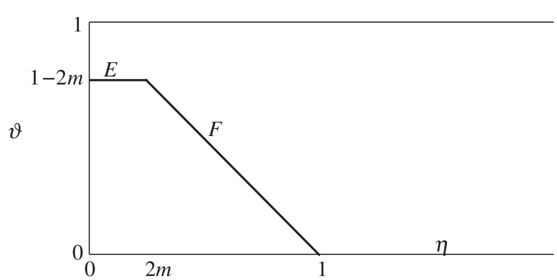
The lines $\vartheta + \eta = 1$ and $\vartheta = 1 - 2m$ give four subsets of the region in which (η, ϑ) varies:



We have

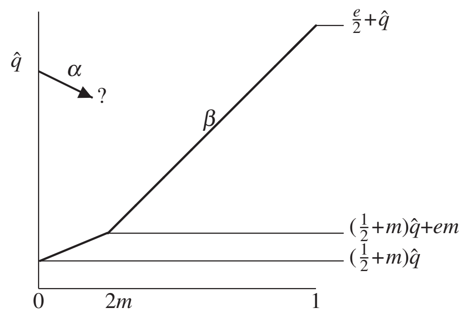
$$M(\eta, \vartheta) = \begin{cases} \max\left(\hat{q} - \eta,\ \frac{e}{2}\eta + \hat{q}(\frac{1}{2} + m)\right) & \text{on } A, \\ \max\left(\hat{q} - 1 + \vartheta,\ \frac{e}{2}\eta + \hat{q}(\frac{1}{2} + m)\right) & \text{on } B \\ \max\left(\hat{q} - \eta,\ \frac{e}{2}\eta + \hat{q}(1 - \frac{\vartheta}{2})\right) & \text{on } C, \\ \max\left(\hat{q} - 1 + \vartheta,\ \frac{e}{2}\eta + \hat{q}(1 - \frac{\vartheta}{2})\right) & \text{on } D. \end{cases}$$

On B and D, these expressions for $M(\eta, \vartheta)$ contain η only once. So in the search for optimal values, we can take η minimal in these cases. This brings them into the cases C and D, respectively. In region C, the variable ϑ occurs only once. So it makes sense to take it optimal, that is, $\vartheta = 1 - 2m$ if $0 \leq \eta \leq 2m$ and $\vartheta = 1 - \eta$ if $2m \leq \eta \leq 1$. This reduces our search for the optimum to the lines E and F in the following figure.



Thus, we are left with a one-dimensional problem: find the minimum for $0 \leq \eta \leq 1$ of the maximum of the two functions $\alpha(\eta) = \hat{q} - \eta$ and

$$\beta(\eta) = \begin{cases} \frac{e}{2}\eta + \hat{q}\left(\frac{1}{2} + m\right) & \text{for } 0 \leq \eta \leq 2m, \\ \frac{1}{2}\left(e + \hat{q}\right) + \frac{1}{2}\hat{q} & \text{for } 2m \leq \eta \leq 1. \end{cases}$$



Since $\big((1/2)+m\big)\hat q \le \hat q \le \hat q + (e/2)$, the graphs of α and β intersect for some value of η in $[0, 1]$. This gives the value of the optimum we look for. This leads to the optimal value

$$\begin{aligned} M\left(\frac{\hat q}{\hat q+e+2}, \frac{e+2}{\hat q+e+2}\right) &= \frac{\hat q+e+1}{\hat q+e+2}\hat q \qquad \text{if } m \le \frac{\hat q}{2(\hat q+e+2)}, \\ M\left(\frac{(1-2m)\hat q}{e+2}, 1-2m\right) &= \frac{e+1+2m}{e+2}\hat q \qquad \text{if } m \ge \frac{\hat q}{2(\hat q+e+2)}. \end{aligned}$$

If $\hat\tau > 0$, we can just replace m by $\hat\tau$ in the results. If $\tau = 0$, we take $0 < c < \#Q/(2(d+2))$. This depends on d, hence on Γ, and on the partition in Q and E. We now have

$$\frac{\hat q}{2(\hat q+e+2)} = \frac{1}{2} - \frac{2+e}{2(\hat q+2+e)} \ge \frac{1}{2} - \frac{2+e}{2(2+e+\#Q)} = \frac{\#Q}{2(d+e)} > c = m.$$

So we can apply the estimate for $m \le \hat q/(2(\hat q+e+2))$.

Thus, we obtain the following estimates for the error terms:

$$\begin{aligned} &\mathrm{O}_{\Gamma,E}\left(n(z)\prod_{j\in Q} V_j^{(\hat q+1+\#E)/(\hat q+2+\#E)}\right) \qquad \text{if } \hat\tau \le \frac{\hat q}{2(\hat q+2+\#E)}, \\ &\mathrm{O}_{\Gamma,E}\left(n(z)\prod_{j\in Q} V_j^{(1+2\hat\tau+\#E)/(2+\#E)}\right) \qquad \text{if } \frac{\hat q}{2(\hat q+2+\#E)} \le \hat\tau < \frac{1}{2}. \end{aligned} \tag{81}$$

Each exceptional term in $\mathcal{E}(U, V; z)$ in (46) contributes at most $\prod_{j\in Q} V_j^{(1/2)+\hat\tau}$ and is absorbed by the error term in (81). We are left with the term corresponding to the constant function. See (92) in Lemma 6.1 for its simple form.

Thus, we have obtained the following asymptotic result:

Theorem 5.5. Let Γ be an irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})^d$ with $d \geq 2$. We partition the set $\{1, \ldots, d\}$ into two disjoint non-empty subsets Q and E. For each $j \in E$, we fix a bounded interval $[U_j, V_j] \subset [0, \infty)$. For each $j \in Q$, we fix a constant $\alpha_j \in [0, 1)$, we put $U_j = \alpha_j V_j$, where V_j is a large quantity tending to infinity. We assume that $\min_{j\in Q} V_j^{\hat{q}} = \prod_{j\in Q} V_j$ for some fixed real number $\hat{q} \geq \#Q$.

Let $z \in \mathfrak{H}$. The number $\mathbf{N}(U, V; z)$ in (13) satisfies

$$\begin{aligned} \mathbf{N}(U, V; z) &= \frac{(4\pi)^d}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)} \prod_{j=1}^{d} (V_j - U_j) \\ &+ \begin{cases} \mathrm{O}_{\Gamma,E,\alpha}\left(n(z) \prod_{j\in Q} V_j^{(\hat{q}+1+\#E)/(\hat{q}+2+\#E)}\right) & \text{if } \hat{\tau} \leq \frac{\hat{q}}{2(\hat{q}+2+\#E)}, \\ \mathrm{O}_{\Gamma,E,\alpha}\left(n(z) \prod_{j\in Q} V_j^{(1+2\hat{\tau}+\#E)/(2+\#E)}\right) & \text{if } \frac{\hat{q}}{2(\hat{q}+2+\#E)} \leq \hat{\tau} < \frac{1}{2}. \end{cases} \end{aligned} \tag{82}$$

The implicit constants in the estimates depend on the discrete group Γ, on the partition $\{1, \ldots, d\} = Q \sqcup E$, on the constants α_j with $j \in Q$, and on the choice of the intervals $[U_j, V_j)$ for $j \in E$. □

As mentioned in the introduction, it is known that $\hat{\tau} < 1/2$ if $d \geq 2$. We note that the presence of totally exceptional eigenvalues for Γ has no explicit influence on this asymptotic formula. The size of he spectral gap does influence the quality of the error term only if it is larger than $\hat{q}/(2(\hat{q}+2+\#E))$.

For Theorem 1.1 in the introduction, we take $\alpha_j = 0$, hence $U_j = 0$ for all $j \in Q$, and all V_j with $j \in Q$ equal. With the relation in (12) between $u\big((\gamma z)_j, z_j\big) \in [U_j, V_j)$ and the hyperbolic distance $\mathrm{dist}\big((\gamma z)_j, z_j\big) \in [A_j, B_j)$, the main term takes the form

$$\frac{4^{\#E}\pi^d}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}^d)}\, e^{dT} \prod_{j\in E} \frac{e^{B_j} + e^{-B_j} - e^{A_j} - e^{-A_j}}{4},$$

in the notations of Theorem 1.1, which leads to the main term in the asymptotic formula in that theorem. For the error terms, we use that for equal V_j for $j \in Q$, the parameter $\hat{q}$ is equal to $q = \#Q$.

6 Estimates of Selberg Transforms

Here we collect and prove the estimates of Selberg transforms that we have used. This can be done factor by factor. So in this section, we work on $\mathfrak{H}$, we do not use an index j, and we denote real numbers by U and V.

6.1 Integral representations

The results we need are given in Lemmas 6.1, 6.2, and 6.3. To derive these lemmas, we start with an arbitrary measurable compactly supported function k on $[0,\infty)$ with values in $[0,1]$. By the definitions in Section 4.1, we have the following integral representation for the Selberg transform h of k:

$$\begin{aligned} h(\tau) &= 2\int_{r=-\infty}^{\infty} e^{r\tau} \int_{u=\sinh^2 r/2}^{\infty} k(u)\,\frac{\mathrm{d}u}{\sqrt{u-\sinh^2 r/2}}\,\mathrm{d}r \\ &= 4\int_0^\infty \cosh r\tau \int_{u=\sinh^2 r/2}^{\infty} k(u)\,\frac{\mathrm{d}u}{\sqrt{u-\sinh^2 r/2}}\,\mathrm{d}r. \end{aligned} \tag{83}$$

The inner integral gives a non-negative compactly supported function. So $h(0)\ge |h(it)|$ for $t\in\mathbb{R}$, which gives

$$|h(it)| \le h(0) \quad (t\in\mathbb{R}), \qquad \tau\mapsto h(\tau) \text{ is increasing on } \left[0,\tfrac12\right]. \tag{84}$$

(For the latter, we use that $\tau\mapsto\cosh r\tau$ is increasing.)

We interchange the order in the double integration, and obtain

$$h(\tau) = 2\int_{u=0}^{\infty} k(u)\int_{r\in\mathbb{R},\ \sinh^2 r/2\le u} e^{r\tau}\left(u-\sinh^2 r/2\right)^{-1/2}\mathrm{d}r\,\mathrm{d}u. \tag{85}$$

Following the approach in [22], we define $x=x(u)\ge 1$ by $2+4u=x+x^{-1}$, hence $x(u)=1+2u+2\sqrt{u+u^2}$. The condition $\sinh^2(r/2)\le u$ amounts to $|r|\le\log x$. With the substitution $e^r=x^{-1}\bigl(1+y(x^2-1)\bigr)$ the inner integral equals

$$\begin{aligned} &= \int_{-\log x}^{\log x} e^{r\tau}\bigl((x+x^{-1}-e^r-e^{-r})/4\bigr)^{-1/2}\,\mathrm{d}r \\ &= 2\,x^{\frac12-\tau}\int_0^1 \bigl(1+y(x^2-1)\bigr)^{\tau-\frac12}\bigl(y(1-y)\bigr)^{-\frac12}\,\mathrm{d}y && (86) \\ &= 2\pi\,x^{\frac12-\tau}\,{}_2F_1\bigl(\tfrac12-\tau,\tfrac12;1;1-x^2\bigr), && (87) \end{aligned}$$

by the standard integral representation of the hypergeometric series in [3], Section 2.1.3, (10). Let us work under the standing assumption that $0\le\operatorname{Re}\tau\le 1/2$.

From (86), we obtain the bound $O\big(x^{\mathrm{Re}\,\tau-(1/2)}\big)$ for the inner integral in (85). Hence if $\mathrm{Supp}(k) \subset [A, B]$, then

$$\begin{aligned} h(\tau) &\ll \int_{x(A)}^{x(B)} x^{2\,\mathrm{Re}\,\tau-\frac{1}{2}}\,\mathrm{d}x \ll \frac{x(B)^{\mathrm{Re}\,\tau+\frac{1}{2}} - x(A)^{\mathrm{Re}\,\tau+\frac{1}{2}}}{\mathrm{Re}\,\tau+\frac{1}{2}} \\ &= (B-A)\left(1+2T+2\sqrt{T^2+T}\right)^{\mathrm{Re}\,\tau-\frac{1}{2}} \qquad \text{with } A \le T \le B \\ &\ll \begin{cases} (B-A)\big(1+A^{\mathrm{Re}\,\tau-\frac{1}{2}}\big) & \text{if } A > 0, \\ 1+B^{\mathrm{Re}\,\tau+\frac{1}{2}} & \text{if } A = 0 \text{ and } B > 0. \end{cases} \end{aligned} \tag{88}$$

Proceeding with (87), we get

$$\begin{aligned} h(\tau) &= \pi \int_1^\infty k\big((x+x^{-1}-2)/4\big)\, x^{-\frac{3}{2}-\tau}\,(x^2-1) \\ &\qquad \cdot {}_2F_1\big(\tfrac{1}{2}-\tau, \tfrac{1}{2}; 1; 1-x^2\big)\,\mathrm{d}x \qquad (89) \\ &= \pi \int_1^2 k\big((x+x^{-1}-2)/4\big)\, x^{-\frac{3}{2}-\tau}\,(x^2-1) \qquad (90) \\ &\qquad \cdot {}_2F_1\big(\tfrac{1}{2}-\tau, \tfrac{1}{2}; 1; 1-x^2\big)\,\mathrm{d}x \\ &\quad + \sqrt{\pi} \int_2^\infty k\big((x+x^{-1}-2)/4\big)\, x^{-\frac{3}{2}}\,(x^2-1)^{\frac{1}{2}} \\ &\qquad \cdot \sum_{\pm} \frac{\Gamma(\pm\tau)}{\Gamma\big(\frac{1}{2}\pm\tau\big)}\, x^{\pm\tau}\, {}_2F_1\big(\tfrac{1}{2}, \tfrac{1}{2}; 1\mp\tau; \tfrac{1}{1-x^2}\big)\,\mathrm{d}x. \end{aligned}$$

(For (90), we have used a Kummer relation, [3], Section 2.9, (34), (10), and (13).)

Let us use (90) in the case when $\mathrm{Supp}(k) \subset [A, B]$. For the part of $[A, B]$ corresponding to a subinterval $x \in [1, 2]$, we have the estimates in (88). We now estimate the integral over an interval $[A, B]$ with $x(A) \ge 2$ by

$$\sum_{\pm} \left| \frac{\Gamma(\pm\tau)}{\Gamma(\frac{1}{2}\pm\tau)} \right| \int_{x(A)}^{x(B)} x^{-\frac{1}{2}\pm\mathrm{Re}\,\tau} \left| {}_2F_1\big(\tfrac{1}{2}, \tfrac{1}{2}; 1\mp\tau; \tfrac{1}{1-x^2}\big)\right| \mathrm{d}x.$$

Uniformly for τ in a compact set T, we find an estimate by

$$\sum_{\pm} \left| \frac{\Gamma(\pm\tau)}{\Gamma(\frac{1}{2}\pm\tau)} \right| (B-A)\, A^{\pm\,\mathrm{Re}\,\tau-\frac{1}{2}}. \tag{91}$$

Note that this estimate is bad if τ is near 0. To handle a neighborhood of $\tau = 0$, we consider the second integral in (90) as a holomorphic function of the complex variable τ.

Table 3 Bounds for $h(\tau)$ under the assumption that $\mathrm{Supp}(k) \subset [A, B]$. These bounds depend implicitly on $c \in (0, 1/2)$

	i) $c \le \tau \le 1/2$	ii) $\lvert\tau\rvert \le c$	iii) $\tau = it \in i\mathbb{R} \setminus (-c, c)$
a) $1 \le A < B$	$(B-A)A^{\tau-(1/2)}$	$(B-A)A^{c-(1/2)}$	$\lvert t\rvert^{-1/2}(B-A)A^{-1/2}$
b) $0 < A < B \le 1$	$(B-A)A^{-1/2}$		
c) $0 = A < B \le 1$	$B^{1/2}$		
d) $0 < A < B$	$(B-A) \cdot \max(A^{\tau-(1/2)}, A^{-1/2})$	$(B-A) \cdot \max(A^{c-(1/2)}, A^{-1/2})$	$(B-A)A^{-1/2} \cdot \max(1, \lvert t\rvert^{-1/2}B)$
e) $0 = A < B$	$1 + B^{\tau+(1/2)}$	$1 + B^{c+(1/2)}$	$1 + \lvert t\rvert^{-1/2}B$

It is holomorphic at $\tau = 0$, since the contributions of $\Gamma(\tau)$ and $\Gamma(-\tau)$ cancel each other. If $\mathrm{Supp}(k)$ is contained in $[A, B]$ with $x(A) \ge 2$, we find for $|\tau| = c$ with a small $c > 0$ by the reasoning that led to the estimate in (91) a bound

$$\mathrm{O}_c\big((B-A)A^{c-\frac{1}{2}}\big).$$

By holomorphy, this bound extends to $|\tau| \le c$. Thus, if $\mathrm{Supp}(k) \subset [A, B]$ with $x(A) \ge 2$, and if $|\tau| \le c$, then

$$h(\tau) \ll_c (B-A)\, A^{c-\frac{1}{2}}.$$

For $\tau = it \in i\mathbb{R}$ with $x \ge 2$, we use that

$$\Big|{}_2F_1\big(\tfrac{1}{2}, \tfrac{1}{2}; 1 \pm it; \tfrac{1}{1-x^2}\big)\Big| \le {}_2F_1\big(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{1-x^2}\big).$$

By Stirling's formula, we get a bound $\big(1+|t|\big)^{-1/2}(B-A)\, A^{-(1/2)}$ uniformly on $|t| \ge c$.

In Table 3, we have combined these results.

We will use these estimates repeatedly in the proofs of the following lemmas. In some cases, we shall return to the integral representations.

6.2 Lemmas for Selberg transforms

First we consider the Selberg transform $\eta(U, V; \tau)$ in (47) of the characteristic function χ of a bounded interval $[U, V) \subset [0, \infty)$.

Lemma 6.1. The map $\tau \mapsto \eta(U, V; \tau)$ is positive and increasing on $\big[0, 1/2\big]$.

$$\eta\big(U, V; \tfrac{1}{2}\big) = 4\pi(V - U), \tag{92}$$

$$\big|\eta(U, V; it)\big| \le \eta(U, V; 0) \qquad (t \in \mathbb{R}). \tag{93}$$

Moreover, if $c \in \big(0, 1/2\big)$ is fixed, then we have uniformly for $c \le \tau \le 1/2$ the estimate

$$\eta(U, V; \tau) = \sqrt{\pi}\,\frac{2^{2\tau+1}\Gamma(\tau)}{\Gamma(\frac{3}{2}+\tau)}\left(V^{\tau+\frac{1}{2}} - U^{\tau+\frac{1}{2}}\right) + \mathrm{O}_c\big(V^{-\tau+\frac{1}{2}}\big) \qquad (V \to \infty). \tag{94}$$

□

If the difference $V - U$ is small, then the O-term may be larger than the explicit term in (94).

Proof. We apply the computations in Section 6.1 to the characteristic function χ of $[U, V)$. In (84), we find the first statement of the lemma, and (93). Taking $\tau = 1/2$ in (86) gives (92).

For (94), we need an asymptotic formula, not an estimate. We use (90). We separate the cases $x(U) \ge 2$ and $x(U) \le 2$, which correspond to $U \ge 1/8$ and $U \le 1/8$, respectively, since $2 + 4u = x(u) + 1/x(u)$.

If $x(U) \ge 2$, we need only the integral over $[2, \infty)$. Writing ${}_2F_1\big((1/2) - \tau, 1/2; 1; 1 - x^2\big) = 1 + \mathrm{O}(x^{-2})$ as $x \to \infty$, we get, uniformly for $\tau \in \big[c, 1/2\big]$:

$$\begin{aligned}\sum_{\pm} \pi \frac{\Gamma(\pm\tau)}{\Gamma(\frac{1}{2} \pm \tau)} \int_{x(U)}^{x(V)} x^{\pm\tau-\frac{1}{2}}\big(1 + \mathrm{O}_c(x^{-2})\big)\,\mathrm{d}x \\ = \sum_{\pm} \pi \frac{\Gamma(\pm\tau)}{\Gamma(\frac{3}{2} \pm \tau)}\Big(x(V)^{\frac{1}{2}\pm\tau} - x(U)^{\frac{1}{2}\pm\tau} + \mathrm{O}_c\big(x(U)^{\pm\tau-\frac{3}{2}}\big)\Big).\end{aligned}$$

For $T \ge 1$, we have $x(T) = 4T + \mathrm{O}(1)$. So the main term with $\pm = +$ gives the explicit term in (94). The other terms give $\mathrm{O}(1) + \mathrm{O}(U^{\tau-(3/2)}) + \mathrm{O}(V^{(1/2)-\tau}) = \mathrm{O}\big(V^{(1/2)-\tau}\big)$.

If $x(U) \le 2$, we get from $x \in [2, \infty)$ the contribution

$$\pi \frac{\Gamma(+\tau)}{\Gamma(\frac{3}{2}+\tau)}\big(V^{\tau+\frac{1}{2}} - \mathrm{O}(1)\big) + \mathrm{O}(V^{\frac{1}{2}-\tau}).$$

We add to it the contribution

$$\int_{x(U)}^{2} x^{-\frac{3}{2}-\tau}(x^2-1)\,{}_2F_1\big(\tfrac{1}{2}-\tau, \tfrac{1}{2}; 1; 1-x^2\big)\,\mathrm{d}x \ll \int_1^2 \mathrm{O}(1) = \mathrm{O}(1),$$

and obtain (94) in this case as well. ■

Next we consider functions approximating the characteristic function of $[U, V)$, satisfying the following conditions.

$$\begin{aligned}
&k \in C_c^\infty[0,\infty)\,, \qquad 0 \le k \le 1,\\
&\exists_{Y>0} \text{ such that } 2Y \le U \text{ if } U > 0,\ 2Y \le V - U, \text{ and}\\
&k = 1 \text{ on } \begin{cases} [U+Y, V-Y] & \text{if } U > 0,\\ [0, V-Y] & \text{if } U = 0,\end{cases}\\
&k = 0 \text{ on } \begin{cases} [0, U-Y] \cup [V+Y, \infty) & \text{if } U > 0,\\ [V+Y, \infty) & \text{if } U = 0.\end{cases}
\end{aligned} \tag{95}$$

Lemma 6.2. The Selberg transform h in (4.1) of a function $k \in C^\infty[0,\infty)$ satisfying the conditions in (95) has the following properties:

a) $4\pi(V - U - 2Y) \le h\big(1/2\big) \le 4\pi(V - U + 2Y)$.

b) If $V < 1$, then

$$\left|h(\tau) - h\left(\frac{1}{2}\right)\right| \ll V^{3/2}\left|\tfrac{1}{2} - \tau\right|$$

for all τ with $0 \le \operatorname{Re}\tau \le 1/2$.

c) For each $c \in \big(0, 1/2\big)$, the difference with the Selberg transform $\eta(U, V; \tau)$ of the characteristic function of $[U, V)$ satisfies the estimate

$$\eta(U, V; \tau) - h(\tau) \ll_c \begin{cases} Y\max\big(V^{\tau-\frac{1}{2}}, U^{-\frac{1}{2}}\big) & \text{if } U > 0,\\ Y\max\big(V^{-\frac{1}{2}}, V^{\tau-\frac{1}{2}}\big) & \text{if } U = 0,\end{cases}$$

uniformly in $\tau \in \big[c, 1/2\big]$. □

Proof. Part a) follows by a comparison of k with the characteristic functions of the intervals $[U + Y, V - Y)$ and $[U, V)$, and an application of (92) in Lemma 6.1.

For b), we use (24), and note that $y^s - 1 = s(y-1)\,(1+\xi(y-1))^{s-1}$ for some $\xi = \xi_{s,y} \in (0,1)$ to obtain

$$\left|h(\tau) - h\!\left(\frac{1}{2}\right)\right| \le \left|\frac{1}{2} - \tau\right| \int_{\mathfrak{H}} k\big(u(z,i)\big)\,|y-1|\,\big(1+\xi(y-1)\big)^{-\frac{1}{2}-\operatorname{Re}\tau}\,\mathrm{d}\mu(z).$$

For small V, the values of y that occur in the integral are between $1 - \mathrm{O}\big(\sqrt{V}\big)$ and $1 + \mathrm{O}\big(\sqrt{V}\big)$. Thus, the integral is bounded by $\mathrm{O}\big(\sqrt{V}\big)\int_{\mathfrak{H}} k\big(u(z,i)\big)\,\mathrm{d}\mu(z)$, which gives b).

For c), we apply the estimate i) d) in Table 3 to a function with support in the union of the intervals $[U-Y, U+Y]$ and $[V-Y, V+Y]$. For $U > 0$, we find:

$$Y\max(U^{\tau-\frac{1}{2}}, U^{-\frac{1}{2}}) + Y\max\big((V-Y)^{\tau-\frac{1}{2}}, (V-Y)^{-\frac{1}{2}}\big) \ll Y\max(V^{\tau-\frac{1}{2}}, U^{-\frac{1}{2}}).$$

If $U = 0$, we have only the contribution of $[V-Y, V+Y]$. ■

Lemma 6.3. The Selberg transform h of a function $k \in C^\infty[0,\infty)$ satisfying the conditions in (95) has the following properties:

a) $h(\tau) = \eta(U, V; \tau) + \mathrm{O}(Y)$ for $\tau \in \big[0, 1/2\big]$.

b) Let $c \in \big(0, 1/2\big)$. Then we have, uniformly for $\tau \in i[-c,c] \cup (0,c]$:

$$h(\tau) \ll_c \begin{cases} (V-U)U^{c-\frac{1}{2}} & \text{if } 1 \le U \le V, \\ V^{c+\frac{1}{2}} & \text{if } U = 0,\ V \ge 1, \end{cases}$$

and, without dependence on c:

$$h(\tau) \ll V.$$

c) Let $c \in \big(0, 1/2\big)$ and take $l \in \mathbb{N}$. Then we have for each $\delta > 0$, uniformly for $t \in \mathbb{R} \smallsetminus (-c, c)$:

$$h(it) \ll_{c,l} \begin{cases} Y\,\|k^{(l)}\|_\infty \max\big(V^{l-\frac{1}{2}}, U^{-\frac{1}{2}}\big)\,|t|^{-l-\frac{1}{2}} & \text{if } \delta \le U < V, \\ Y\,\|k^{(l)}\|_\infty \max\big(V^{-\frac{1}{2}}, V^{l-\frac{1}{2}}\big)\,|t|^{-l-\frac{1}{2}} & \text{if } U = 0,\ V > \delta. \end{cases}$$ □

We note that we have stated what we need, not the best estimate one might prove by separating more cases. In the proof, we will see that in c) we have to avoid intervals with $U \in (0, \delta)$, to be able to apply an asymptotic estimate for hypergeometric functions.

Proof. Equation (24) shows that at real values of τ the Selberg transform is monotonous on real-valued functions. In the case $U > 0$, both the characteristic function of $[U, V]$ and the function k are between the characteristic functions χ_1 of $[U+Y, V-Y]$ and $[U-Y, V+Y]$. The statement of part a) follows by applying Lemma 6.1 to the difference $\chi_2 - \chi_1$, which is the sum of the characteristic functions of two disjoint intervals of length $\mathrm{O}(Y)$. The case $U = 0$ is proved in a similar way.

The first estimate in b) can be read off from Table 3. The bound $h(\tau) \ll V$ follows from a) and (92) in Lemma 6.1.

For d), we modify the discussion in Section 6.1. By the smoothness of k, we find in (4.1)

$$q(p) = \frac{(-1)^l \sqrt{\pi}}{\Gamma(l+\frac{1}{2})} \int_p^\infty k^{(l)}(u)\,(u-p)^{l-\frac{1}{2}}\,\mathrm{d}u \tag{96}$$

for each $l \in \mathbb{N}$. Proceeding as in (83), (85)–(87), and (89)–(90), we obtain

$$\begin{aligned} h(\tau) &= \frac{\sqrt{\pi}(-1)^l}{2^{2l}\Gamma(l+\frac{1}{2})} \int_{x=1}^\infty k^{(l)}\big((x+x^{-1}-2)/4\big) \int_{y=0}^1 x^{-\frac{3}{2}-\tau-l} \\ &\quad \cdot (x^2-1)^{2l+1} \big(1+y(x^2-1)\big)^{\tau-l-\frac{1}{2}} \big(y(1-y)\big)^{l-\frac{1}{2}}\,\mathrm{d}y\,\mathrm{d}x \qquad (97) \\ &= \frac{\pi(-1)^l}{2^{4l}\,l!} \int_1^\infty k^{(l)}\big((x+x^{-1}-2)/4\big)\, x^{-\frac{3}{2}-\tau-l}\,(x^2-1)^{2l+1} \\ &\qquad \cdot {}_2F_1\big(l+\tfrac{1}{2}-\tau, l+\tfrac{1}{2}; 2l+1; 1-x^2\big)\,\mathrm{d}x \\ &= \frac{\pi(-1)^l}{2^{4l}\,l!} \int_1^2 k^{(l)}\big((x+x^{-1}-2)/4\big)\, x^{-\frac{5}{2}-3l-\tau} \qquad (98) \\ &\qquad \cdot (x^2-1)^{2l+1}\,{}_2F_1\big(l+\tfrac{1}{2}+\tau, l+\tfrac{1}{2}; 2l+1; 1-x^{-2}\big)\,\mathrm{d}x \\ &\quad + \frac{\sqrt{\pi}(-1)^l}{2^{2l}} \int_2^\infty k^{(l)}\big((x+x^{-1}-2)/4\big) \sum_{\pm} \frac{\Gamma(\pm\tau)}{\Gamma(l+\frac{1}{2}\pm\tau)} x^{-\frac{3}{2}\pm\tau-l} \\ &\qquad \cdot (x^2-1)^{l+\frac{1}{2}}\,{}_2F_1\big(\tfrac{1}{2}-l, \tfrac{1}{2}+l; 1\mp\tau; (1-x^2)^{-1}\big)\,\mathrm{d}x. \end{aligned}$$

We apply this for $\tau = it \in i\mathbb{R}$ with $|t| \geq c$. Consider first an interval $[U_1, V_1]$ with $1 < x(U_1) < x(V_1) \leq 2$. Then we use the first integral in (98) to get a bound

$$\ll_l \|k^{(l)}\|_\infty \int_{x(U_1)}^{x(V_1)} \big|{}_2F_1\big(l+\tfrac{1}{2}+it, l+\tfrac{1}{2}; 2l+1; 1-x^{-2}\big)\big|\,\mathrm{d}x.$$

For $x \geq 1+\delta > 1$, we have by formulas (14) and (15) in Section 2.3.2 of [3]

$$ {}_2F_1\big(l+\tfrac{1}{2}, l+\tfrac{1}{2}+it; 2l+1; 1-x^{-2}\big) \ll_{l,\delta} |t|^{-\frac{1}{2}-l}. $$

This gives a bound

$$ \begin{aligned} &\mathrm{O}_{l,\delta}\big(\|k^{(l)}\|_\infty\big(x(V_1)-x(U_1)\big)\big)\\ &\qquad \ll \|k^{(l)}\|_\infty\,(V_1-U_1)\big(1+\frac{1}{\sqrt{V_1+V_1^2}+\sqrt{U_1+U_1^2}}\big)\\ &\qquad \ll \|k^{(l)}\|_\infty\,(V_1-U_1)V_1^{-\frac{1}{2}}. \end{aligned} $$

We stress that the use of δ is critical for the application of the asymptotic behavior from *loc. cit.* If we allow x to get down to 1, the implicit constant blows up.

For an interval $[U_2, V_2]$ with $x(U_2) \geq 2$, we can use the second integral in (98). The hypergeometric series shows that

$$ \big|{}_2F_1\big(\tfrac{1}{2}-l, \tfrac{1}{2}+l; 1\mp it; (1-x^2)^{-1}\big)\big| \leq {}_2F_1\big(\tfrac{1}{2}+l, \tfrac{1}{2}+l; 1; (1-x^2)^{-1}\big), $$

which is $\mathrm{O}_l(1)$ for $x \geq 2$. By Stirling's formula, we get an estimate by

$$ \mathrm{O}_l\left(\|k^{(l)}\|_\infty\,|t|^{-l-\frac{1}{2}}\int_{x(U_2}^{x(V_2)} x^{-\frac{1}{2}+l}\,\mathrm{d}x\right) \ll \|k^{(l)}\|_\infty\,|t|^{-\frac{1}{2}-l}(V_2-U_2)V_2^{l-\frac{1}{2}}. $$

If $U=0$, we have only to estimate the integral over $[V-Y, V+Y]$. This gives the bound $Y\|k^{(l)}\|_\infty \max(V^{-1/2}, V^{l-(1/2)})$. If $U \geq \delta$, we get from the interval $[U-Y, U+Y]$ the bound $Y\|k^{(l)}\|_\infty \max(U^{l+(1/2)}, U^{-1/2})$. Together with the bound for the interval $[V-Y, V+Y]$ we get the other bound in c) of the lemma. ∎

7 Spectral Theory

7.1 Spectral expansion

The pointwise convergence of the spectral expansion of sufficiently differentiable elements of $L^2(\Gamma\backslash\mathfrak{H}^d)$ in Theorem 4.1 is similar to well-known facts for the case $d=1$ (e.g., Theorems 4.7 and 7.4 in [13]), and for rank-one Lie groups (Lemma 2.2 in [21]). We sketch how to obtain the pointwise convergence in Theorem 4.1 in the present context.

We consider first the mechanism of the Selberg transform on $\mathfrak{H}$, given in Section 4.1. We replace $k \in C_c^\infty[0,\infty)$ by

$$r_s(u) \;=\; \frac{\Gamma(2s)}{4\pi\,\Gamma(s)} u^{-s} {}_2F_1(s,s;2s;-1/u), \tag{99}$$

with $\mathrm{Re}\, s > 1$. It has a logarithmic singularity at $u = 0$ and its support is not compact. Nevertheless, it determines a kernel function $(z,w) \mapsto r_s(u(z,w))$ on $\mathfrak{H}$ such that the corresponding convolution operator $R_s : f \mapsto R_s f$ is well defined for bounded $f \in C^\infty(\mathfrak{H})$. It is in fact the free space resolvent on $\mathfrak{H}$, and satisfies

$$R_s\big(\Delta - s + s^2\big) f \;=\; f. \tag{100}$$

See Section 1.9 of [13]. If $a, s \in \mathbb{C}$ both have real part larger than 1, the difference $r_{s,a} = r_s - r_a$ has no singularity at $u = 0$, and has the Selberg transform

$$h_{s,a}(t) \;=\; \frac{s - s^2 - a + a^2}{\big(t^2 + (s-\frac12)^2\big)\big(t^2 + (a-\frac12)^2\big)} \tag{101}$$

for $|\,\mathrm{Im}\, t| < \mathrm{Re}\, s - (1/2)$. Moreover, the resolvent equation gives on bounded functions in $C^\infty(\mathfrak{H})$ such that Δf and $\Delta^2 f$ are also bounded:

$$\begin{aligned} L_{r_{s,a}}(\Delta - s + s^2)(\Delta - a + a^2) f \;&=\; (R_s - R_a)(\Delta - s + s^2)(\Delta - a + a^2) f \\ &= (s - s^2 - a + a^2) f. \end{aligned} \tag{102}$$

Taking $s, a \in \mathbb{C}^d$ with $\mathrm{Re}\, s_j > 1$, $\mathrm{Re}\, a_j > 1$, $s_j \neq a_j$ for all j, we form

$$k_{s,a}\big(u(z,w)\big) = \prod_j r_{s_j,a_j}\big(u(z_j,w_j)\big) \;=\; \prod_j \Big(r_{s_j}\big(u(z_j,w_j)\big) - r_{a_j}\big(u(z_j,w_j)\big)\Big),$$

and obtain a kernel operator $\mathcal{K}_{s,a}$ on $\Gamma\backslash\mathfrak{H}^d$ given by the kernel function

$$R_{s,a}(z,w) \;=\; \sum_{\gamma\in\Gamma} k_{s,a}(u(\gamma z, w)).$$

We apply this operator to differentiable bounded functions f on $\Gamma\backslash\mathfrak{H}^d$ for which the derivatives $\Delta_1^{b_1}\cdots\Delta_d^{b_d} f$ are bounded for all choices $b_j \in \{0,1,2\}$.

$$f = \prod_j \frac{1}{(s_1-s_1^2-a_1+a_1^2)\cdots(s_d-s_d^2-a_d+a_d^2)}\,\mathcal{K}_{s,a} f_1,$$
$$f_1 = (\Delta_1 - s_1 + s_1^2)\cdots(\Delta_d - s_d + s_d^2)(\Delta_1 - a_1 + a_1^2)\cdots(\Delta_d - a_d + a_d^2)\ f.$$

Now we note that the values $\mathcal{K}_{s,a} f(z)$ are given by a scalar product in $L^2(\Gamma\backslash\mathfrak{H}^d)$:

$$\mathcal{K}_{s,a} f_1(z) = \left\langle f_1, \overline{R_{s,a}(z,\cdot)}\right\rangle = \left\langle f_1, R_{\bar{s},\bar{a}}(z,\cdot)\right\rangle.$$

Taking the scalar product is a continuous operation $L^2(\Gamma\backslash\mathfrak{H}^d) \longrightarrow \mathbb{C}$. Thus, using (38) and (101) we conclude that the L^2-expansion of f_1 is transformed in a pointwise expansion:

$$\begin{aligned}\mathcal{K} f_1(z) &= \sum_\ell h_{s,a}(t_\ell)\,\psi_\ell(z)\,a_\ell^{(1)} \\ &\quad + \sum_\kappa 2c_\kappa \sum_{\mu\in\mathcal{L}_\kappa}\int_0^\infty \overline{h_{s,a}(t+\mu)}\,E(\kappa;it,i\mu;z)\,b_{\mu,\kappa}^{(1)}(t)\,\mathrm{d}t; \\ h_{s,a}(t) &= \prod_j \frac{s_j - s_j^2 - a_j + a_j^2}{\left(t_j^2 + (s_j-\frac12)^2\right)\left(t_j^2+(a_j-\frac12)^2\right)},\end{aligned} \tag{103}$$

where $a_\ell^{(1)} = \langle f_1, \psi_\ell\rangle$ and $b_{\kappa,\mu}^{(1)}(t) = \int_{\Gamma\backslash\mathfrak{H}^d} f_1(z)\overline{E(\kappa;it,i\mu;z)}\,\mathrm{d}\mu(z)$. Since $R_{s,a}(z,w)$ is bounded for $z\in\mathfrak{H}^d$ uniformly in z in compact sets, the convergence of the expansion (103) is also uniform on compact sets.

For two times differentiable functions in $L^2(\Gamma\backslash\mathfrak{H}^d)$, application of Δ_j changes a_ℓ in the spectral expansion into $\left((1/4)-\mu_{\ell,j}^2\right)a_\ell$, and $b_{\kappa,\mu}(t)$ into $\left((1/4)+(t+\mu_j)^2\right)b_{\kappa,\mu}(t)$. Taking this into account, the pointwise spectral expansion of f takes the form

$$f(z) = \sum_\ell \psi_\ell(z)\,a_\ell + \sum_\kappa 2c_\kappa \sum_{\mu\in\mathcal{L}_\kappa}\int_0^\infty E(\kappa;it,i\mu;z)\,b_{\mu,\kappa}(t)\,\mathrm{d}t. \tag{104}$$

Now we turn to the sum in (30) defining $K(z,w)$. The sum is locally finite in w, uniform for z in a fixed compact set, and defines a smooth bounded differentiable function

$f : w \mapsto K(z, w)$ with compact support modulo Γ. Its derivatives are bounded, uniform for z in compact sets, hence its spectral expansion in w converges pointwise.

7.2 Spectral measure

To prove Theorem 4.2, we use the estimate of the counting function in Lemma 3.2, and apply the mechanism of the Selberg transform.

We take $k_j \in C_c^\infty[0, \infty)$ that satisfy $0 \le k_j \le 1$, $k_j = 1$ on $[0, \eta_j]$, and $k_j = 0$ on $[\delta_j, \infty)$ for quantities $\eta = (\eta_j)_j$ and $\delta = (\delta_j)_j$ with $0 < \eta_j < \delta_j < 1/2$ to be chosen later. We form k and K as in (27) and (30). We have seen in (31) that $K(z, \cdot) \in L^2(\Gamma\backslash\mathfrak{H}^d)$. We shall give two inequalities in which the norm $\|K(z, \cdot)\|_2$ occurs.

We have

$$
\begin{aligned}
\|K(z,\cdot)\|_2^2 &= \int_{\Gamma\backslash\mathfrak{H}^d} |K(z,w)|^2 \, \mathrm{d}\mu(w) \\
&= \sum_{\gamma,\delta\in\Gamma} \int_{\mathfrak{F}} k(\gamma z, w)\, k(\delta z, w)\, \mathrm{d}\mu(w) \\
&= \sum_{\gamma,\delta\in\Gamma} \int_{\mathfrak{F}} k(\delta^{-1}\gamma z, \delta^{-1} w)\, k(z, \delta^{-1} w)\, \mathrm{d}\mu(w) \\
&= \sum_{\gamma} \int_{\mathfrak{H}^d} k(\gamma z, w)\, k(z, w)\, \mathrm{d}\mu(w).
\end{aligned}
$$

The second factor restricts the domain of integration to w with $u(z_j, w_j) \le \delta_j$ for all j, and the first factor to w with $u\big((\gamma z)_j, w_j\big) \le \delta_j$ for all j. For the hyperbolic distances, this means that $\mathrm{dist}\big((\gamma z)_j, z_j\big) \le 2\upsilon_j$, where υ_j corresponds to δ_j according to the relation (12). For small values, we have $\delta_j \sim (1/4)\upsilon_j^2$. Hence, $u(\gamma_j z_j, z_j) \le \tilde\delta_j$ with $\tilde\delta_j \sim 4\delta_j$ as $\delta_j \downarrow 0$. Hence, with $\tilde\delta = (\tilde\delta_j)_j$:

$$
\|K(z,\cdot)\|_2^2 \le \mathbf{N}(z; 0, \tilde\delta) \int_{\mathfrak{H}^d} k(z,w)\, \mathrm{d}\mu(w) = \mathbf{N}(z; 0, \tilde\delta) \prod_j h\big(\tfrac{1}{2}\big). \tag{105}
$$

Note that Lemma 6.2 a) implies that $\prod_j h\big(1/2\big)$ is a positive quantity between $(4\pi)^d \prod_j(\delta_j - \eta_j)$ and $(4\pi)^d \prod_j \delta_j$.

Let $X \in [1, \infty)^d$. We recall that in (40) we have given a bounded subset of the spectral set depending on X. Theorem 4.1 implies that

$$\begin{aligned}
\|K(z, \cdot)\|_2^2 &= \sum_{\ell} |h(\tau_\ell)|^2 \, |\psi_\ell(z)|^2 \\
&\quad + \sum_{\kappa} 2c_\kappa \sum_{\mu \in \mathcal{L}_\kappa} \int_0^\infty |h(it + i\mu)|^2 \, |E(\kappa; it, i\mu; z)|^2 \, \mathrm{d}t \\
&\geq \min\{|h(it)|^2 \ : \ t \in Y(X)\} \Bigg(\sum_{\ell\,,\ t_\ell \in Y(X)} |\psi_\ell(z)|^2 \\
&\quad + \sum_{\kappa} 2c_\kappa \sum_{\mu \in \mathcal{L}_\kappa} \int_{t \geq 0\,,\ (t+\mu_j)_j \in Y(X)} |E(\kappa; it, i\mu; z)|^2 \, \mathrm{d}t \Bigg).
\end{aligned} \tag{106}$$

To get a hold on a lower bound of h on $Y(X)$, we use Lemma 6.2 b). For $\tau \in Y(X)$, it gives

$$\begin{aligned}
\left| h_j(\tau_j) - h_j\!\left(\frac{1}{2}\right) \right| &\ll \delta_j^{3/2} \, X_j, \\
|h_j(\tau_j)| \geq h_j\!\left(\frac{1}{2}\right) - \mathrm{O}\!\left(\delta_j^{3/2} X_j\right) &= 4\pi(\delta_j - \eta_j) - \mathrm{O}\!\left(\delta_j^{3/2} X_j\right).
\end{aligned}$$

We take $\eta_j = (1/2)\delta_j$, and $\delta_j = \varepsilon X_j^{-2}$ with $\varepsilon > 0$ sufficiently small to have $|h_j(\tau_j)| \geq \delta_j$. This gives

$$\min\left\{|h(\tau)|^2 \ : \ \tau \in Y(X)\right\} \ \geq \ \frac{1}{(4\pi)^d} \prod_j h_j\!\left(\tfrac{1}{2}\right)^2.$$

Thus, we obtain from (106) the inequality

$$\left\| K(z, \cdot) \right\|_2^2 \geq c_1 \, S(X; z, z) \prod_j h_j\!\left(\tfrac{1}{2}\right)^2, \tag{107}$$

for some positive constant c_1, which does not depend on Γ. If the X_j are sufficiently large, the δ_j and the $\tilde{\delta}_j$ are sufficiently small to apply Lemma 3.2. By (107) and (105), we get

$$\begin{aligned}
S(X; z, z) &\leq \frac{1}{c_1 \, \prod_j h_j\!\left(\frac{1}{2}\right)^2} \, \mathbf{N}(z; 0, \tilde{\delta}) \prod_j h_j\!\left(\tfrac{1}{2}\right) \\
&\ll_\Gamma \ \frac{1}{\prod_j (\delta_j/2)} \, n(\tilde{\delta}^{-1/2}, z) \ \ll \ n(X, z) \prod_j X_j^2.
\end{aligned}$$

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