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Fourier Coefficients of Cusp Forms

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Abstract. Let $\psi_1, \psi_2, \psi_3, \ldots$ be an orthonormal basis of the space of cusp forms of weight zero for the full modular group. Let

$$\psi_j(z) = \sum_{n \neq 0} \gamma_{jn} W_{s_j}(4\pi |n| y) e^{2\pi i n x}$$

be the Fourier series expansion. The following theorem is proved:

Let $\sigma \in (\frac{1}{4}, \frac{1}{2})$; let f be a holomorphic function on the strip $|\operatorname{Re} s| \leqq \sigma$, satisfying $f(-s) = f(s)$ and

$$f(s) = \mathcal{O}(|\tfrac{1}{4} - s^2|^{-2} |\cos \pi s|^{-1})$$

on this strip; let m and n be non-zero integers, then

$$\sum_{j=1}^{\infty} f(s_j) \bar{\gamma}_{jm} \gamma_{jn}$$

converges and is equal to

$$\begin{aligned} &-(2\pi i)^{-1} \int_{\operatorname{Re} s = 0} f(s) c_{00}(-s) c_{0|m|}(s) c_{0|n|}(s) \, ds \\ &+(2\pi i)^{-1} (4\pi |m|)^{-1} \int_{\operatorname{Re} s = \sigma} f(s) c_{mn}(s) 2s \, ds \\ &-\delta_{mn} (2\pi i)^{-1} (4\pi |m|)^{-1} \int_{\operatorname{Re} s = 0} f(s) \sin \pi s \; 2s \, ds. \end{aligned}$$

The functions $c_{00}(s)$ and $c_{0|m|}(s)$ are coefficients occurring in the Fourier series expansion of the Eisenstein series; the function $c_{mn}(s)$ is a coefficient in the Fourier series expansion of a Poincaré series.

The theorem is applied to obtain some asymptotic results concerning the Fourier coefficients γ_{jn}. Under additional conditions on the function f the formula in the theorem is modified in such a way that the Fourier coefficients of holomorphic cusp forms appear.

Key words: Cusp forms – Fourier coefficients – Modular group

The main object of this paper is to derive a formula containing the Fourier coefficients of cusp forms with weight zero for the full modular group. This formula is similar to the Selberg trace formula (see [4, appendix], [1, 3]), as it relates a certain sum over the spectrum of the Laplace operator to a number of integrals. The difference is that it is not obtained by computing in two ways the trace of a certain operator, but by doing the same thing for a double-sided Fourier coefficient of the kernel of the operator.

The integrals in the formula contain Fourier coefficients of Eisenstein and Poincaré series, which are better known than the Fourier coefficients of cusp forms.

Section 1 of this paper contains general results about automorphic forms of weight zero. In Section 2 a connection is studied between Poincaré series and the resolvent of the Laplace operator. In Section 3 the main theorem is derived. In the last two sections the theorem is used to get some information about the Fourier coefficients. Section 4 contains asymptotic results and in Section 5 a link is made with the Fourier coefficients of holomorphic cusp forms.

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1. Preliminaries

This section serves mainly to fix notations and to recall results. References are [4], [5, Ch. XIV], [8].

On the upper half plane H the group $\Gamma = SL_2(\mathbf{Z})$ acts. The operator $L = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ is invariant under Γ. The measure $dz = y^{-2}\,dx\,dy$ is also invariant under Γ.

Γ-invariant eigenfunctions of L with eigenvalue $\frac{1}{4}-s^2$ have a Fourier series expansion

(1.1) $\sum_{n\in\mathbf{Z}} a_n(y)\, e^{2\pi i n x}$,

where a_n satisfies the differential equation

(1.2) $a_n'' - (4\pi^2 n^2 - (\frac{1}{4}-s^2)\, y^{-2})\, a_n = 0$.

So for $s \neq 0$ the function a_0 is a linear combination of $y^{\frac{1}{2}+s}$ and $y^{\frac{1}{2}-s}$. For $n \neq 0$ all solutions are linear combinations of $W_s(4\pi|n|y)$ and $V_s(4\pi|n|y)$, where

(1.3) $W_s(y) = \pi^{-\frac{1}{2}}\, y^{\frac{1}{2}}\, K_s(y/2)$,

(1.4) $V_s(y) = \pi^{\frac{1}{2}}\, y^{\frac{1}{2}}\, I_s(y/2)$.

K_s and I_s are modified Bessel functions, see [14]. I'll use the following asymptotic behaviour for $y \to \infty$

(1.5) $W_s(y) \sim e^{-y/2}$ for $\operatorname{Re} s > -\frac{1}{2}$, uniform for s in compact sets,

(1.6) $V_s(y) \sim e^{y/2}$ for $\operatorname{Re} s > -\frac{1}{2}$, uniform for s in compact sets.

Examples of Γ-invariant eigenfunctions are the Eisenstein series η_s and the Poincaré series P_{ms} with $m \in \mathbf{Z}$, $m \neq 0$. For $\operatorname{Re} s > \frac{1}{2}$ they are obtained as a sum of translates of the function $y^{s+\frac{1}{2}}$, resp. $V_s(4\pi|m|y)\, e^{2\pi i m x}$. These functions are eigenfunctions of L invariant under the subgroup of Γ consisting of the upper triangular matrices in Γ. Therefore one takes the sum of all translates by a set of representatives of Γ modulo that subgroup. The functions η_s and P_{ms} so obtained have a meromorphic continuation as functions of s and they satisfy a functional equation for $s \mapsto -s$.

The Eisenstein series has a Fourier series expansion

$$(1.7)\qquad \eta_s(z)=y^{\frac{1}{2}+s}+c_{00}(s)\,y^{\frac{1}{2}-s}+\sum_{n\neq 0} c_{0|n|}(s)\,W_s(4\pi|n|\,y)\,e^{2\pi inx},$$

$$(1.8)\qquad c_{00}(s)=\pi^{s-\frac{1}{2}}\Gamma(\tfrac{1}{2}-s)\,\zeta(1-2s)/(\pi^{-\frac{1}{2}-s}\Gamma(s+\tfrac{1}{2})\,\zeta(1+2s)),$$

$$(1.9)\qquad c_{0|n|}(s)=|n|^{-\frac{1}{2}-s}\sigma_{2s}(|n|)/(\pi^{-\frac{1}{2}-s}\Gamma(s+\tfrac{1}{2})\,\zeta(1+2s))\qquad \text{for } |n|\neq 0,$$

$$(1.10)\qquad \sigma_{2s}(|n|)=\sum_{d\,|\,|n|} d^{2s}.$$

(1.11) **Lemma.** *On the line* $\operatorname{Re}s=0$ *one has*

$$|c_{00}(s)|=1$$

and for all $\varepsilon>0$

$$c_{0|n|}(s)=\mathcal{O}(|\operatorname{Im}s|^{\varepsilon}\,|\Gamma(s)|^{-1})\qquad |\operatorname{Im}s|\to\infty.$$

Proof. The first assertion is clear; the second one follows from [13, p. 114].

The Fourier series expansion of the Poincaré series is

$$(1.12)\qquad \begin{aligned} P_{ms}(z)=&\,c_{|m|0}(s)\,y^{\frac{1}{2}-s}+V_s(4\pi|m|\,y)\,e^{2\pi imx}\\ &+\sum_{n\neq 0} c_{mn}(s)\,W_s(4\pi|n|\,y)\,e^{2\pi inx},\end{aligned}$$

$$(1.13)\qquad c_{|m|0}(s)=2\pi^2 s^{-1}\,|m|\,c_{0|m|}(s),$$

$$(1.14)\qquad c_{mn}(s)=\begin{cases} 2\pi\,|m/n|^{\frac{1}{2}}\sum\limits_{c=1}^{\infty} r_{mn}(c)\,c^{-1}J_{2s}(4\pi c^{-1}|mn|^{\frac{1}{2}}) & \text{if } mn>0\\ 2\pi\,|m/n|^{\frac{1}{2}}\sum\limits_{c=1}^{\infty} r_{mn}(c)\,c^{-1}I_{2s}(4\pi c^{-1}|mn|^{\frac{1}{2}}) & \text{if } mn<0.\end{cases}$$

Here J_{2s} and I_{2s} are Bessel functions (see [14]) and $r_{mn}(c)$ denotes a Kloosterman sum:

$$(1.15)\qquad r_{mn}(c)=\sum_{\substack{d=0\\(d,c)=1}}^{c-1} e^{2\pi i(md'+nd)/c}\qquad \text{with } dd'\equiv 1 \bmod c.$$

It satisfies for all $\varepsilon>0$

$$(1.16)\qquad r_{mn}(c)=\mathcal{O}(c^{\frac{1}{2}+\varepsilon})\qquad c\to\infty;$$

see [11] and [15]. Hence the sum for $c_{mn}(s)$ converges for $\operatorname{Re}s>\frac{1}{4}$. For a general Fuchsian group one would be sure of convergence only for $\operatorname{Re}s>\frac{1}{2}$.

For a proof of (1.12–14) see [9].

It is interesting to look at the Poincaré series for $s=\frac{1}{2}$. As $W_{\frac{1}{2}}(y)=e^{-y/2}$ and $V_{\frac{1}{2}}(y)=e^{y/2}-e^{-y/2}$, one has for $m\geqq 1$

$$(1.17)\quad P_{-m,\frac{1}{2}}(z)=e^{-2\pi imz}$$
$$+24\sigma_1(m)+\sum_{n=1}^{\infty} c_{-m,n}(\tfrac{1}{2})\,e^{2\pi inz}$$
$$-e^{-2\pi im\bar{z}}+\sum_{n=1}^{\infty} c_{-m,-n}(\tfrac{1}{2})\,e^{-2\pi in\bar{z}}.$$

The term $e^{-2\pi imz}$ is the only term which is not square integrable on the standard fundamental domain of $\Gamma\backslash H$. There is a meromorphic function $p(z)$, a polynomial in the modular invariant $j(z)$, such that $p(z)-e^{-2\pi imz}$ is also square integrable on the standard fundamental domain. As the only square integrable eigenfunctions of L with eigenvalue 0 which are Γ-invariant as well, are the constant functions, it is clear that $P_{-m,\frac{1}{2}}-p$ is a constant. Hence

$$(1.18)\quad c_{-m,-n}(\tfrac{1}{2})=\delta_{mn}\qquad \text{for } m,n>0.$$

(1.14) gives a series for the Fourier coefficients of p. Taking $m=1$ one gets the formula of Petersson for the Fourier coefficients of j, see [10, p. 202]. From (1.14) it is clear that also

$$(1.19)\quad c_{mn}(\tfrac{1}{2})=\delta_{mn}\qquad \text{for } m,n>0.$$

Lemma. *Let $\sigma\in(\frac{1}{4},\frac{1}{2})$, let m and n be non-zero integers, then*

$$(1.20)\quad c_{mn}(s)=\mathcal{O}((4\pi^2|m n|)^{\operatorname{Re}s}|\Gamma(2s+1)|^{-1})\qquad \textit{uniform for } \operatorname{Re}s\geqq\sigma,$$

$$(1.21)\quad c_{mn}(s)=\mathcal{O}(e^{\pi|\operatorname{Im}s|}|\operatorname{Im}s|^{-\frac{1}{2}-2\operatorname{Re}s})\qquad \textit{for } |Im\,s|\to\infty,$$

$\operatorname{Re}s\geqq\sigma$ *fixed.*

Proof. Let $\operatorname{Re}s>\sigma$.

$$\begin{aligned}(1.22)\quad c_{mn}(s)&\ll\sum_{c=1}^{\infty}|r_{mn}(c)|\,c^{-1}\sum_{k=0}^{\infty}(2\pi c^{-1}|m n|^{\frac{1}{2}})^{2\operatorname{Re}s+2k}/(k!\,|\Gamma(2s+k+1)|)\\ &\ll\left(\sum_{c=1}^{\infty}|r_{mn}(c)|\,c^{-1-2\operatorname{Re}s}\right)(2\pi|m n|^{\frac{1}{2}})^{2\operatorname{Re}s}|\Gamma(2s+1)|^{-1}\\ &\quad\times\sum_{k=0}^{\infty}\frac{(2\pi|m n|^{\frac{1}{2}})^{2k}}{k!\,|2s+1|\,|2s+2|\dots|2s+k|}\\ &\ll(4\pi^2|m n|)^{\operatorname{Re}s}|\Gamma(2s+1)|^{-1}.\end{aligned}$$

The second assertion follows from Stirling's formula.

The functions $c_{mn}(s)$ are connected with the Fourier coefficients of holomorphic Poincaré series. Take $m\geqq1$ and $k\geqq2$; denote

$$(1.23)\quad g_{km}(z)=\sum e^{2\pi imAz}(cz+d)^{-2k};$$

$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ runs through a set of representatives of Γ modulo the subgroup of upper triangular matrices. One has:

$$(1.24)\quad g_{km}(z)=\sum_{n=1}^{\infty}(\delta_{mn}+(-n/m)^k c_{mn}(k-\tfrac{1}{2}))\,e^{2\pi inz};$$

see [6, p. 298].

There are no values of s for which P_{ms} or η_s are square integrable functions on $\Gamma\backslash H$ with respect to $dz=y^{-2}\,dx\,dy$. Yet in some way the Eisenstein series and the Poincaré series are sufficient to get the whole of $\mathscr{L}^2(\Gamma\backslash H)$. The space $\mathscr{L}^2(\Gamma\backslash H)$ is the orthogonal sum of subspaces $\mathscr{L}_e^2(\Gamma\backslash H)$ and $\mathscr{L}_d^2(\Gamma\backslash H)$. The Eisenstein series η_s with $\operatorname{Re} s=0$ and $\operatorname{Im} s>0$ span $\mathscr{L}_e^2(\Gamma\backslash H)$ as a continuous direct sum. $\mathscr{L}_d^2(\Gamma\backslash H)$ is spanned by square integrable eigenfunctions of L. A one-dimensional subspace of $\mathscr{L}_d^2(\Gamma\backslash H)$ is the space of constant functions; a nontrivial one of them can be obtained as the residue of η_s in $s=\frac{1}{2}$. All other square integrable eigenfunctions have eigenvalues larger than $\frac{1}{4}$; they are called cusp forms. The corresponding values of s satisfy $\operatorname{Re} s=0$, $s\neq 0$. It is a special feature of $SL_2(\mathbf{Z})$ that there are no eigenvalues in $(0,\frac{1}{4}]$, see [7, thm. 32, p. 203]. All cusp forms are linear combinations of residues of Poincaré on the line $\operatorname{Re} s=0$.

Let $\psi_1,\psi_2,\psi_3,\dots$ be a complete orthonormal system of cusp forms, with eigenvalues $\lambda_1\leqq\lambda_2\leqq\lambda_3\leqq\cdots$. I take $\lambda_j=\frac{1}{4}-s_j^2$, with $s_j=i\,t_j$, $t_j>0$.

From the Selberg trace formula follows that

$$(1.25)\quad \sum_{j=1}^{\infty}e^{-v\lambda_j}=\frac{1}{12}v^{-1}+\mathcal{O}(v^{-\frac{1}{2}-\varepsilon})\qquad\text{for } v\searrow 0.$$

This may be derived by taking $h(r)=e^{-vr^2}$ in [4, appendix]. (1.25) implies

$$(1.26)\quad \lambda_j\sim 12j\qquad\text{for } j\to\infty.$$

One may assume that the cusp forms ψ_j all are eigenfunctions of all Hecke operators T_n ($n\geqq 1$) and T_{-1}. These operators I normalize in this paper as follows:

$$(1.27)\quad T_{-1}f(z)=f(-\bar{z}),$$

$$(1.28)\quad T_n f(z)=\sum_{\substack{ad=n,\,b \bmod d\\ (a,b,d)=1}} f((az+b)/d)\qquad\text{for } n\in\mathbf{N}.$$

The Fourier series expansions of the ψ_j are of the form

$$(1.29)\quad \psi_j(z)=\sum_{n\neq 0}\gamma_{jn}\,W_{s_j}(4\pi|n|\,y)\,e^{2\pi inx}.$$

I'll call the γ_{jn} Fourier coefficients although properly $\gamma_{jn}\,W_{s_j}(4\pi|n|\,y)$ would deserve that name.

Let $n\in\mathbf{N}$ and let λ_{jn} be the eigenvalue of T_n with respect to ψ_j. Then an easy computation shows that

$$(1.30)\quad \gamma_{jn}=\gamma_{j1}\,n^{-1}\sum_{d^2|n}\lambda_{jn/d^2}.$$

From the Selberg trace formula follows for $n\geqq 2$

$$(1.31)\quad \sum_{j=1}^{\infty}\lambda_{jn}\,e^{-v\lambda_j}=\mathcal{O}(v^{-\frac{1}{2}-\varepsilon})\qquad\text{for } v\searrow 0.$$

This can be found by taking $h(r)=e^{-vr^2}$ in [3, (11, 10)]. The term $\frac{1}{12}v^{-1}$ in (1.25) is due to the identity of Γ; in the double Γ-coset corresponding to T_n the identity does not occur; all other terms give contributions of the order $\mathcal{O}(v^{-\frac{1}{2}-\varepsilon})$. From (1.31) follows

$$(1.32)\quad \sum_{j=1}^{k} \lambda_{jn}=o(k) \qquad \text{for } k\to\infty.$$

2. The Resolvent of L

There exists a connection between Poincaré series and the resolvent of the unbounded operator L on $\mathscr{L}^2(\Gamma\backslash H)$. This connection will be used in Section 3.

For $\frac{1}{4}-s^2$ outside the spectrum of L one has in $\mathscr{L}^2(\Gamma\backslash H)$ the bounded operator

$$(2.1)\quad R(s)=[L-(\tfrac{1}{4}-s^2)]^{-1}.$$

Faddeev uses this operator to define the function η_s directly for $\operatorname{Re} s\in(-\frac{1}{2},\frac{3}{2})$, see [2], [5, Ch. XIV]. (The variable s in [5] corresponds to $s+\frac{1}{2}$ in this paper.)

Let $m\in\mathbf{Z}$, $m\neq 0$ and $\alpha\in C_c^\infty(1,\infty)$ be given; by taking $\alpha(y)=0$ for $y<1$, one may consider α as a function on $(0,\infty)$. The function $\vartheta_{\alpha m}\in C_c^\infty(\Gamma\backslash H)$ is defined by prescribing its values on the standard fundamental domain $F=\{z\in H\,|\,|z|>0,\ |\operatorname{Re} z|\leqq\frac{1}{2}\}$:

$$(2.2)\quad \vartheta_{\alpha m}(z)=\alpha(y)\,e^{2\pi imx} \qquad \text{for } z\in F.$$

The function

$$(2.3)\quad \eta_{\alpha ms}=R(s)\,\vartheta_{\alpha m}$$

is a square integrable function on $\Gamma\backslash H$ for $\operatorname{Re} s>0$, $s\neq\frac{1}{2}$. Moreover it satisfies

$$(2.4)\quad L\eta_{\alpha ms}=(\tfrac{1}{4}-s^2)\,\eta_{\alpha ms}+\vartheta_{\alpha m}.$$

Let for $y>0$:

$$(2.5)\quad b_{\alpha|m|s}(y)=(4\pi|m|)^{-1}\,V_s(4\pi|m|\,y)\int_0^y W_s(4\pi|m|\,y')\,\alpha(y')\,y'^{-2}\,dy'$$
$$-(4\pi|m|)^{-1}\,W_s(4\pi|m|\,y)\int_0^y V_s(4\pi|m|\,y')\,\alpha(y')\,y'^{-2}\,dy'$$

and

$$(2.6)\quad l_{\alpha ms}(z)=b_{\alpha|m|s}(y)\,e^{2\pi imx} \qquad \text{for } z\in F,$$

then

$$(2.7)\quad Ll_{\alpha ms}(z)=(\tfrac{1}{4}-s^2)\,l_{\alpha ms}-\vartheta_{\alpha m}.$$

So $\eta_{\alpha ms}+l_{\alpha ms}$ is an eigenfunction of L with eigenvalue $\frac{1}{4}-s^2$.

(2.8) **Proposition.** *Let*

$$w_{\alpha|m|}(s)=\int_0^\infty W_s(4\pi|m|y)\,\alpha(y)\,y^{-2}\,dy,$$

then

$$\eta_{\alpha ms}+l_{\alpha ms}=(4\pi|m|)^{-1}\,w_{\alpha|m|}(s)\,P_{ms}$$

for all s with $\operatorname{Re}s>0$, $s\neq\frac{1}{2}$.

Proof. As $\frac{1}{4}-s^2$ is not an element of the spectrum of L, it is sufficient to show that the left and right hand sides of the equality differ by a function in $\mathscr{L}^2(\Gamma\backslash H)$. I prove this for $\frac{1}{2}<\operatorname{Re}s<\frac{3}{2}$; then it is true elsewhere as both sides are meromorphic in s. The following lemma will be proved:

(2.9) **Lemma.** *If* $\frac{1}{2}<\operatorname{Re}s<\frac{3}{2}$, *then* $\eta_{\alpha ms}(z)=\mathcal{O}(y^{\frac{1}{2}-\operatorname{Re}s})$ *for* $y\to\infty$.

Hence $\eta_{\alpha ms}$ is square integrable and its contribution to all Fourier coefficients is bounded. As $l_{\alpha ms}$ does only contribute to the m-th Fourier coefficient, one needs only look at this coefficient. It has the form

$$(2.10)\quad u\,W_s(4\pi|m|y)+v\,V_s(4\pi|m|y).$$

As the W_s-term is square integrable on F, it is sufficient to determine v. This can be done by taking $\lim_{y\to\infty}$ of this Fourier coefficient multiplied by $e^{-2\pi|m|y}$, see (1.5) and (1.6). By the lemma the contribution of $\eta_{\alpha ms}$ is zero.

$$(2.11)\quad v=\lim_{y\to\infty}e^{-2\pi|m|y}\,b_{\alpha|m|s}(y)=(4\pi|m|)^{-1}\,w_{\alpha|m|}(s).$$

So

$$(2.12)\quad \eta_{\alpha ms}+l_{\alpha ms}\equiv(4\pi|m|)^{-1}\,w_{\alpha|m|}(s)\,V_s(4\pi|m|y)\,e^{2\pi imx}$$

modulo $\mathscr{L}^2(F)$. By (1.12) one gets the proposition.

Proof of Lemma (2.9). I'll use several results from [5, Ch. XIV, §8 and §10]; the variable s in [5, Ch. XIV] corresponds with $s+\frac{1}{2}$ in this paper.

The resolvent can be expressed in other operators Q and B, acting on functions on F; for the meaning of Q, ω and B see [5, Ch. XIV]:

$$(2.13)\quad R(s)=Q(s+\tfrac{1}{2})+(I+\omega(s+\tfrac{1}{2})\,Q(s+\tfrac{1}{2}))\,B(s+\tfrac{1}{2})(I+\omega(s+\tfrac{1}{2})\,Q(s+\tfrac{1}{2})).$$

The proof of Lemma (2.9) now is completed by checking the following steps:

$$(2.14)\quad Q(s+\tfrac{1}{2})\,\vartheta_{\alpha ms}=0,$$

$$(2.15)\quad B(s+\tfrac{1}{2})\,\vartheta_{\alpha ms}(z)=\mathcal{O}(y^{-1})\qquad\text{for } y\to\infty,\ z\in F,$$

$$(2.16)\quad Q(s+\tfrac{1}{2})\,f(z)=\mathcal{O}(y^{\frac{1}{2}-\operatorname{Re}s})\qquad y\to\infty,\ z\in F,$$

for any function f satisfying $f(z)=\mathcal{O}(y^{-1})$ on F.

(2.17) **Proposition.** *Let $n \in \mathbf{Z}$, $n \neq 0$. The n-th Fourier coefficient of $\eta_{\alpha m s}$ is equal to*

$$(4\pi|m|)^{-1} w_{\alpha|m|}(s)\, c_{mn}(s)\, W_s(4\pi|m|y) - \delta_{mn} b_{\alpha|m|s}(y)$$
$$+\delta_{mn}(4\pi|m|)^{-1} w_{\alpha|m|}(s)\, V_s(4\pi|m|y).$$

Proof. See (1.12) and (2.8).

3. A Formula Containing Fourier Coefficients of Cusp Forms

In this section I derive a formula containing the γ_{jn} of (1.29). The idea is to take two simple functions φ_1 and φ_2 in $\mathscr{L}^2(\Gamma\backslash H)$ and a holomorphic function g such that the operator $g(L)$ makes sense. Then I express $\langle g(L)\varphi_1, \varphi_2\rangle$ in two ways. The first way is based on the decomposition $\mathscr{L}^2(\Gamma\backslash H) = \mathscr{L}^2_e(\Gamma\backslash H) + \mathscr{L}^2_d(\Gamma\backslash H)$ and gives an expression of $\langle g(L)\varphi_1, \varphi_2\rangle$ in terms of Eisenstein series and the basis $\{\psi_j\}$ of the cusp forms. The other way is based on the description of $g(L)$ as an integral around the spectrum of L. In the equality so obtained the functions φ_1 and φ_2 occur. In two steps the dependence on the special form of φ_1 and φ_2 is removed.

In this section two non-zero integers m and n and a real number $\sigma \in (\frac{1}{4}, \frac{1}{2})$ are kept fixed.

Let f be a holomorphic function on a neighbourhood of the strip $|\mathrm{Re}\, s| \leqq \sigma$, satisfying

(3.1) $f(-s) = f(s)$

(3.2) $f(s) = \mathcal{O}((\frac{1}{4} - s^2)^{-4})$ on the strip $|\mathrm{Re}\, s| \leqq \sigma$.

The function g defined by $g(\frac{1}{4} - s^2) = f(s)$ is bounded on the spectrum of L restricted to the orthogonal complement in $\mathscr{L}^2(\Gamma\backslash H)$ of the constant functions. Therefore there exists a bounded operator $g(L)$ on this subspace of $\mathscr{L}^2(\Gamma\backslash H)$.

Take $\alpha, \beta \in C_c^\infty(1, \infty)$ with

(3.3) $1 < x < y$ for all $x \in \mathrm{supp}(\alpha)$ and $y \in \mathrm{supp}(\beta)$.

Let

(3.4) $\varphi_1 = \vartheta_{\alpha m}$ and $\varphi_2 = \vartheta_{\beta n}$,

then $\varphi_1, \varphi_2 \in \mathscr{L}^2(\Gamma\backslash H)$ and φ_1, φ_2 are orthogonal to the constant functions. I consider the inner product

(3.5) $$\langle g(L)\varphi_1, \varphi_2\rangle = \int_{\Gamma\backslash H} g(L)\varphi_1(z)\overline{\varphi_2(z)}\, dz.$$

It is equal to the sum of a "cuspidal part" and a "Eisenstein part". The cuspidal part is

(3.6) $$\sum_{j=1}^{\infty} g(\lambda_j)\langle\varphi_1, \psi_j\rangle\overline{\langle\varphi_2, \psi_j\rangle};$$

by (1.29), the definition of $w..(\cdot)$ in (2.8) and the fact that $\overline{W_{s_j}(y)} = W_{s_j}(y)$, it is equal to

$$(3.7)\quad \sum_{j=1}^{\infty} f(s_j)\,\overline{\gamma_{jm}}\,\gamma_{jn}\, w_{\alpha|m|}(s_j)\, w_{\bar\beta|n|}(s_j).$$

The Eisenstein part is

$$(3.8)\quad (2\pi i)^{-1} \int\limits_{\substack{\operatorname{Re} s=0\\ \operatorname{Im} s>0}} f(s)\, E\varphi_1(s)\, \overline{E\varphi_2(s)}\, ds,$$

where for any $\varphi \in C_c^\infty(\Gamma\backslash H)$

$$(3.9)\quad E\varphi(s) = \int\limits_{\Gamma\backslash H} \varphi(z)\, \eta_s(z)\, dz.$$

The Eisenstein part equals

$$\begin{aligned}
(3.10)\quad (2\pi i)^{-1} &\int\limits_{\substack{\operatorname{Re} s=0\\ \operatorname{Im} s>0}} f(s)\, E\varphi_1(s)\, E\overline{\varphi_2}(-s)\, ds \\
&= (2\pi i)^{-1} \int\limits_{\substack{\operatorname{Re} s=0\\ \operatorname{Im} s>0}} f(s)\, c_{00}(-s)\, E\varphi_1(s)\, E\overline{\varphi_2}(s)\, ds \\
&= (4\pi i)^{-1} \int\limits_{\operatorname{Re} s=0} f(s)\, c_{00}(-s)\, E\varphi_1(s)\, E\overline{\varphi_2}(s)\, ds \\
&= (4\pi i)^{-1} \int\limits_{\operatorname{Re} s=0} f(s)\, c_{00}(-s)\, c_{0|m|}(s)\, c_{0|n|}(s)\, w_{\alpha|m|}(s)\, w_{\bar\beta|n|}(s)\, ds.
\end{aligned}$$

The sum of the expressions in (3.7) and (3.10) equals $\langle g(L)\varphi_1, \varphi_2\rangle$.

To express the same expression in another way the resolvent $R(s)$ can be used. Let C_1 be a contour in the λ-plane going around $[\frac{1}{4}, \infty)$ in negative direction at a distance δ with $\delta \in (0, \frac{1}{2})$. Then

$$(3.11)\quad \langle g(L)\varphi_1, \varphi_2\rangle = (2\pi i)^{-1} \int\limits_{C_1} g(\lambda) \langle (L-\lambda)^{-1} \varphi_1, \varphi_2\rangle\, d\lambda.$$

Let C_2 be the contour in the s-plane with $\operatorname{Re} s > 0$ corresponding to C_1 under $\lambda = \frac{1}{4} - s^2$. Then

$$\begin{aligned}
(3.12)\quad \langle g(L)\varphi_1, \varphi_2\rangle &= (2\pi i)^{-1} \int\limits_{C_2} f(s) \langle R(s)\varphi_1, \varphi_2\rangle (-2s)\, ds \\
&= (2\pi i)^{-1} \int\limits_{\operatorname{Re} s=\sigma} f(s) \langle R(s)\varphi_1, \varphi_2\rangle\, 2s\, ds.
\end{aligned}$$

Let

$$(3.13)\quad v_{\alpha|m|}(s) = \int_0^\infty V_s(4\pi|m|y)\, \alpha(y)\, y^{-2}\, dy,$$

then by Proposition (2.17) the expression in (3.12) equals

$$(3.14)\quad (2\pi i)^{-1}(4\pi|m|)^{-1}\int\limits_{\operatorname{Re}s=\sigma} f(s)\, w_{\alpha|m|}(s)\, w_{\beta|n|}(s)\, c_{mn}(s)\, 2s\, ds$$
$$+\delta_{mn}(2\pi i)^{-1}\{(4\pi|m|)^{-1}\int\limits_{\operatorname{Re}s=\sigma} f(s)\, w_{\alpha|m|}(s)\, v_{\beta|m|}(s)\, 2s\, ds$$
$$-\int\limits_{\operatorname{Re}s=\sigma} f(s)\int\limits_{1}^{\infty} b_{\alpha m s}(y)\,\bar{\beta}(y)\, y^{-2}\, dy\, 2s\, ds\}.$$

In the terms with δ_{mn} the path of integration may be moved to the line $\operatorname{Re}s=0$. Keeping (3.3) in mind one sees that these terms are equal to

$$(3.15)\quad \delta_{mn}(2\pi i)^{-1}(4\pi|m|)^{-1}\int\limits_{\operatorname{Re}s=0} f(s)\{w_{\alpha|m|}(s)\, v_{\beta|m|}(s)$$
$$-w_{\alpha|m|}(s)\, v_{\bar{\beta}|m|}(s)+v_{\alpha|m|}(s)\, w_{\bar{\beta}|m|}(s)\}\, 2s\, ds.$$

As $V_{-s}=V_s+2\sin\pi s\, W_s$ and $f(-s)=f(s)$ one gets

$$(3.16)\quad \delta_{mn}(2\pi i)^{-1}(4\pi|m|)^{-1}\int\limits_{\operatorname{Re}s=0} f(s)\tfrac{1}{2}(v_{\alpha|m|}(s)-v_{\alpha|m|}(-s))\, w_{\beta|m|}(s)\, 2s\, ds$$
$$=-\delta_{mn}(2\pi i)^{-1}(4\pi|m|)^{-1}\int\limits_{\operatorname{Re}s=0} f(s)\sin\pi s\; w_{\alpha|m|}(s)\, w_{\bar{\beta}|m|}(s)\, 2s\, ds.$$

From (3.7), (3.10), (3.14) and (3.16) follows:

$$(3.17)\quad \sum_{j=1}^{\infty} f(s_j)\,\overline{\gamma_{jm}}\,\gamma_{jn}\, w_{\alpha|m|}(s_j)\, w_{\bar{\beta}|n|}(s_j)$$
$$=-(4\pi i)^{-1}\int\limits_{\operatorname{Re}s=0} f(s)\, c_{00}(-s)\, c_{0|m|}(s)\, c_{0|n|}(s)\, w_{\alpha|m|}(s)\, w_{\beta|n|}(s)\, ds$$
$$+(2\pi i)^{-1}(4\pi|m|)^{-1}\int\limits_{\operatorname{Re}s=\sigma} f(s)\, c_{mn}(s)\, w_{\alpha|m|}(s)\, w_{\beta|n|}(s)\, 2s\, ds$$
$$-\delta_{mn}(2\pi i)^{-1}(4\pi|m|)^{-1}\int\limits_{\operatorname{Re}s=0} f(s)\sin\pi s\; w_{\alpha|m|}(s)\, w_{\beta|n|}(s)\, 2s\, ds.$$

In each term of (3.17) occurs

$$(3.18)\quad w_{\alpha|m|}(s)\, w_{\bar{\beta}|n|}(s)$$
$$=\iint\limits_{1\leq y<y'} W_s(4\pi|m|y)\, W_s(4\pi|n|y')\,\alpha(y)\,\overline{\beta(y')}\, y^{-2}\, dy\, y'^{-2}\, dy'.$$

To remove the dependence of (3.17) on α and β I first take the integral over y and y' outside the sum and integrals in (3.17). For each term one has to check whether this is allowed.

For the first term it is sufficient to prove that

$$(3.19)\quad \sum_{j=1}^{\infty} |f(s_j)\,\overline{\gamma_{jm}}\,\gamma_{jn}\, W_{s_j}(4\pi|m|y)\, W_{s_j}(4\pi|n|y')|$$

converges.

(3.20) **Lemma.** $\gamma_{jn}\, W_{s_j}(4\pi|n|y)=\mathcal{O}(\lambda_j)$ $j\to\infty$, *uniform for* $y>0$.

Proof. Given $\kappa > \frac{5}{2}$ Faddeev defines two operators T and V such that $R(\kappa) = T + V$, as can be found in [5, Ch. XIV, § 8]. Directly from the definition of T follows that $T\psi = 0$ for all cusp forms ψ; hence $\psi_j = (\lambda_j - \frac{1}{4} + \kappa^2) V\psi_j$. Now V is a bounded operator $\mathscr{L}^2(\Gamma\backslash H) \to \mathscr{L}^\infty(\Gamma\backslash H)$, see [5, Ch. XIV, § 8]. Hence

$$(3.21) \quad \|\psi_j\|_\infty \ll (\lambda_j - \tfrac{1}{4} + \kappa^2) \|\psi_j\|_2 \ll \lambda_j.$$

As $|\gamma_{jn} W_{s_j}(4\pi |n| y)| \leqq \|\psi_j\|_\infty$ the assertion of the lemma follows.

Lemma (3.20) and (1.26) ensure the convergence in (3.19), uniform in y and y'.

For the second term one needs

$$(3.22) \quad W_s(y) = \pi^{-1} 2^{2s+2} \Gamma(s + \tfrac{3}{2}) y^{-s-\frac{1}{2}} \int_0^\infty u \sin(\tfrac{1}{2} y u)(u^2+1)^{-s-\frac{3}{2}} du,$$

which can be derived from Basset's formula for K_s, see [14, p. 172]. From this and (1.11) follows that

$$(3.23) \quad |f(s) c_{00}(-s) c_{0|m|}(s) c_{0|n|}(s) W_s(4\pi |m| y) W_s(4\pi |n| y')|$$
$$= \mathcal{O}(|\operatorname{Im} s|^{2\varepsilon - 6}) \qquad \text{for } |\operatorname{Im} s| \to \infty, \quad \operatorname{Re} s = 0,$$

uniform for $y \geqq 1$, $y' \geqq 1$. Hence the integral of the expression in (3.23) converges on the line $\operatorname{Re} s = 0$, uniform in y and y'.

For the last terms in (3.17) one proceeds analogously, using (1.21) for the third term.

Having moved the integrals over y and y' outside one may consider (3.17) as an equation relating four distributions on the region $y < y'$. These distributions are given by continuous functions, hence the corresponding equation is valid for these functions:

(3.24) **Proposition.** *Let f and σ be as in* (3.1), *let m and n be non-zero integers and let $1 < y < y'$, then*

$$\sum_{j=1}^\infty f(s_j) \overline{\gamma_{jm}} \gamma_{jn} W_{s_j}(4\pi |m| y) W_{s_j}(4\pi |n| y')$$
$$= -(4\pi i)^{-1} \int_{\operatorname{Re} s = 0} f(s) c_{00}(-s) c_{0|m|}(s) c_{0|n|}(s)$$
$$\cdot W_s(4\pi |m| y) W_s(4\pi |n| y') ds$$
$$+ (2\pi i)^{-1} (4\pi |m|)^{-1} \int_{\operatorname{Re} s = \sigma} f(s) c_{mn}(s)$$
$$\cdot W_s(4\pi |m| y) W_s(4\pi |n| y') 2s\, ds$$
$$- \delta_{mn} (2\pi i)^{-1} (4\pi |m|)^{-1} \int_{\operatorname{Re} s = 0} f(s) \sin \pi s$$
$$\cdot W_s(4\pi |m| y) W_s(4\pi |n| y') 2s\, ds.$$

In the equality in Proposition (3.24) the variables y and y' occur. To be able to remove them, one has to add conditions for f.

(3.25) **Lemma.** *Let the function f in* (3.24) *satisfy the additional condition*

$$f(s) = \mathcal{O}(|\tfrac{1}{4} - s^2|^{-2} |\cos \pi s|^{-1})$$

for $|\mathrm{Re}\, s| \leqq \sigma$; *then for each term T in the right hand side of the equality in* (3.24) *the limit*

$$\lim_{y \to \infty} \lim_{y' \to \infty} e^{2\pi(|m| y + |n| y')} T$$

exists and can be evaluated by taking the limit inside the integral.

Proof. From (3.22) follows that

(3.26) $|W_s(y)| \leqq W_{\mathrm{Re}\, s}(y)$.

Using this, the asymptotic behaviour of $W_{\mathrm{Re}\, s}$ in (1.5) and the estimates made in the proof of (3.24), it is easy to apply the dominated convergence theorem of Lebesgue. The factor $|\cos \pi s|^{-1}$ in the bound of f counteracts the Γ-factors in $c_{0|n|}(s)\, c_{0|m|}(s)$ and $c_{mn}(s)$ and the factor $\sin \pi s$.

The first term in (3.24) cannot be handled in the same way, as the order of growth of $\overline{\gamma_{jm}}\, \gamma_{jn}$ is not known. Instead of dominated convergence Fatou's lemma will be applied.

(3.27) **Lemma.** *Let* $\tilde{f}(s) = (\tfrac{1}{4} - s^2)^{-2} (\cos \pi s)^{-1}$, *then*

$$\sum_{j=1}^{\infty} \tilde{f}(s_j)\, |\gamma_{jn}|^2$$

converges.

Proof. By (3.25) the limit

$$\text{(3.28)} \quad \lim_{y \to \infty} \lim_{y' \to \infty} \sum_{j=1}^{\infty} \tilde{f}(s_j)\, |\gamma_{jn}|^2\, e^{2\pi |n| (y + y')} W_{s_j}(4\pi |n|\, y)\, W_{s_j}(4\pi |n|\, y')$$

exists. As the terms are positive the assertion of the lemma follows by Fatou's lemma.

(3.29) **Theorem.** *Let* $\sigma \in (\tfrac{1}{4}, \tfrac{1}{2})$; *let f be a holomorphic function on the strip* $|\mathrm{Re}\, s| \leqq \sigma$, *satisfying* $f(-s) = f(s)$ *and*

$$f(s) = \mathcal{O}(|\tfrac{1}{4} - s^2|^{-2} |\cos \pi s|^{-1})$$

on this strip; let m and n be non-zero integers; then

$$\sum_{j=1}^{\infty} f(s_j)\, \overline{\gamma_{jm}}\, \gamma_{jn}$$

converges, and

$$\sum_{j=1}^{\infty} f(s_j)\overline{\gamma_{jm}}\,\gamma_{jn} = -(2\pi i)^{-1} \int_{\mathrm{Re}\, s=0} f(s)\, c_{00}(-s)\, c_{0|m|}(s)\, c_{0|n|}(s)\, ds$$
$$+(2\pi i)^{-1}(4\pi|m|)^{-1} \int_{\mathrm{Re}\, s=\sigma} f(s)\, c_{mn}(s)\, 2s\, ds$$
$$-\delta_{mn}(2\pi i)^{-1}(4\pi|m|)^{-1} \int_{\mathrm{Re}\, s=0} f(s) \sin\pi s\; 2s\, ds.$$

Remark. The Fourier coefficients γ_{jn} are defined in (1.29); the functions $c_{..}(s)$ can be found in (1.8), (1.9), (1.13) and (1.14).

Proof. The only thing left to do is to take the limits inside the sum in

$$\lim_{y\to\infty} \lim_{y'\to\infty} \sum_{j=1}^{\infty} f(s_j)\overline{\gamma_{jm}}\,\gamma_{jn}\, e^{2\pi(|m|y+|n|y')}\, W_{s_j}(4\pi|m|y)\, W_{s_j}(4\pi|n|y').$$

But now we can use dominated convergence, for by Lemma (3.27) a multiple of

$$\tilde{f}(s_j)(|\gamma_{jm}|^2+|\gamma_{jn}|^2)$$

is summable; so it can be used as majorant.

4. Some Asymptotic Results on Fourier Coefficients of Cusp Forms

Theorem (3.29) is used as a starting point for the derivation of asymptotic estimates for the γ_{jn}.

Throughout this section t_j denotes $\mathrm{Im}\, s_j$.

(4.1) **Proposition.**

$$\sum_{j=1}^{\infty} e^{-v\lambda_j}\, \overline{\gamma_{jm}}\,\gamma_{jn}/\cosh\pi t_j = \delta_{mn}(4\pi^2|m|)^{-1} v^{-1} + \mathcal{O}(v^{-\frac{1}{2}-\varepsilon})$$

for $v\searrow 0$, *with* $\varepsilon>0$.

Proof. Take in (3.29)

(4.2) $f(s)=e^{-v(\frac{1}{4}-s^2)}/\cos\pi s.$

The second and third term in (3.29) are $\mathcal{O}(v^{-\frac{1}{2}-\varepsilon})$ and $\mathcal{O}(v^{-\frac{1}{2}})$ as follows from (1.11) and (1.21). The last term gives

(4.3) $$\delta_{mn}(2\pi^2|m|)^{-1} e^{-\frac{1}{4}v} \int_0^{\infty} t\, e^{-vt^2}(1+\mathcal{O}(e^{-2\pi t}))\, dt$$
$$=\delta_{mn}(4\pi^2|m|)^{-1} v^{-1} + \mathcal{O}(1) \qquad \text{for } v\searrow 0.$$

(4.4) **Corollary.**

$$\sum_{j=1}^{\infty} e^{-v\lambda_j}\, |\gamma_{jn}|^2/\cosh\pi t_j = (4\pi^2|n|)^{-1} v^{-1} + \mathcal{O}(v^{-\frac{1}{2}-\varepsilon})$$

for $v \searrow 0$ *and hence*

$$\sum_{j=1}^{N} |\gamma_{jn}|^2/\cosh \pi t_j \sim 3(\pi^2 |n|)^{-1} N \qquad N \to \infty.$$

For the last assertion (1.26) is used.

One may interpret Corollary (4.4) as saying that on the average $|\gamma_{j1}|$ equals $\pi^{-1}(3\cosh \pi t_j)^{\frac{1}{2}}$. It will turn out that the exact equality $|\gamma_{j1}| = \pi^{-1}(3\cosh \pi t_j)^{\frac{1}{2}}$ cannot be valid for all j.

In the following p is a fixed prime number. By (1.30) one has for all integral $n \geqq 1$:

(4.5) $$\lim_{v \searrow 0} v \sum_{j=1}^{\infty} p^{-n}\Big(\sum_{\substack{0 \leqq l \leqq n \\ n-l \text{ even}}} \lambda_{jp^l}\Big) |\gamma_{j1}|^2 (\cosh \pi t_j)^{-1} e^{-v\lambda_j} = 0.$$

Now $\sum_{\substack{0 \leqq l \leqq n \\ n-l \text{ even}}} \lambda_{jp^l}$ is a polynomial in λ_{jp}, not depending on j. This polynomial h_n is determined by

(4.6) $$h_n(p^{\frac{1}{2}}(u+u^{-1})) = p^{n/2}(u^{-n} + u^{-n+2} + \cdots + u^n).$$

So

(4.7) $$\Phi_p(h) = \lim_{v \searrow 0} v \sum_{j=1}^{\infty} h(\lambda_{jp}) |\gamma_{j1}|^2 (\cosh \pi t_j)^{-1} e^{-v\lambda_j}$$

defines a functional Φ_p on the polynomials in one variable, satisfying

(4.8) $\Phi_p(1) = (4\pi^2)^{-1}$,

(4.9) $\Phi_p(h_n) = 0 \qquad$ for $n \geqq 1$.

This functional can be described as a measure on $[-2p^{\frac{1}{2}}, 2p^{\frac{1}{2}}]$ given by a continuous function:

(4.10) **Proposition.** *Let* Φ_p *be the functional given in* (4.7). *Then*

$$\Phi_p(h) = \int_{-2p^{\frac{1}{2}}}^{2p^{\frac{1}{2}}} h(\lambda)(8\pi^3 p)^{-1}(4p - \lambda^2)^{\frac{1}{2}}\, d\lambda.$$

Proof. Φ_p is determined by (4.8) and (4.9). These equalities are easily checked making the substitution $\lambda = 2p^{\frac{1}{2}} \cos \vartheta$ and using (4.6).

It is known that $|\lambda_{jp}| \leqq p+1$ for all j. The Ramanujan conjecture for non-holomorphic cusp forms amounts to

(4.11) $|\lambda_{jp}| \leqq 2p^{\frac{1}{2}} \qquad$ for all j;

see [12]. Proposition (4.10) suggests that the majority of the λ_{jp} satisfy (4.11).

From (1.25) and (1.31) I'll derive a similar proposition; from it will follow that the Ramanujan conjecture is asymptotically true.

Let Ψ'_p be the following functional on the polynomials

(4.12) $$\Psi'_p(h)=\lim_{v\searrow 0} v\sum_{j=1}^{\infty} h(\lambda_{jp})\,e^{-v\lambda_j}.$$

Then Ψ'_p satisfies:

(4.13) $\Psi'_p(1)=1/12$,

(4.14) $\Psi'_p(h_1)=0$ and

(4.15) $\Psi'_p(h_n)=\Psi'_p(h_{-2})$ for $n\geqq 2$.

(4.16) **Proposition.** *The functional* Ψ'_p *described in* (4.12) *satisfies*

$$\Psi'_p(h)=\int_{-2p^{\frac{1}{2}}}^{2p^{\frac{1}{2}}} h(\lambda)(p+1)(24\pi)^{-1}((p+1)^2-\lambda^2)^{-1}(4p-\lambda^2)^{\frac{1}{2}}\,d\lambda.$$

Proof. Along the same lines as that of (4.10); take

$$((p+1)^2-\lambda^2)^{-1}=(p-e^{2i\vartheta})^{-1}(p-e^{-2i\vartheta})^{-1}$$

and expand it as a series.

(4.17) **Proposition.** *Let k be the characteristic function of* $[-p-1,\,-2p^{\frac{1}{2}})\cup(2p^{\frac{1}{2}},p+1]$, *then*

$$\lim_{N\to\infty} N^{-1}\sum_{j=1}^{N} k(\lambda_{jp})=0.$$

This means that the λ_{jp} not satisfying (4.11) have density zero.

Proof. Firstly remark that (4.12) and (4.16) can be extended to all continuous functions on $[-p-1,p+1]$. The next step is to approximate the function k by continuous functions to get

(4.18) $$\lim_{v\searrow 0} v\sum_{j=1}^{\infty} k(\lambda_{jp})\,e^{-v\lambda_j}=0.$$

Here one has to use that Ψ'_p has no points with non-zero measure. Finally one obtains the statement in the lemma by a Tauberian argument.

Combining (4.10) and (4.16) one gets:

(4.19) **Corollary.** *For all polynomials h*

$$\lim_{v\searrow 0} v\sum_{j=1}^{\infty} h(\lambda_{jp})\,e^{-v\lambda_j}(3\pi^{-2}-|\gamma_{j1}|^2(\cosh\pi t_j)^{-1})$$
$$=\int_{-2p^{\frac{1}{2}}}^{2p^{\frac{1}{2}}} h(\lambda)(8\pi^3 p)^{-1}(4p-\lambda^2)^{\frac{1}{2}}(\lambda^2-p-1)((p+1)^2-\lambda^2)^{-1}\,d\lambda.$$

Clearly $|\gamma_{j1}|=\pi^{-1}(3\cosh\pi t_j)^{\frac{1}{2}}$ cannot be valid for all j. Actually I don't even know whether the $|\gamma_{j1}|^2/\cosh\pi t_j$ are bounded.

It is remarkable that the difference $3\pi^{-2}-|\gamma_{j1}|^2/\cosh \pi t_j$ and the eigenvalues of the T_p are dependent. On the other hand there seems to be no dependence between $|\gamma_{j1}|^2/\cosh \pi t_j$ and the eigenvalues of T_{-1}:

(4.20) **Corollary** *of* (4.1). *Let* $T_{-1}\psi_j=\varepsilon_j\psi_j$, *then*

$$\sum_{j=1}^{\infty} e^{-v\lambda_j}\tfrac{1}{2}(1\pm\varepsilon_j)|\gamma_{j1}|^2/\cosh \pi t_j=(2\pi^2)^{-1}v^{-1}+\mathcal{O}(v^{-\frac{1}{2}-\varepsilon}) \qquad \textit{for } v\searrow 0.$$

5. A Connection with the Holomorphic Cusp Forms

For $k\geqq 2$ the number $c_{mn}(k-\frac{1}{2})$ is up to a factor the Fourier coefficient of a Poincaré series of weight $2k$, see (1.23) and (1.24). In this Section I take the function f in (3.29) in such a way that the path of integration in the $c_{nm}(s)$-term can be moved to the right. The result is a formula in which the Fourier coefficients γ_{jn} are connected with the Fourier coefficients of the holomorphic cusp forms.

(5.1) Let $k\in\mathbf{N}$, $k\geqq 2$; I denote by $\psi_{k1},\psi_{k2},\dots,\psi_{kr_k}$ an orthonormal basis of the holomorphic cusp forms for the modular group with weight $2k$. For $2\leqq k\leqq 5$ the dimension r_k is zero. Let

$$(5.2)\quad \psi_{kj}=\sum_{n=1}^{\infty}\gamma_{kjn}e^{2\pi i n z}$$

be the Fourier series expansion ψ_{kj}. Let $m\geqq 1$; in (1.23) is given the corresponding Poincaré series of weight $2k$. Now the inner product of ψ_{kj} and g_{km} equals

$$(5.3)\quad \gamma_{jkm}(4\pi m)^{1-2k}(2k-2)!$$

see [6, p. 286]. From this and (1.24) one easily deduces that

$$(5.4)\quad \delta_{mn}+(n/m)^k(-1)^k c_{mn}(k-\tfrac{1}{2})=(2k-2)!(4\pi m)^{1-2k}\sum_{j=1}^{r_k}\overline{\gamma_{kjm}\gamma_{kjn}}$$

for $m,n\geqq 1$.

Now let $u_1,u_2,\dots,u_q$ be points in the region $\operatorname{Re}s\geqq\frac{1}{2}$ satisfying $u_r+\frac{1}{2}\notin\mathbf{Z}$, and $A_1,A_2,\dots,A_q$ complex numbers with $\sum_{r=1}^{q}A_r=0$. Set

$$(5.5)\quad g(s)=\sum_{r=1}^{q}A_r(s^2-u_r^2)^{-1}.$$

Then

$$(5.6)\quad g(s)=\mathcal{O}((\tfrac{1}{4}-s^2)^{-2}) \qquad \text{for } |s| \text{ large.}$$

Now let

$$(5.7)\quad f(s)=g(s)/\cos\pi s;$$

then f satisfies the conditions of (3.29). In the terms

$$(5.8)\quad (2\pi i)^{-1}(4\pi|m|)^{-1} \int_{\mathrm{Re}\, s=\sigma} f(s)\, c_{mn}(s)\, 2s\, ds$$
$$- \delta_{mn}(2\pi i)^{-1}(4\pi|m|)^{-1} \int_{\mathrm{Re}\, s=0} f(s) \sin \pi s\ 2s\, ds$$

the line of integration can be moved to the right. Let $l \in \mathbf{N}$ be larger than $\mathrm{Re}\, u_1, \ldots, \mathrm{Re}\, u_q$. Then the sum of the terms in (5.8) equals

$$(5.9)\quad (2\pi i)^{-1}(4\pi|m|)^{-1} \int_{\mathrm{Re}\, s=l} f(s)\, c_{mn}(s)\, 2s\, ds$$
$$- \delta_{mn}(2\pi i)^{-1}(4\pi|m|)^{-1} \int_{\mathrm{Re}\, s=l} f(s) \sin \pi s\ 2s\, ds$$
$$-(4\pi|m|)^{-1} \sum_{k=1}^{l} g(k-\tfrac{1}{2})(-1)^k(2k-1)\, c_{mn}(k-\tfrac{1}{2})$$
$$- \delta_{mn}(4\pi|m|)^{-1} \sum_{k=1}^{l} g(k)(2k-1)$$
$$-(4\pi|m|)^{-1} \sum_{r=1}^{q} A_r(\cos \pi u_r)^{-1}\, c_{mn}(u_r)$$
$$+ \delta_{mn}(4\pi|m|)^{-1} \sum_{r=1}^{q} A_r \tan \pi u_r;$$

see the estimate of $c_{mn}(s)$ in (1.20). Furthermore the integrals over $\mathrm{Re}\, s = l$ tend to zero for $l \to \infty$. Hence one has the following proposition.

(5.10) **Proposition.** *Let m and n be non-zero integers; let g be as in* (5.5). *Then*

$$\sum_{j=1}^{\infty} g(s_j)\, \overline{\gamma_{jm}}\, \gamma_{jn}/\cosh \pi t_j$$
$$= -(2\pi i)^{-1} \int_{\mathrm{Re}\, s=0} g(s)(\cos \pi s)^{-1}\, c_{00}(-s)\, c_{0|m|}(s)\, c_{0|n|}(s)\, ds$$
$$-(4\pi|m|)^{-1} \sum_{k=1}^{\infty} g(k)(-1)^k(2k-1)\, c_{mn}(k-\tfrac{1}{2})$$
$$- \delta_{mn}(4\pi|m|)^{-1} \sum_{k=1}^{\infty} (2k-1)\, g(k)$$
$$-(4\pi|m|)^{-1} \sum_{r=1}^{q} A_r\, c_{mn}(u_r)/\cos \pi u_r$$
$$+ \delta_{mn}(4\pi|m|)^{-1} \sum_{r=1}^{q} A_r \tan \pi u_r.$$

Take m and n positive and use (1.19). Together with (5.4) this gives:

(5.11) **Corollary.** *Let m and n be positive integers and let g be as in* (5.5). *Then*

$$\sum_{j=1}^{\infty} g(s_j)\,\overline{\gamma_{jm}}\,\gamma_{jn}/\cosh \pi t_j$$

$$+(2\pi i)^{-1} \int_{\mathrm{Re}\, s=0} g(s)(\cos \pi s)^{-1}\, c_{00}(-s)\, c_{0|m|}(s)\, c_{0|n|}(s)\, ds$$

$$+\sum_{k=2}^{\infty} g(k)(2k-1)!(16\pi^2 m n)^{-k} \sum_{j=1}^{r_k} \overline{\gamma_{kjm}}\,\gamma_{kjn}$$

$$=(4\pi m)^{-1} \sum_{r=1}^{q} A_r(-c_{mn}(u_r)/\cos \pi u_r + \delta_{mn} \tan \pi u_r).$$

The terms in the left hand side of the formula in (5.11) all have the same structure. They correspond to several components of the representation of $SL_2(\mathbf{R})$ in $\mathscr{L}^2(SL_2(\mathbf{Z})\backslash SL_2(\mathbf{R}))$.

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