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Modular Forms of Varying Weight. I

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1. Introduction

1.1. Real analytic modular forms have been studied for the first time by Maass; he came to them from the study of Dirichlet series, [8]. Especially in the work of Selberg, [10], their significance for number theoretic questions has become clear. In the trace formula of Selberg one quantity is expressed in two different ways. On one side one has geometrical data on the quotient of the upper half plane modulo a discrete group of transformations. If this group is e.g. the modular one, these data have arithmetical nature. On the other side one has spectral data for the Laplace operator on the upper half plane acting in the Hilbert space of functions invariant under the transformations from the discrete group and square integrable on the quotient. The eigenfunctions of the resulting self adjoint operator are modular forms.

More general formulas of the same nature have been discovered. A general treatment is given by Good, [3].

It is possible to do this, not only for invariant functions, but also for functions transforming under the discrete group in a way determined by so called multiplier systems. The dependence of the modular spectrum on the multiplier-system is in general unknown. So we do not know how the spectral side of Selberg's trace formula and similar sum formulas depend on the multiplier system.

The modular forms which are holomorphic on the upper half plane are well known; the other real analytic ones are rather mysterious. The only general mechanism of "constructing" them is the technique of Eisenstein and Poincaré series. One builds families of eigenfunctions of the Laplace operator on the upper half plane, transforming in the prescribed way, meromorphically depending on a parameter, which is essentially the eigenvalue of the Laplace operator. But in general their growth is large, and they are not square integrable. But all

In this note and its successors we consider the following problems, for the case of the full modular group:

i) Give analytic continuation and functional equations for Eisenstein and Poincaré series in two variables parametrizing eigenvalue and multiplier system.

ii) Give, as far as possible, all square integrable modular forms as values of combinations of Eisenstein and Poincaré series in two variables.

iii) Get as much information as possible on the dependence of the modular spectrum on the multiplier system.

1.2. The full modular group admits real analytic automorphic forms for each real weight; see e.g. [9], [4] Chap. 9. For each weight one has 6 possible multiplier systems, which one may choose continuously depending on the weight.

Using the differential operators given in [9], 3.1, 3.2, one may change the weight by multiples of 2 without essentially changing the spectrum. In this way one may obtain a reduction to weights between 0 and 12, and to only one family of multiplier systems.

In this note we use methods from perturbation theory of linear operators. For holomorphic families of self-adjoint operators in Hilbert space rather strong results are available. For weights in (0, 12) this gives the holomorphy of the modular spectrum; see Proposition 2.5.

For weight 0 the Laplacian has also a continuous spectrum; here perturbation theory gives only results after restriction of the Laplacian to a smaller space. The ideas in the work of Colin the Verdière, [2], are an essential ingredient to cover weight 0. Proposition 2.8 describes a part of the spectrum which is holomorphic at weight 0. Proposition 2.6 will in the next note turn out to be the key to the analytic continuation of Eisenstein and Poincaré series in two variables.

Once this continuation will be available we shall get more precise information on the dependence of the modular spectrum on the weight.

1.3. Perturbation theory of linear operators is not easily extended to more than one complex variable. So our methods will probably not work if one considers groups for which the variety of multiplier systems has complex dimension larger than one.

2. Formulation of Results

2.1. Let $\mathfrak{h}$ be the upper half plane, on it acts $G=\mathrm{SL}_2(\mathbb{R})$ by $\begin{pmatrix} a b \\ c d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$; this action leaves invariant the measure $d\mu(z)=dx\,dy/y^2$, here and later on $x=\mathrm{Re}\,z$, $y=\mathrm{Im}\,z$.

$\Gamma=\mathrm{SL}_2(\mathbb{Z})$ is the full modular group, with $F=\{z\in\mathfrak{h}\colon |x|\leqq\frac{1}{2},\ |z|\geqq 1\}$ its standard fundamental domain.

compare e.g. the discussion in [1], 4.3, 4.4. The multiplier system of the Dedekind eta function is $v_{\frac{1}{2}}$, see [6], Chap. IX. The multiplier system v_r is suitable for all weights $q \equiv r(2)$. If $r \in 2\mathbb{Z}$, then v_r is a character of Γ.

2.2. In the theory of automorphic forms one is interested in functions $f\colon \mathfrak{h} \to \mathbb{C}$ satisfying

(α) $f(\gamma \cdot z) = v_r(\gamma) e^{iq \arg(cz+d)} f(z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
with $-\pi < \arg \leqq \pi$.

(β) $L_q f = \lambda f \quad$ for some $\lambda \in \mathbb{C}$
with $L_q = -y^2 \frac{\partial^2}{\partial x^2} - y^2 \frac{\partial^2}{\partial y^2} + iqy \frac{\partial}{\partial x}$.

(γ) some growth condition.

2.3. If $r, q \in \mathbb{R}$ then $H(r, q)$ is the Hilbert space of classes of functions f on $\mathfrak{h}$ satisfying (α) and for which

$$\int_F |f|^2 \, d\mu < \infty.$$

The differential operator L_q determines a self-adjoint operator $A_q(r)$ in $H(r, q)$; for $0 < r < 12$ it has a discrete spectrum, if $r = 0$ also a continuous spectrum is present.

2.4. Denote by $A^{sq}(r, q, \lambda)$ the λ-eigenspace of $A_q(r)$ in $H(r, q)$. Let

$$E_q^{\pm} = \pm 2iy \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} \pm q.$$

One may show that

$$E_q^{\pm}\colon A^{sq}(r, q, \lambda) \to A^{sq}(r, q \pm 2, \lambda)$$

and

$$L_q = -\tfrac{1}{4} E_{q \mp 2}^{\pm} E_q^{\mp} - \tfrac{1}{4} q^2 \pm \tfrac{1}{2} q.$$

(See e.g. [1], 2.5, 2.6, 4.5, 10.3, 10.4.)

This implies that $E_q^{\pm}$ is bijective, except for special values of λ. These exceptions correspond to the discrete series case, fully described by holomorphic modular forms. The unknown part of the spectrum of $A_q(r)$ is the same for all $q \equiv r(2)$. So we restrict ourselves to the case $q = r$.

In this note will be proved the following propositions:

2.5. **Proposition.** *There are real analytic functions*

$$\mu_j\colon (0, 12) \to \mathbb{R}$$

$$\phi_j\colon (0, 12) \times \mathfrak{h} \to \mathbb{C}$$

for $j = 1, 2, \ldots$ such that for

is a complete orthonormal system in $H(r, r)$,

$$L_r \phi_j(r) = \mu_j(r)\,\phi_j(r);$$

if $r_0 \in (0, 12)$ *and* $N \notin \{\mu_j(r_0): j \geqq 1\}$, *then there exists a neighbourhood* U *of* r_0 *in* $(0, 12)$ *such that for all* $r \in U$ *the set* $\{j \geqq 1: \mu_j(r) < N\}$ *is finite and does not depend on* $r \in U$.

2.6. **Proposition.** *Let* $a > 5$, $B \subset \mathbb{Z}$, B *finite*, $0 \in B$, $\omega_\nu \in \mathbb{C}$ *for each* $\nu \in B$.

There is an open neighbourhood W *of* $(-12, 12)$ *in* $\mathbb{C}$ *and an analytic set* $S \subset W \times \mathbb{C}$ *such that* $S \cap (\{r\} \times \mathbb{C})$ *is discrete for each* $r \in W$, *and there exists for each* $(r, s) \in W \times \mathbb{C} \setminus S$

$$P(r, s): \mathfrak{h} \to \mathbb{C}$$

such that

i) $P(r, s)$ *satisfies* (α) *in* 2.2 *with* $q = r$,

ii) $L_r P(r, s) = (\frac{1}{4} - s^2)\, P(r, s)$,

iii) *for each* $z \in \mathfrak{h}$ *the function* $(r, s) \mapsto P(r, s; z)$ *extends to a meromorphic function on* $W \times \mathbb{C}$ *holomorphic outside* S,

iv) $(r, s) \mapsto \int_{\mathfrak{h}} P(r, s; z)\,\overline{k(z)}\,d\mu(z)$ *extends to a meromorphic function on* $W \times \mathbb{C}$, *holomorphic outside* S *for each* $k \in C_c^\infty(\mathfrak{h})$; *moreover for each* $\omega \in S$ *there is a holomorphic function* $g \neq 0$ *on a neighbourhood* U *of* ω *such that*

$$(r, s) \mapsto g(r, s) \int_{\mathfrak{h}} P(r, s; z)\,\overline{k(z)}\,d\mu(z)$$

is holomorphic on U *for all* $k \in C_c^\infty(\mathfrak{h})$.

v) *For each* $(r, s) \in W \times \mathbb{C} \setminus S$ *put*

$$P(r, s; z) = \sum_{\nu \in \mathbb{Z}} e^{2\pi i(\nu + \frac{1}{12} r)x} f_\nu(y)$$

then

$$f_\nu(a) = \omega_\nu a^{\frac{1}{2} + s} e^{-2\pi(\nu + \frac{1}{12} r)a} \cdot {}_1F_1\left[\begin{matrix} \frac{1}{2} + s - \frac{1}{2} r \\ 1 + 2s \end{matrix} \middle| 4\pi(\nu + \tfrac{1}{12} r)a\right] \quad \text{for } \nu \in B$$

and

$$z \mapsto \begin{cases} P(r, s; z) - \sum_{\nu \in B} e^{2\pi i(\nu + \frac{1}{12} r)x} f_\nu(y) & \text{if } y > a \\ P(r, s; z) & \text{if } y \leqq a \end{cases}$$

defines an element of $L^2(F, d\mu)$ *meromorphic in* (r, s) *in Hilbert space sense.*

${}_1F_1$ is a confluent hypergeometric function, see e.g. [11], §1.

2.7. Remark on the *proof*. If P is given as a distribution on $(W \times \mathbb{C} \setminus S) \times \mathfrak{h}$ satisfying ii) and iv), then iii) follows. For P is an eigendistribution of the elliphence it is real analytic in (r, s, z). So one may interchange the integrations in the right hand side of

$$\int_{\mathfrak{h}} P(r_0, s_0)\bar{k}\,d\mu = \int_{C_1}\int_{C_2}\int_{\mathfrak{h}} P(r, s)\bar{k}\,d\mu(r-r_0)^{-1}(s-s_0)^{-1}\frac{dr}{2\pi i}\frac{ds}{2\pi i};$$

this gives the pointwise holomorphy.

2.8. **Proposition.** *Let $B \subset \mathbb{Z}$, B finite, $0 \in B$. Denote by $S_r^B[\lambda]$ the set of functions $f: \mathfrak{h} \to \mathbb{C}$ satisfying* (α) *and* (β) *in* 2.2 *with $q=r$ and*

$$f(z) = \sum_{\nu \in \mathbb{Z} \setminus B} f^{\nu}(y)\, e^{2\pi i(\nu + \frac{1}{12}r)x}$$

with

$$f^{\nu}(y) = \mathcal{O}(e^{-\beta y}) \quad \text{for } y \to \infty$$

for some $\beta > 0$.

i) *The sets W and S in Proposition* 2.6 *may be chosen such that*

$$\{(r, s) \in W \times \mathbb{C}: S_r^B[\tfrac{1}{4}-s^2] \neq 0\} \subset S \quad \text{and} \quad \dim S_r^B[\tfrac{1}{4}-s^2] < \infty$$

for all $(r, s) \in W \times \mathbb{C}$.

ii) *If $r \in (0, 12)$ and $s \in \mathbb{C}$ satisfy $S_r^B[\frac{1}{4}-s^2] \neq 0$, then $\frac{1}{4}-s^2 \in \mathbb{R}$ and $\frac{1}{4}-s^2 \geqq -\frac{1}{4}r^2 + \frac{1}{2}|r|$.*

iii) *If for some interval $J \subset (-12, 12)$ there are analytic*

$$\mu: J \to \mathbb{R}$$

$$\psi: J \times \mathfrak{h} \to \mathbb{C}$$

such that $\psi \neq 0$ and

$$\psi(r, .) \in S_r^B[\mu(r)] \quad \text{for each } r \in J,$$

then μ may be extended holomorphically to a simply connected neighbourhood U of $(-12, 12)$, $U \subset W$, and there are $\psi_1, \ldots, \psi_m: U \times \mathfrak{h} \to \mathbb{C}$ such that

a) *ψ_j is analytic on $U \times \mathfrak{h}$,*

b) *$\psi_j(r, z)$ is holomorphic in $r \in U$ for $z \in \mathfrak{h}$,*

c) *$\psi_j(r, .) \in S_r^B[\mu(r)]$ for each $r \in U$,*

d) *there is a neighbourhood U_1 of J, $U_1 \subset U$, and there are holomorphic functions $f_1, \ldots, f_m$ on U_1 such that*

$$\psi(r, z) = \sum_{i=1}^{m} f_i(r)\psi_i(r, z) \quad \text{for all } r \in J,\ z \in \mathfrak{h}.$$

3. Transformation

To study the effect of changes in ...

3.1. Lemma. *There exists a function* $\alpha: \Gamma \to \frac{1}{6}\pi\mathbb{Z}$ *such that* $v_r(\gamma) = e^{ir\alpha(\gamma)}$ *for all* $\gamma \in \Gamma$.

Proof. Let η be the Dedekind eta function, and $\log\eta$ a holomorphic logarithm of it. Then for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \tfrac{1}{2} i\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \log\eta\left(\frac{az+b}{cz+d}\right) - \log\eta(z) - \tfrac{1}{2}\log(cz+d)$$

defines α such that $v_r = e^{ir\alpha}$. See [6], Chap. IX.

3.2. Lemma. *Let* $Y > 3$. *There exist* $t \in C^\infty(\mathfrak{h})$, *real valued, satisfying*

i) $t(z) = \frac{1}{6}\pi x$ *for* $y \geqq Y$,

ii) $t(\gamma z) = t(z) + \alpha(\gamma) + \arg(cz+d)$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Proof. $2\,\mathrm{Im}\log\eta$ satisfies ii), it is even analytic on $\mathfrak{h}$.

Define b on $\mathfrak{h}$ by taking for $z \in F$, $\gamma \in \Gamma$:

$$b(\gamma z) = \begin{cases} 0 & \text{if } y < 2 \\ \frac{1}{6}\pi x + \alpha(\gamma) + \arg(cz+d) & \text{if } y \geqq 2; \end{cases}$$

on the boundary of F no ambiguities arise.

Take further a cut off function $\psi_0 \in C^\infty(0, \infty)$,

$$\psi_0(y) = \begin{cases} 1 & \text{for } y \leqq Y-1 \\ 0 & \text{for } y \geqq Y \end{cases} \qquad 0 \leqq \psi_0 \leqq 1;$$

and define for $z \in F$, $\gamma \in \Gamma$

$$\psi(\gamma z) = \psi_0(y).$$

Then $\psi \in C^\infty(\Gamma \backslash \mathfrak{h})$, and

$$t = \psi \cdot 2\,\mathrm{Im}\log\eta + (1-\psi) \cdot \mathrm{b}$$

satisfies the requirements of the lemma.

3.3. We shall keep $Y < 5$, t and ψ as in Lemma 3.2 fixed throughout this paper.

3.4. For a function f on $\mathfrak{h}$ put $g = e^{-irt} \cdot f$. Then from the properties of t follows that condition (α) on f with $q = r$ is equivalent to the following condition on g:

($\tilde{\alpha}$) $\quad g(\gamma z) = g(z) \quad$ for all $\gamma \in \Gamma$.

Furthermore, the differential operator $L(r)$ defined by

$$L(r)\, e^{-irt} = e^{-irt}\, L$$

$$L(r) = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+r\left\{i(y-2y^2t_x)\frac{\partial}{\partial x}-2iy^2t_y\frac{\partial}{\partial y}\right.$$
$$\left.-iy^2(t_{xx}+t_{yy})\right\}+r^2\{-yt_x+y^2(t_x^2+t_y^2)\}$$

is elliptic with coefficients in $C^\infty(\mathfrak{h})$.

3.5. Under the correspondence $f\mapsto g=e^{-irt}\cdot f$ the Hilbert space $H(r,r)$, with $r\in\mathbb{R}$, is mapped isomorphically onto $H:=L^2(\Gamma\backslash\mathfrak{h}, d\mu)$. We shall prove:

3.6. **Proposition.** *For $j=1,2,\ldots$ there are a simply connected neighbourhood U_j of $(0,12)$ and*

$$\mu_j\colon U_j\to\mathbb{C} \text{ holomorphic}$$
$$\phi_j\colon U_j\to H \text{ holomorphic}$$

such that

i) $L(r)\phi_j(r)=\mu_j(r)\phi_j(r)$ *for* $r\in U_j$, $j\geqq 1$.

ii) $\{\phi_j(r)\colon j\geqq 1\}$ *is a complete orthonormal system in H for each* $r\in(0,12)$.

iii) $\mu_j(r)$ *is real, and* $\mu_j(r)\geqq\frac{1}{2}r(1-\frac{1}{2}r)$ *for* $r\in(0,12)$.

iv) *Let $r_0\in(0,12)$ and $\lambda_0\in\mathbb{C}$. There are neighbourhoods U of r_0 and X of λ_0 such that:*

$$Z=\{j\geqq 1\colon \mu_j(r_0)\in X\} \quad \text{is finite},$$
$$U\subset U_j \quad \text{for each } j\in Z,$$

if $\lambda\in X$ and $r\in U$ then $\{\phi_j(r)\colon j\in Z,\ \mu_j(r)=\lambda\}$ is a basis of $\{f\in H\colon L(r)f=\lambda f\}$.

3.7. This proposition implies Proposition 2.5. For ϕ_j determines a distribution on $U_j\times\mathfrak{h}$ given by an analytic function in (r,z) by a similar argument as in 2.7.

3.8. Proposition 2.6 will be proved by taking $P(r,s)=Q(r,s)\cdot e^{irt}$, with $Q(r,s)$ a family of distributions on $\Gamma\backslash\mathfrak{h}$, satisfying

ii) $L(r)Q(r,s)=(\frac{1}{4}-s^2)Q(r,s)$

iv) $(r,s)\mapsto\int_{\mathfrak{h}} Q(r,s)e^{irt}\bar{k}\,d\mu$

for $k\in C_c^\infty(\mathfrak{h})$ determines a meromorphic function in the sense of that proposition.

v) The corresponding condition on the Fourier coefficients.

3.9. Let for $l\in 2\mathbb{Z} \bmod 12\mathbb{Z}$:

D_l: the space of distributions on $\mathfrak{h}$, transforming under translations by $\gamma\in\Gamma$ according to the character v_{-l} of Γ

A_l: the subspace of D_l of distributions $k\mapsto\int_{\mathfrak{h}} f\bar{k}\,d$ with $f\in C^\infty(\mathfrak{h})$

H_l: the subspace of D_l of distributions $k\mapsto\int_{\mathfrak{h}} f\bar{k}\,d\mu$ with f measurable and

H_l is a Hilbert space, with K_l as a dense subspace. The scalar product is given by $\langle f, g\rangle = \int_{\Gamma\backslash\mathfrak{h}} f\bar{g}\, d\mu$.

3.10. D is orthogonal to the kernel of the map

$$C_c^\infty(\mathfrak{h}) \ni k \mapsto \tfrac{1}{2}\sum_{\gamma\in\Gamma} k\circ\gamma \in K;$$

this map is in fact surjective onto K, and D may be identified with the antidual of K.

3.11. $L(r)D\subset D$, and its adjoint is given by $L(r)^* = L(\bar{r})$. $L(r)$ leaves the subspaces A and K of D invariant.

It is not possible to transform $E_r^\pm$ completely within the framework of Γ-invariant functions; here we need the cases $l=2$ and -2.

Define the differential operators $E^\pm(r)$ and $F^\pm(r)$ by the following scheme (for condition (α) see 2.2):

$$\begin{array}{ccc}
\left\{\begin{matrix} f\in C^\infty(\mathfrak{h}), \text{ satisf.} \\ (\alpha) \text{ with } q=r \end{matrix}\right\} & \underset{E_r^\pm}{\overset{E_{r\pm2}^\mp}{\rightleftarrows}} & \left\{\begin{matrix} f\in C^\infty(\mathfrak{h}), \text{ satisf.} \\ (\alpha) \text{ with } q=r\pm2 \end{matrix}\right\} \\
\downarrow \cdot e^{-irt} & & \downarrow \cdot e^{-i(r\pm2)t} \\
A_0 & \underset{E^\pm(r)}{\overset{F^\mp(r)}{\rightleftarrows}} & A_{\pm2}
\end{array}$$

So $E^\pm(r) = e^{\mp 2it}\left\{\pm 2iy\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y} + r(2iyt_y \mp 2yt_x \pm 1)\right\}$

$$F^\pm(r) = E^\pm(r) \mp 2e^{\mp 2it}(2iyt_y \mp 2yt_x \pm 1).$$

The adjoints are given by $E^\pm(r)^* = -F^\mp(\bar{r})$. $E^\pm(r)$ and $F^\mp(r)$ act on D_0, resp. $D_{\pm2}$ and preserve A_l and K_l. We have

$$L(r) = -\tfrac{1}{4}F^\pm(r)E^\mp(r) - \tfrac{1}{4}r^2 \pm \tfrac{1}{2}r.$$

4. Fourier Expansion

Each $f\in A$ satisfies $f(z+1)=f(z)$, and hence may be expanded in a Fourier series. Only above the level $y=Y$ this expansion corresponds to the usual Fourier series expansion for functions satisfying condition (α) in 2.2.

4.1. If $\nu\in\mathbb{Z}$, $\varphi\in C_c^\infty(Y,\infty)$ then

$\left\{\begin{matrix} 0 & \text{if } y\le Y \end{matrix}\right\}$ for $z\in F$

For $d \in D$ and $\nu \in \mathbb{Z}$ define the distribution $F_\nu d$ on (Y, ∞) by

$$\langle F_\nu d, \varphi \rangle = \langle d, h_{\varphi,\nu} \rangle \quad \text{for all } \varphi \in C_c^\infty(Y, \infty).$$

If $f \in A$, then $F_\nu f \in C^\infty(Y, \infty)$; if $f \in K$, then $F_\nu f(y) = 0$ for y sufficiently large, and if $f \in H$ then $F_\nu f \in L^2((Y, \infty), dy/y^2)$ and

$$\sum_{\nu \in \mathbb{Z}} \|F_\nu f\|^2 \leqq \|f\|^2.$$

4.2. $F_\nu L(r) = l_\nu(r) F_\nu$, with $l_\nu(r)$ the differential operator on (Y, ∞) given by

$$l_\nu(r) = -y^2 \frac{\partial^2}{\partial y^2} + 2\pi y(\nu + \tfrac{1}{12} r)(2\pi y(\nu + \tfrac{1}{12} r) - r).$$

4.3. The space of solutions of $(l_\nu(r) - \frac{1}{4} + s^2) f = 0$ has dimension two. We single out the following solution:

$$\begin{aligned}
\mu_\nu(r, s; y) &= y^{\frac{1}{2}+s} e^{-2\pi y(\nu + \frac{1}{12} r)} \\
&\quad \cdot {}_1F_1 \left[\begin{matrix} \frac{1}{2} + s - \frac{1}{2} r \\ 1 + 2s \end{matrix} \middle| 4\pi y(\nu + \tfrac{1}{12} r) \right] \quad \text{for } s \notin \{-\tfrac{1}{2}, -1, -\tfrac{3}{2}, \ldots\} \\
&= y^{\frac{1}{2}+s} \qquad\qquad \text{if } \nu + \tfrac{1}{12} r = 0 \\
&= (4\pi(\nu + \tfrac{1}{12} r))^{-\frac{1}{2}-s} M_{\frac{1}{2}r, s}(4\pi(\nu + \tfrac{1}{12} r) y) \\
&\qquad\qquad \text{if } \nu + \tfrac{1}{12} r \neq 0;
\end{aligned}$$

see e.g [11], §1 for the confluent hypergeometric function ${}_1F_1$ and the Whittaker function $M_{\frac{1}{2}r, s}$. In the expression for the case $\nu + \frac{1}{12} r \neq 0$ we have to choose the argument of $\nu + \frac{1}{12} r$ prudently. $\mu_\nu(r, s)$ is an analytic function on $(0, \infty)$; its importance is its holomorphy in $(r, s) \in \mathbb{C} \times (\mathbb{C} \setminus \{-\frac{1}{2}, -1, \ldots\})$, pointwise and tested against $\varphi \in C_c^\infty(0, \infty)$. The singularity along the line $s + \frac{1}{2} l = 0$ ($l = 1, 2, \ldots$), may be removed by multiplication by $s + \frac{1}{2} l$.

4.4. Let $(a, b) \subset (0, \infty)$. If Ω is open in $\mathbb{C}^2$ and $h: \Omega \times (a, b) \to \mathbb{C}$ satisfies

i) $(l_\nu(r) - \frac{1}{4} + s^2) h = 0$ for $(r, s) \in \Omega$,

ii) $(r, s) \mapsto h(r, s; y)$ is holomorphic for $y \in (a, b)$,

then h may be extended to a map $\Omega \times (0, \infty) \to \mathbb{C}$ with the same properties.

Outside lines of the form $s = \frac{1}{2} l$, $l \in \mathbb{Z}$, one may see this by writing $h(r, s) = a(r, s) \mu_\nu(r, s) + b(r, s) \mu_\nu(r, -s)$. At the exceptional lines one uses another basis of the space of solutions.

5. Self-adjoint Extension of $L(r)$

Take $r \in (-12, 12)$. It is known that the method of Friedrichs extension gives a

5.1. Put $D(r)=\{f\in H: E^+(r)f\in H_2,\ E^-(r)f\in H_{-2}\}$ and define on $D(r)$ the sesquilinear form

$$s(r)[f,g]=\tfrac{1}{8}\langle E^+(r)f,\ E^+(r)g\rangle+\tfrac{1}{8}\langle E^-(r)f,\ E^-(r)g\rangle-\tfrac{1}{4}r^2\langle f,g\rangle.$$

This form is symmetric and bounded from below by $-\frac{1}{4}r^2$. On $K\subset D(r)$ it is given by

$$s(r)[f,g]=\langle L(r)f,g\rangle,$$

see 3.11.

One may easily check that $s(r)$ is closed. Now a representation theorem for sesquilinear forms implies that there exists a self-adjoint operator $A(r)$ in H with domain dense in $D(r)$ such that

$$s(r)[f,g]=\langle A(r)f,g\rangle \qquad \text{for } f\in \mathrm{dom}(A(r)),\ g\in D(r);$$

see e.g. [5], Chap. VI, Theorem 2.1, §2.1 and §2.3. This operator is an extension of $L(r)$ on K, and on its domain it is given by the differential operator $L(r)$.

5.2. If λ is an eigenvalue of $A(r)$, then all its eigenfunctions are given by C^∞-functions, square integrable on F.

Conversely, if $f\in C^\infty(\Gamma\backslash\mathfrak{h})$, $(L(r)-\lambda)f=0$ and $f\in H$, then $f\in\mathrm{domain}(A(r))$ and $A(r)f=\lambda f$. To see this one may work with $e^{ir\mathfrak{t}}\cdot f=f_1$ which satisfies $L_r f_1=\lambda f_1$. By inspecting its Fourier series, or by following the reasoning in the proof of Lemma 10.2.5 of [1], one sees that $E_r^\pm f_1$ are square integrable. This implies $f\in D(r)$, and then $A(r)f=\lambda f$ follows.

5.3. The bound $s(r)\geqq -\frac{1}{4}r^2$ implies the same bound for the spectrum of $A(r)$.

One has even $\lambda\geqq -\frac{1}{4}r^2+\frac{1}{2}|r|$ for each eigenvalue λ of $A(r)$. For if $A(r)f=\lambda f$ then

$$\langle\lambda f,f\rangle=\tfrac{1}{4}\langle E^\pm(r)f,\ E^\pm(r)f\rangle+(-\tfrac{1}{4}r^2\pm\tfrac{1}{2}r)\langle f,f\rangle.$$

(To see that in this situation $-\langle F^\mp(r)E^\pm(r)f,f\rangle=\|E^\pm(r)f\|^2$ use again the reasoning in §10.2 of [1].)

6. Truncation

Now we apply the ideas in [2] and combine them with analytic perturbation theory as described e.g. in [5], Chap. VII, §4.

6.1. We fix a finite set $B\subset\mathbb{Z}$ and a number $a>Y$. A lot of objects depending on this choice of B and a will be defined; the dependence on B will not show in the notation.

6.2. Define ${}^aD=\{d\in D:\ F_\nu d=0 \text{ on } (a,\infty) \text{ for all } \nu\in B\}$; ${}^aH={}^aD\cap H$, ${}^aK={}^aD\cap K$.

so ${}^aH$ is closed in H.

${}^aK$ is dense in ${}^aH$.

6.3. For $r\in(-12, 12)$ put

$$\mu(r)=\inf_{\nu\in\mathbb{Z},\, \nu\notin B} |\nu+\tfrac{1}{12}r|.$$

6.4. We fix $r_0\in(-12, 12)$, such that $\mu(r_0)=\mu>0$. Let ${}^aD(r_0)$ denote $D(r_0)\cap{}^aH$ and denote by ${}^as(r_0)$ the restriction of $s(r_0)$ to ${}^aD(r_0)$. This restriction is a closed, symmetric form, bounded from below, densely defined in ${}^aH$.

6.5. Define the positive sesquilinear form $[\cdot,\cdot]$ on ${}^aD(r_0)$ by

$$[f, g]={}^as(r_0)[f, g]+\tfrac{1}{4}r_0^2\langle f, g\rangle.$$

6.6. **Lemma.** *For all $\alpha\geqq a$, α sufficiently large, one has for all $f\in{}^aD(r_0)$:*

$$\pi^2\mu^2 \int_{\substack{z\in F\\ y\geqq\alpha}} y^2\,|f(z)|^2\,d\mu(z)\leqq[f, f].$$

Proof. $E^{\pm}(r)=e^{\mp 2it}\left(\pm 2iy\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+rb_{\pm}\right)$ with $b_{\pm}=2iyt_y\mp 2yt_x\pm 1$. For $y>Y$: $b_{+}(z)=\pm(1-\frac{1}{3}\pi y)$. As $E^{\pm}(r)f\in H_{\pm 2}$, we know that $2yf_y+2ir_0yt_yf$ and $2iyf_x-2r_0yt_xf+r_0f$ are elements of $L^2(F, d\mu)$. So we get

$$\begin{aligned}[f, f]&=\int_F\{\tfrac{1}{4}|2yf_y+2ir_0yt_yf|^2\\ &\qquad+\tfrac{1}{4}|2iyf_x-2r_0yt_xf+r_0f|^2\}\,d\mu(z)\\ &\geqq\int_{\substack{z\in F\\ y\geqq\alpha}}\tfrac{1}{4}|2iyf_x+r_0(1-\tfrac{1}{3}\pi y)f|^2\,d\mu(z)\\ &=\sum_{\nu\notin B}\int_\alpha^\infty\tfrac{1}{4}(-4\pi\nu y+r_0(1-\tfrac{1}{3}\pi y))^2\,|F_\nu f(y)|^2\,y^{-2}\,dy.\end{aligned}$$

Now take $\alpha>r_0/(2\pi\mu)$, then for all $y\geqq\alpha$:

$$(-4\pi\nu y+r_0(1-\tfrac{1}{3}\pi y))^2\geqq(2\pi|\nu+\tfrac{1}{12}r_0|\,y)^2,$$

so

$$\begin{aligned}[f, f]&\geqq\sum_{\nu\notin B}\pi^2\mu^2\int_\alpha^\infty y^2\,|F_\nu f(y)|^2\cdot y^{-2}\,dy\\ &=\pi^2\mu^2\int_{z\in F,\, y\geqq\alpha} y^2\,|f(z)|^2\cdot d\mu(z).\end{aligned}$$

6.7. **Lemma.** *The form $f\mapsto\|f\|^2$ on ${}^aD(r_0)$ is compact with respect to the form $f\mapsto[f, f]$.*

6.8. Lemma. *For $f\in {}^aD(r_0)$ we have $b_\pm f\in L^2(F,d\mu)$ and the form $f\mapsto \|b_\pm f\|^2$ on ${}^aD(r_0)$ is relatively bounded with respect to $f\mapsto [f,f]$.*

Proof. On $\{z\in F;\, y\leqq \alpha\}$ the functions b_+ and b_- are bounded, and on $y\geqq\alpha$ we have $|b_\pm|\ll y$. So with help of Lemma 6.6 we see that

$$\|b_\pm f\|^2\leqq c_1[f,f]+c_2\|f\|^2$$

for some $c_1, c_2>0$.

6.9. Now put for $f, g\in {}^aD(r_0)$:

$$\begin{aligned} s^1[f,g]=&\tfrac18\langle e^{-2i\theta}b_+f,\, E^+(r_0)g\rangle+\tfrac18\langle e^{2i\theta}b_-f,\, E^-(r_0)g\rangle\\ &+\tfrac18\langle E^+(r_0)f,\, e^{-2i\theta}b_+g\rangle+\tfrac18\langle E^-(r_0)f,\, e^{2i\theta}b_-g\rangle\\ &-\tfrac12 r_0\langle f,g\rangle\\ s^2[f,g]=&\tfrac18\langle b_+f, b_+g\rangle+\tfrac18\langle b_-f, b_-g\rangle-\tfrac14\langle f,g\rangle. \end{aligned}$$

Then for $h, k\in {}^aK$, and $r\in\mathbb{C}$:

$$\begin{aligned}\langle L(r)h,k\rangle=&s(r_0)[h,k]+(r-r_0)s^1[h,k]\\ &+(r-r_0)^2s^2[h,k].\end{aligned}$$

The forms s^1 and s^2 are relatively bounded with respect to ${}^as(r_0)$ by Lemma 6.8. In this situation [5], Chap. VII, §4.3, Theorem 4.8 gives:

6.10. Lemma. *There is a disk $U(r_0)$ in $\mathbb{C}$ with center r_0 such that for all $r\in U(r_0)$ the form*

$$t(r)[f,g]={}^as(r_0)[f,g]+(r-r_0)s^1[f,g]+(r-r_0)^2s^2[f,g]$$

on ${}^aD(r_0)$ has a sectorial closure $\tilde t(r)$. The family of forms $r\mapsto \tilde t(r)$ on $U(r_0)$ is holomorphic with domain not depending on r.

Remark that $\tilde t(r_0)=t(r_0)={}^as(r_0)$, as ${}^as(r_0)$ is closed; so $\operatorname{dom}\tilde t(r)={}^aD(r_0)$.

6.11. Lemma. *For $r\in U(r_0)$ and $h,k\in {}^aK$*

$$\tilde t(r)[h,k]=\langle L(r)h,k\rangle.$$

For $r\in U(r_0)\cap(-12,12)$ such that $\mu(r)>0$:

$$\tilde t(r)={}^as(r).$$

See 6.3 for the definition of $\mu(r)$.

Proof. The first assertion follows from 6.9. For r as in the second assertion we have ${}^aD(r)={}^aD(r_0)$ by Lemma 6.8, for

$$E^\pm(r)=E^\pm(r_0)+(r-r_0)e^{\mp 2i\theta}b_\pm.$$

6.12. Proposition. *Let a and B be as in 6.1. There exist an open neighbourhood W of $(-12, 12)$ in $\mathbb{C}$ if $0 \in B$, or an open neighbourhood W of $(0, 12)$ in $\mathbb{C}$ if $0 \notin B$, and on W a holomorphic family ${}^a s$ of closed sectorial sesquilinear forms with constant domain, such that for $r \in (-12, 12)$ if $0 \in B$, or $r \in (0, 12)$ if $0 \notin B$: ${}^a s(r) = s(r) | D(r) \cap {}^a H$.*

Proof. Clear from 6.10 and 6.11 by analytic continuation.

6.13. Proposition. *Let a, B and W be as in Proposition 6.12. There exists on W a self-adjoint holomorphic family of operators ${}^a A$ in ${}^a H$ with compact resolvent, extending the restriction of $L(r)$ to ${}^a K$.*

Proof. See [5], Chap. VII, §4.2, Theorem 4.2 and Remark 4.7 to get the self-adjoint family ${}^a A$. From Lemma 6.11 we see that ${}^a A(r) k = L(r) k$ for $k \in {}^a K$. To prove the compactness of the resolvent, it is sufficient to consider one r; see [5], Chap. VII, §4.2, Theorem 4.3. Take $r \in (0, 12)$. Lemma 6.7 shows that the identity

$$I \colon {}^a D(r) \text{ with } [.,.] \to {}^a H \text{ with } \langle .,. \rangle$$

is compact, so it has a compact adjoint I^*. Let $f \in {}^a H$, $g \in {}^a D(r)$. Put $h = I^* f \in {}^a D(r)$. Then

$$\begin{aligned} {}^a s(r)[h, g] &= [h, g] - \tfrac{1}{4} r^2 \langle h, g \rangle \\ &= \langle f - \tfrac{1}{4} r^2 h, g \rangle. \end{aligned}$$

So $h \in \operatorname{dom} {}^a A(r)$ and ${}^a A(r) h = f - \frac{1}{4} r^2 h$, hence

$$({}^a A(r) + \tfrac{1}{4} r^2) \cdot I^* f = f,$$

which shows that $({}^a A(r) + \frac{1}{4} r^2)^{-1}$ is compact.

6.14. Proposition. *Let W be as in Proposition 6.12. Take $r \in W$. For each $f \in \operatorname{dom} {}^a A(r)$ there exists a map $c_f \colon B \to \mathbb{C}$ such that for $k \in K$:*

$$\langle L(r) f - {}^a A(r) f, k \rangle = \sum_{v \in B} c_f(v) \overline{F_v k(a)}.$$

Proof. The distribution $(L(r) - {}^a A(r)) f$ is clearly orthogonal to ${}^a K$. For $v \in B$:

$$F_v(L(r) f - {}^a A(r) f) = l_v(r) F_v f - F_v {}^a A(r) f = 0 \quad \text{on } y > a$$

and also

$$F_v(L(r) f - {}^a A(r) f) = 0 \quad \text{on } y < a,$$

for $L(r) f - {}^a A(r) f$ is orthogonal to ${}^a K$. So $F_v(L(r) f - {}^a A(r) f)$ is a distribution supported by a. Now $F_v {}^a A(r) f \in L^2((Y, \infty), dy/y^2)$ and

$$\begin{aligned} \langle L(r) f, h_{\varphi, v} \rangle = \tfrac{1}{8} \langle E^-(r) f, E^-(r) h_{\varphi, v} \rangle \\ + \tfrac{1}{8} \langle E^+(r) f, E^+(r) h_{\varphi, v} \rangle - \tfrac{1}{4} r^2 \langle f, h_{\varphi, v} \rangle, \end{aligned}$$

6.15. *Proof* of Proposition 3.6. Take $B=\emptyset$, then ${}^aH=H$. Now i) and ii) are given in [5], Chap. VII, §3.5, Theorem 3.9. As ${}^aA$ is a self-adjoint family, iii) is clear from 5.1. To obtain iv) take a closed curve Γ in $\mathbb{C}$ encircling λ_0 once. The reasoning in [5], Chap. VII, §3.1 reduces everything to a finite dimensional situation, and only the μ_j with $\mu_j(r_0)$ inside Γ need to be taken into account.

7. Proof of Propositions 2.6 and 2.8

We use the method by which Colin de Verdière gives the analytic continuation of Eisenstein series.

7.1. We take a, B and ω as in Proposition 2.6. Proposition 6.13 gives a neighbourhood W of $(-12, 12)$ and on it the holomorphic family ${}^aA$. The resolvent $R(r,s)=({}^aA(r)-\frac{1}{4}+s^2)^{-1}$ is bounded holomorphic on $(W\times\mathbb{C})\setminus S_0$ where $S_0 =\{(r,s): \frac{1}{4}-s^2$ in the spectrum of ${}^aA(r)\}$, see [5], Chap. VII, §1.2, Theorem 1.3. Remark that $S_0\cap(\{r\}\times\mathbb{C})$ is discrete for each $r\in W$.

7.2. **Lemma.** *S_0 is a principal analytic set in $W\times\mathbb{C}$.*

Proof. We have to show that locally S_0 is the zero set of a holomorphic function. Take $(r_0, s_0)\in S_0$, put $\lambda_0=\frac{1}{4}-s_0^2$. The procedure in [5], Chap. VII, §1.3 enables us to split up ${}^aH=M_1(r)\oplus M_2(r)$ holomorphically in r near r_0, such that ${}^aA(r)M_i(r)\subset M_i(r)$ and $\dim M_1(r)<\infty$. All eigenvalues of ${}^aA(r)$ in $M_2(r)$ stay away from λ_0. So near (r_0, s_0) the description of S_0 depends on the spectrum of ${}^aA(r)$ in $M_1(r)$. In [5], Chap. VII, §3.1 it is also shown that for r near r_0 all $M_1(r)$ are isomorphic, with the isomorphism depending holomorphically on r. So finite dimensional theory shows that near (r_0, s_0) the set S_0 is the zero set of $(r,s)\mapsto P_r(\frac{1}{4}-s^2)$, where P_r is the characteristic polynomial of ${}^aA(r)$ in $M_1(r)$.

7.3. **Lemma.** *$R(r,s)$ is bounded meromorphic on $W\times\mathbb{C}$.* Meromorphic means that locally there are non-zero functions g, such that $g(r,s)\cdot R(r,s)$ is holomorphic; in this case bounded holomorphic, i.e. given by power series converging in operator norm.

Proof. As in the proof of the previous lemma we have locally ${}^aH =M_1(r)\oplus M_2(r)$. On $M_2(r)$ the resolvent is bounded holomorphic near (r_0, s_0), on $M_1(r)$ it is bounded meromorphic by finite dimensional theory.

7.4. Put $S=S_0\cup\{(r,s): r\in W, 2s\in\mathbb{Z}, s<0\}$. Then S is an analytic set in $W\times\mathbb{C}$, and $S\cap(\{r\}\times\mathbb{C})$ is discrete for all $r\in W$. Let $X=(W\times\mathbb{C})\setminus S$.

7.5. Put for $(r,s)\in X$, $z\in F$:

$$h(r,s;z)=\sum_{v\in B}\omega_v\cdot(1-\psi(z))\cdot e^{2\pi i v x}\cdot\mu_v(r,s;y)$$

with ψ as in the proof of Lemma 3.2. It extends to a Γ-invariant function on $\mathfrak{h}$ and satisfies $(L(r)-\frac{1}{4}+s^2)h(r,s)=0$, on $y>Y$. For each $k\in K$ the function extends to a meromorphic function on $W\times\mathbb{C}$, holomorphic on X. For each $(r_0, s_0)\in S$ a function $g\neq 0$ such that $g\cdot\langle h, k\rangle$ is holomorphic near (r_0, s_0) may be chosen independent of k; in fact, take $g=1$ if $s_0\notin S\setminus S_0$, and $g(r,s)=s-s_0$ if $s_0\in\frac{1}{2}\mathbb{Z}$, $s_0<0$.

7.6. $H(r,s)=-(L(r)-\frac{1}{4}+s^2)h(r,s)$ has the same meromorphic behaviour on $W\times\mathbb{C}$ as h. It is a family of elements of ${}^aK\subset{}^aH$, its L^2-norm is bounded on compact subsets of X, so it is a holomorphic map $X\to{}^aH$; and similarly, for g as in 7.5, $g\cdot H$ gives holomorphic maps into ${}^aH$ locally on S. So H defines a meromorphic map $W\times\mathbb{C}\to{}^aH$ with singularities only on S.

7.7. $G(r,s)=R(r,s)H(r,s)$ also defines a meromorphic map $G\colon W\times\mathbb{C}\to{}^aH$, holomorphic on X. Put $g_\nu(r,s)=F_\nu G(r,s)$ for $\nu\in B$. Then g_ν is a meromorphic map $W\times\mathbb{C}\to L^2((Y,\infty), dy/y^2)$, holomorphic on X.

7.8. **Lemma.** *For each $\nu\in B$ and each $(r,s)\in X$ there exists a solution $G_\nu(r,s)$ of $(l_\nu(r)-\frac{1}{4}+s^2)G_\nu=0$ such that*

$$G_\nu(r,s;a)=0$$
$$G_\nu(r,s)=g_\nu(r,s)\quad \text{on } (Y,a).$$

For each $\varphi\in C_c^\infty(Y,\infty)$ the map

$$(r,s)\mapsto\int_Y^\infty G_\nu(r,s)(y)\,\overline{\varphi(y)}\,y^{-2}\,dy$$

is meromorphic on $W\times\mathbb{C}$, holomorphic on X; for each point of S a holomorphic function $g\neq 0$ such that $g\cdot\int_Y^\infty G_\nu\cdot\bar\varphi\, y^{-2}\,dy$ becomes holomorphic near that point, may be chosen independent of φ.

Proof. On (Y,a) we have $(l_\nu(r)-\frac{1}{4}+s^2)g_\nu(r,s)=0$ (see Proposition 6.14), so we may apply 4.4 to define $G_\nu(r,s)$ for $(r,s)\in X$. At a point $(r_0,s_0)\in S$ we may apply 4.4 to $g\cdot g_\nu$, for the condition $(l_\nu(r)-\frac{1}{4}+s^2)g\cdot g_\nu=0$ may be proved weakly, and hence is satisfied on S too.

For $\varphi\in C_c^\infty(Y,\infty)$ we find by applying Proposition 6.14 and by partial integration:

$$\begin{aligned}
c_{G(r,s)}(\nu)\cdot\overline{\varphi(a)}&=\langle (L(r)-\tfrac{1}{4}+s^2)\,G(r,s), h_{\nu,\varphi}\rangle\\
&\quad+\langle(\tfrac{1}{4}-s^2-{}^aA(r))\,G(r,s), h_{\nu,\varphi}\rangle\\
&=\langle G(r,s), (L(\bar r)-\tfrac{1}{4}+\bar s^2)\,h_{\nu,\varphi}\rangle\\
&\quad+\langle(L(r)-\tfrac{1}{4}+s^2)\,h(r,s), h_{\nu,\varphi}\rangle\\
&=\int_a^Y g_\nu(r,s;y)\overline{(l_\nu(\bar r)-\tfrac{1}{4}+\bar s^2)\,\varphi(y)}\,y^{-2}\,dy+0\\
&=g_\nu(r,s;a)\,\overline{\varphi'(a)}-\left(\frac{\partial}{\partial y}g_\nu\right)(r,s;a)\,\overline{\varphi(a)}.
\end{aligned}$$

7.9. Define for $z \in F$, $\gamma \in \Gamma$, $(r,s) \in X$:

$$\tilde{F}(r,s;\gamma z) = \begin{cases} 0 & \text{if } y \leqq a \\ \sum_{v \in B} e^{2\pi i v x} G_v(r,s;y) & \text{if } y \geqq a. \end{cases}$$

Then $\tilde{F}(r,s)$ is again meromorphic on $W \times \mathbb{C}$ with singularities only on S when tested against $k \in K$, with denominator not depending on k.

7.10. Put for $(r,s) \in X$

$$Q(r,s) = G(r,s) + \tilde{F}(r,s) + h(r,s);$$

as distribution it is meromorphic on $W \times \mathbb{C}$, holomorphic on X, with denominator not depending on the test function. It is Γ-invariant.

$$\begin{aligned}(L(r) - \tfrac{1}{4} + s^2) Q(r,s) &= ({}^a A(r) - \tfrac{1}{4} + s^2) G(r,s) + (L(r) - {}^a A(r)) G(r,s) \\ &\quad + (L(r) - \tfrac{1}{4} + s^2) \tilde{F}(r,s) - H(r,s) \\ &= H(r,s) - H(r,s) + \sum_{v \in B} \left(- \left(\frac{\partial}{\partial y} g_v \right)(r,s;a) \overline{\delta_a} \right. \\ &\quad \left. + (l_v(r) - \tfrac{1}{4} + s^2)(G_v(r,s) \cdot \chi_{[a,\infty)}) \right) e^{2\pi i v x} = 0\end{aligned}$$

as

$$(l_v(r) - \tfrac{1}{4} + s^2)(G_v(r,s) \chi_{[a,\infty)}) = \left(\frac{\partial}{\partial y} G_v \right)(r,s;a) \overline{\delta_a}$$

by partial integration as in the proof of 7.8.

(By $\overline{\delta_a}$ is meant $\varphi \mapsto \overline{\varphi(a)}$ and $\chi_{[a,\infty)}$ is the characteristic function of $[a,\infty)$.)

7.11. As indicated in 3.8 we take $P(r,s) = Q(r,s) \cdot e^{irt}$ to prove Proposition 2.6, and S as constructed in 7.4. Now condition ii) is satisfied, hence $z \mapsto P(r,s;z)$ is an analytic function and i) is valid pointwise.

If $k \in C_c^\infty(\mathfrak{h})$ then k_1 defined by

$$k_1(z) = \tfrac{1}{2} \sum_{\gamma \in \Gamma} k(\gamma z)$$

is an element of K.

$$\langle Q(r,s), k_1 \rangle = \int_{\mathfrak{h}} Q(r,s;z) \cdot \overline{k(z)}\, d\mu(z)$$

has the desired meromorphy in (r,s); this is not destroyed if we multiply the integrand by $e^{irt(z)}$, as we see by expanding e^{irt} in its power series. This takes care of iv) and iii).

7.12. For $v \in B$ and $(r,s) \in X$, $y > Y$

Finally, the family of elements in $L^2(F, d\mu)$ which we have to prove to be meromorphic in (r, s), is given by

$$e^{irt} \cdot G(r, s).$$

We know that $G(r, s)$ has the right behaviour in $L^2(F, d\mu)$.

As t and e^{irt} are bounded on F as long as r stays in a compact set, the family of operators in $L^2(F, d\mu)$

$$f \mapsto e^{irt} f$$

is bounded holomorphic in r. This finishes the proof of condition v).

7.13. *Proof* of Proposition 2.8. Take W as chosen above. If $f \in S_r^B[\lambda]$, $r \in W$, then f corresponds to $h \in H$ with

$$L(r)h = \lambda h.$$
$$F_v h = 0 \quad \text{on} \quad (Y, \infty) \quad \text{for } v \in B.$$

So $h \in {}^aH$ and if we know that $E^{\pm}(r)h \in H_{\pm 2}$, then for $k \in K$

$$\langle \lambda h, k \rangle = {}^a s(r)[h, k],$$

hence $\lambda h = {}^a A(r) h$.

If we use some properties of Whittaker functions, then we may show $E^{\pm}(r)h \in H_{\pm 2}$ by inspection of the Fourier series. In a more intrinsic way one looks at the function on the universal covering group $\hat{G}$ of G, corresponding to f, and proceeds as in [1], § 10.2.

So $f \in S_r^B[\lambda]$ corresponds to an eigenfunction of ${}^aA(r)$ with eigenvalue λ. This implies i) and ii) of the proposition.

As in 6.15 we have eigenvalues ν_j and eigenfunctions $\tilde{\psi}_j$ of ${}^aA$, each holomorphic on a simply connected neighbourhood of $(-12, 12)$, with similar properties as the μ_j and ϕ_j in Proposition 2.5. Take the numbering in such a way that $\nu_1, \ldots, \nu_n$ restrict to μ on J as in iii) of Proposition 2.8 and $\nu_j \neq \mu$ on J for $j > n$. Take $\psi_j(r) = e^{irt} \tilde{\psi}_j(r)$. Then a) and b) are clear.

The graphs of different ν_i intersect only in a discrete set of points. So for almost all $r \in (-12, 12)$ the $\tilde{\psi}_j(r)$, $j = 1, \ldots, n$, form a basis of $\ker({}^aA(r) - \nu(r))$. So for a dense set of $r \in J$ we may express

$$\tilde{\psi}(r) = e^{-irt} \cdot \psi(r) = \sum_{j=1}^{n} f_j(r) \tilde{\psi}_j(r).$$

By considering $z_1, \ldots, z_n \in \mathfrak{H}$ such that $\det((\tilde{\psi}_j(r, z_l)))_{j,l} \neq 0$ locally, we see that the f_j are holomorphic on a neighbourhood of J, and that $\tilde{\psi} = \sum_{j=1}^{n} f_j \tilde{\psi}_j$ is valid on J.

After omitting some of the ν_j we may assume that $f_j \neq 0$ for $j = 1, \ldots, n$. Suppose ψ_l does not satisfy c) in the proposition. Then is nonzero for some (r, y) and some $v \in B$. As ψ_l^v is analytic in (r, y), we may assume $\psi_l^v(r_0, y_0) \neq 0$ with $r_0 \in J$ and $Y < y_0 < a$. This means $\langle \tilde{\psi}_l, h_{\varphi,v} \rangle \neq 0$ for some φ with support inside (Y, a). Now Lemma 7.14 shows that we may rearrange the $\tilde{\psi}_1, \ldots, \tilde{\psi}_n$ such that $\langle \tilde{\psi}_i, h_{\varphi,v} \rangle = 0$ for $i = 1, \ldots, n-1$, and $\langle \tilde{\psi}_n, h_{\varphi,v} \rangle \neq 0$.

After this rearrangement we see that $f_n = 0$, so $\tilde{\psi}_n$ may be omitted. This process may be continued till all ψ_l satisfy c) in the proposition.

7.14. **Lemma.** *Let $I \subset \mathbb{R}$ be an interval and $X \subset \mathbb{C}$ a simply connected open set such that $I = X \cap \mathbb{R}$. Let $a: X \to \mathbb{C}^t$, $t \geq 1$, be a holomorphic map. Then there exist a simply connected open set $X_1 \subset X$ with $I = X_1 \cap \mathbb{R}$ and a holomorphic map U from X_1 into the $t \times t$-matrices with complex elements such that*

$\det U(r) \neq 0$	*for all* $r \in X_1$
$U(r)$ *is unitary*	*for all* $r \in I$
$U(r)\,a(r) \in \mathbb{C} \cdot (0, 0, \ldots, 0, 1)$	*for all* $r \in X_1$.

Remark. To apply this in the proof of Proposition 2.8 take $t = n$, $a(r)_j = \langle \tilde{\psi}_j, h_{\varphi,v} \rangle$. Put

$$\tilde{\chi}_l(r) = \sum_{i=1}^{n} U(r)_{li} \tilde{\psi}_i.$$

Then $\tilde{\chi}_1, \ldots, \tilde{\chi}_n$ are the rearranged $\tilde{\psi}_l$.

Proof. If $a = 0$, we may take $U = Id$. So suppose $a \neq 0$. Define a symmetric holomorphic family T of linear operators in $\mathbb{C}^t$ by $T(r)_{il} = a(r)_i \overline{a(\bar{r})_l}$. If $\sum_{l=1}^{t} x_l \overline{a(\bar{r})_l} = 0$, then $(x_1, \ldots, x_t) \in \ker T(r)$. So $\dim \ker T(r) \geq t-1$, and equality holds for a dense set of r in X. By [5], II, §6.1, Theorem 6.1 there are a simply connected set $X_1 \subset \mathbb{C}$, $I \subset X_1$, and a holomorphic family P of projection operators with rank $t-1$ such that $P(r)\mathbb{C}^t \subset \ker T(r)$ for all $r \in X_1$. As T is symmetric, we have $P(r)^* = P(\bar{r})$. Take $r_0 \in I$. By [5], II, §4.2 and §6.2 there is a family U_1 of operators satisfying

$r \mapsto U_1(r)$ and $r \mapsto U_1(r)^{-1}$ are holomorphic on X_1,

$U_1(r)\,P(r_0)\,U_1(r)^{-1} = P(r)$ for all $r \in X_1$

$U_1(r)$ is unitary for all $r \in I$.

Take a unitary matrix U_0 such that $U_0(\mathrm{Id} - P(r_0))\,U_0^{-1}$ is the orthogonal projection onto $\mathbb{C}(0, \ldots, 0, 1)$. Put $U(r) = U_0\,U_1(r)^{-1}$. The lemma has been proved if we show that $U_0\,P(r_0)\,U_0^{-1} \cdot U(r)\,a(r) = 0$. But

$$U_0\,P(r_0)\,U_0^{-1}\,U(r) = U_0\,U_1(r)^{-1}\,P(r),$$

and $P(r)\,a(r) = 0$, at first for r in a dense subset of I, and then for all $r \in X_1$ by

List of Notations

A, A_l	3.9
^aA	6.13
$A_q(r)$	2.3
$A(r)$	5.1
$A^{\pm q}(r, q, \lambda)$	2.4
a	6.1
B	6.1
$b_{\pm}$	6.6 (proof)
D, D_l	3.9
^aD	6.2
$D(r)$	5.1
$^aD(r)$	6.4
$E_q^{\pm}$	2.4
$E^{\pm}(r)$	3.11
F	2.1
$F^{\pm}(r)$	3.11
F_v	4.1
$\tilde{F}(r, s)$	7.9
f_v	2.6
G	2.1
G_v	7.8
$G(r, s)$	7.7
g_v	7.7
H	3.5, 3.9
H_l	3.9
^aH	6.2
$H(r, q)$	2.3
$H(r, s)$	7.6
$h_{\varphi, v}$	4.1
h	7.5
$\mathfrak{h}$	2.1
K, K_l	3.9
^aK	6.2
L_q	2.2
$L(r)$	3.4
$l_v(r)$	4.2
P	2.6
Q	3.8, 7.10
$R(r, s)$	7.1
S	2.6, 7.4
S_0	7.1
$s(r)$	5.1
$^as(r)$	6.4, 6.12
t	3.2, 3.3
v_r	2.1
W	2.6
X	7.4
Y	3.2, 3.3
α	3.1
Γ	2.1
$d\mu$	2.1
μ_v	4.3
$\mu(r)$	6.3
ψ	3.2, 3.3
$[\,,\,]$	6.5
$\langle\,,\,\rangle$	3.9

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