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Dedekind Sums and Fourier Coefficients of Modular Forms

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We prove the following result on the distribution of Dedekind sums:

$$\lim_{M\to\infty} \frac{\log M}{M} \sum_{c=1}^{M} \frac{1}{c} \sum_{d \bmod c}^{*} g\left(S(d,c), \frac{d}{c}\right) = \frac{12}{\pi^2} \int_{-\infty}^{\infty} \int_{\mathbf{R}/\mathbf{Z}} g(x,y)\, dy\, dx,$$

for each compactly supported continuous function g on $\mathbf{R} \times (\mathbf{R}/\mathbf{Z})$. The proof uses Kuznetsov's sum formula in the modular case for varying real weight.

1. INTRODUCTION

1.1. *Dedekind Sums.* The Dedekind sum $S(d,c)$, with $d, c \in \mathbf{Z}$, $c \geqslant 1$, $(d,c)=1$, is defined by

$$S(d,c) = \sum_{m=1}^{c-1} \left\{\frac{m\,d}{c}\right\} \left\{\frac{m}{c}\right\} \tag{1.1.1}$$

with

$$\{x\} = \begin{cases} 0 & \text{if } x \in \mathbf{Z} \\ y - \frac{1}{2} & \text{if } x \equiv y \bmod 1 \text{ and } 0 < y < 1 \end{cases}$$

Dedekind sums occur in the transformation formula of the eta function of Dedekind (see, e.g., [11]). More generally, one finds Dedekind sums in the multiplier systems of the full modular group for real weights.

1.2. In this paper we study the quantity

$$K_\kappa(u; f) = \sum_{c=1}^{\infty} \frac{1}{c} f(4\pi \sqrt{|u(u+\kappa)|}/c) \sum_{d \bmod c}^{*} e^{24\pi i u S(d,c) + 2\pi i \kappa d/c}, \tag{1.2.1}$$

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where $\kappa \in \mathbf{Z}$, $u \in \mathbf{R}$ such that $u(u+\kappa) \neq 0$. The test function f is continuous on $(0, \infty)$ and satisfies some growth condition at 0 to ensure absolute convergence of the sum. $\sum^*_{d \bmod c}$ means summation over $d \bmod c$, $(c, d) = 1$.

The study of $K_\kappa(u; f)$ with f suitably chosen leads to the following result:

1.3. THEOREM. *For* $g \in C_c(\mathbf{R} \times (\mathbf{R}/\mathbf{Z}))$

$$\lim_{M \to \infty} \frac{\log M}{M} \sum_{c=1}^{M} \frac{1}{c} \sum^*_{d \bmod c} g\left(S(d, c), \frac{d}{c}\right)$$

$$= \frac{12}{\pi^2} \int_{-\infty}^{\infty} \int_{\mathbf{R}/\mathbf{Z}} g(x, y)\, dy\, dx.$$

In [11, p. 28] the question of whether the points $(S(d, c), d/c)$ are dense in the plane is raised. In [6] a positive answer has been given. Reference [9] shows that the sequence of $(d/c, rS(d, c))$ is uniformly distributed modulo 1 for each nonzero $r \in \mathbf{R}$. The result given here describes the distribution a bit more precisely.

1.4. In [3] we studied the distribution of $(S(d, c)/c, d/c)$. We found that the majority of $S(d, c)/c$ is concentrated near 0, independently of d/c. For the minority staying away from 0 we found a discrete limiting distribution. One may view Theorem 1.3 as taking a closer look at the distribution of the majority near 0.

1.5. For a general test function g it is difficult to get an estimate of the error term in Theorem 1.3. More precise statements for special cases may be found in Proposition 5.4 and Lemma 5.5; but there the main term is much more complicated.

1.6. *Sum Formula.* The quantity $K_\kappa(u; f)$, as defined in (1.2.1), with $u \in \mathbf{Z} \setminus \{0\}$, occurs in Kuznetsov's sum formula (cf. Theorem 1 in [7]). To see that this is indeed the case, remark that

$$\sum^*_{d \bmod c} e^{2\pi i(12uS(d, c) + \kappa\, d/c)} = \sum^*_{d \bmod c} e^{2\pi i(ua + (u+\kappa)\, d)/c}$$

(cf. [8, p. 142]); a satisfies $a\, d \equiv 1 \bmod c$. The sum formula states that $K_\kappa(u; f)$ is equal to the sum of some other expressions, most of which contain products of Fourier coefficients of real analytic modular forms of weight zero.

1.7. This sum formula has been generalized from $SL_2(\mathbf{Z})$ to other discrete subgroups of $SL_2(\mathbf{R})$ and other weights by Proskurin [10] and Bruggeman [1]. Deshouillers and Iwaniec [4] have considered the case of congruence subgroups $\Gamma_0(N)$ and have given estimates of sums of

Kloosterman sums and of sums and integrals of products of Fourier coefficients of modular forms.

This paper uses many of the ideas in [4]. We work with automorphic forms for the full modular group only, but we vary the weight continuously. The sums over the order of Fourier coefficients in [4] become integrals here.

In Corollary 3.2 we obtain an estimate for integrals of products of Fourier coefficients that may be compared with Theorem 2 in [4]. It gives less than [4], as we do not integrate over the order of the Fourier coefficients at this point.

1.8. *Overview.* Section 2 states the sum formula used in this paper. It reformulates the results in [1] and also incorporates into the sum formula a result from [2] on the almost complete absence of exceptional eigenvalues in the modular case.

The idea of the sum formula is to relate $K_{n-m}(12n; f)$ to $\int \psi \, d\sigma_{n,m}$, where $d\sigma_{n,m}$ is a measure on $\mathbf{R} \cup i\mathbf{R}$ containing information on Fourier coefficients of modular forms and ψ is related to f by a Bessel transformation.

Section 3 estimates the growth of $|d\sigma_{n,m}|$. This is the basis for estimates in Section 4 of $\int \psi \, d\sigma_{n,m}$ for a special choice of f and ψ. Under this choice the main contribution to $\int \psi \, d\sigma_{n,m}$ is given by the Fourier coefficients of a power of the eta function of Dedekind. The ideas in these sections come from [4]. The transition from weight zero to arbitrary real weight makes it necessary to redo the computations of Deshouillers and Iwaniec in Section 7.1 of [4]. I could not completely recover their results (see the occurrence of logarithms in parts (ii) and (iii) of Lemma 4.1). I thank the referee for a hint that enabled me to handle case (i)(b) of that lemma in a better way.

The proof in Section 5 of Theorem 1.3 is based on the sum formula and the estimates in Section 4.

2. Sum Formula

We rewrite for the modular case the sum formula in [1, 16.4.1], in a way suitable for our present purpose. The result may be formulated as:

2.1. Proposition. *Let* $n, m \in \mathbf{R} \setminus \{0\}$, $n \equiv m \bmod 1$; *put* $\varepsilon = \operatorname{sign}(nm)$. *There are measures* $d\sigma_{n,m}$ *and* $d\delta_{n,n}$ *on* $\mathbf{R} \cup i\mathbf{R}$, *a class* $\mathcal{T}_{\varepsilon,n}$ *of test functions and an integral transform*

$$b_n^{\varepsilon} : \mathcal{T}_{\varepsilon,n} \to \{\textit{functions on } (0, \infty)\}$$

such that for each $\psi \in \mathscr{T}_{\varepsilon,n}$:

(i) $\psi \in L^1(d\sigma_{n,m})$,

(ii) $\psi \in L^1(d\delta_{n,n})$ *if* $n=m$,

(iii) $K_{m-n}(12n, f)$ *as defined in* 1.2.1, *with* $f = b_n^\varepsilon \psi$, *converges absolutely, and*

$$\int \psi(s)\, d\sigma_{n,m}(s) = \delta_{n,m} \int \psi(s)\, d\delta_{n,n}(s) + K_{m-n}(n, f). \tag{2.1.1}$$

The measure $d\sigma_{n,m}$ is described in 2.16, the measure $d\delta_{n,n}$ in 2.5, the class $\mathscr{T}_{\pm 1,n}$ in 2.3, and the integral transform in 2.7.

2.2. *Notation.* By $\int \psi(s)\, d\sigma_{n,m}(s)$ we mean integration over $\mathbf{R} \cup i\mathbf{R}$; similarly for $d\delta_{n,n}$.

2.3. Definition. Let $a > 2$, $\sigma > \frac{1}{2}$, $n \in \mathbf{R} \setminus \{0\}$.

$\mathscr{T}_+(a, \sigma, n)$ is the space of functions ψ on

$$\{s \in \mathbf{C}\colon |\mathrm{Re}\, s| \leqslant \sigma\} \cup \left\{\frac{b-1}{2} > -\frac{1}{2} : b \equiv 12\,|n| \bmod 2\right\}$$

such that

(i) ψ is holomorphic on $|\mathrm{Re}\, s| \leqslant \sigma$,

(ii) ψ is even on $|\mathrm{Re}\, s| \leqslant \sigma$,

(iii) $\psi(s) \ll (1 + |\mathrm{Im}\, s|)^{-a}$,

(iv) $\sum_{b>1,\, b \equiv 12|n| \bmod 2} (b-1)\, |\psi(b-1)/2| < \infty$.

$\mathscr{T}_-(a, \sigma, n)$ is the space of functions ψ on $\{s \in \mathbf{C}\colon |\mathrm{Re}\, s| \leqslant \sigma\}$ such that

(i) $s \mapsto \psi(s)/\cos \pi(s + 6\,|n|)$ is even and holomorphic on $|\mathrm{Re}\, s| \leqslant \sigma$,

(ii) $\psi(s) \ll (1 + |\mathrm{Im}\, s|)^{-a}$.

Put

$$\mathscr{T}_{\pm 1,n} = \bigcup_{a>2,\, \sigma>1/2} \mathscr{T}_{\pm}(a, \sigma, n).$$

2.4. *Relation* with [1, 14.2.7]:

$$\mathscr{T}_{\pm 1}(a, \sigma, n) = {}_{\pm 1}^{\ 1}F^{a}_{12|n|,\sigma}.$$

2.5. Definition. The measure $d\delta_{n,n}$ is defined by

$$\int \phi(s)\, d\delta_{n,n}(s) = \frac{-1}{2\pi i} \int_{\mathrm{Re}\, s=0} \phi(s)\, \frac{4s \sin 2\pi s\, ds}{\pi(\cos 2\pi s + \cos 12\pi n)} + \frac{2}{\pi^2} \sum_{b>1,\, b \equiv 12|n| \bmod 2} (b-1)\, \phi\left(\frac{b-1}{2}\right). \tag{2.5.1}$$

Remark. Clearly $\psi \in L^1(d\delta_{n,n})$ if $\psi \in \mathscr{T}_{1,n}$.

2.6. *Relation* with [1, 14.2.12]:

$$\int \psi(s)\, d\delta_{n,n}(s) = \frac{4}{\pi^2} \langle \psi, 1 \rangle.$$

2.7. DEFINITION. For $n \in \mathbf{R}\backslash\{0\}$ and $\psi \in \mathscr{T}_{\pm 1,n}$ define $b_n^{\pm 1}\psi\colon (0, \infty) \to \mathbf{C}$ by

$$b_n^1\psi(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=0} 8\psi(s)\, J_{2s}(y) \frac{s\, ds}{\cos \pi(s-6\,|n|)}$$
$$+ \sum_{b>1,\, b \equiv 12\,|n| \bmod 2} \frac{4}{\pi}(-1)^{(b-12|n|)/2}(b-1)\,\psi\left(\frac{b-1}{2}\right) J_{b-1}(y) \tag{2.7.1}$$

$$b_n^{-1}\psi(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=0} 8\psi(s)\, I_{2s}(y) \frac{s\, ds}{\cos \pi(s+6\,|n|)}, \tag{2.7.2}$$

where J_u and I_u are the Bessel functions

$$J_u(y) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}y)^{u+2n}}{n!\, \Gamma(u+1+n)} \tag{2.7.3}$$

$$I_u(y) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}y)^{u+2n}}{n!\, \Gamma(u+1+n)}. \tag{2.7.4}$$

Remarks. These integrals converge absolutely and define $b_n^{\pm 1}\psi$ as functions on $(0, \infty)$. The estimate of $\psi(s)$ for $|\mathrm{Im}\, s| \to \infty$ allows moving the line of integration to $\mathrm{Re}\, s = \sigma_1$ if $\psi \in \mathscr{T}_\pm(a, \sigma, n)$ and $\sigma_1 \leqslant \sigma$. In the case of b_n^1 one should take $2\sigma_1 + 1 \not\equiv 12\,|n| \bmod 2$; then the terms with $b < 2\sigma_1 + 1$ in the sum disappear.

If we take $\sigma_1 > \frac{1}{2}$ we obtain an estimate on $b_n^{\pm 1}\psi(y)$ for $y \downarrow 0$ from which the absolute convergence of $K_{m-n}(12n, b_n^{\pm 1}\psi)$ follows.

2.8. *Relation* with [1, 14.2.8]:

$$b_n^{\pm 1}\psi = \frac{4}{\pi} e^{\mp 6\pi i |n|} (b_{12|n|}^{\pm 1})^{\leftarrow} \psi.$$

2.9. PROPOSITION [1, 14.2.1, 14.2.3, 14.2.6]. (i) $C_c^\infty(0, \infty) \subset b_n^\pm(\mathscr{T}_{\pm 1,n})$ *for each* $n \in \mathbf{R}\backslash\{0\}$.

(ii) *On* $C_c^\infty(0, \infty)$ *the inverse of* b_n^+ *is for generic* s *given by*

$$(b_n^+)^{-1} f(s) = \frac{-\pi/2}{\sin 2\pi s}\left\{\cos \pi(s+6\,|n|) \int_0^\infty f(y)\, J_{2s}(y) \frac{dy}{y}\right.$$
$$\left. - \cos \pi(s-6\,|n|) \int_0^\infty f(y)\, J_{-2s}(y) \frac{dy}{y}\right\} \tag{2.9.1}$$

and for $b \equiv 12\,|n| \bmod 2$ *by*

$$(b_n^+)^{-1} f\left(\frac{b-1}{2}\right) = \frac{\pi}{2}(-1)^{(b-12|n|)/2} \int_0^\infty f(y)\, J_{b-1}(y)\, \frac{dy}{y}. \qquad (2.9.2)$$

(iii) *On* $C_c^\infty(0, \infty)$ *the inverse of* b_n^- *is given by*

$$(b_n^-)^{-1} f(s) = \cos \pi(s - 6\,|n|) \int_0^\infty f(y)\, K_{2s}(y)\, \frac{dy}{y} \qquad (2.9.3)$$

with $K_{2s} = (\pi/(2 \sin \pi s))\{I_{-2s} - I_{2s}\}$.

(iv) *If* $\psi \in \mathscr{T}_{1,n}$ *satisfies* $b_n^+ \psi \in C_c^\infty(0, \infty)$, *then*

$$\int \psi(s)\, d\delta_{n,n}(s) = 0.$$

2.10. *Proofs.* Propositions 2.1 and 2.9 have been proved in [1]; we only need to reconstruct Proposition 2.1 in the present notation. This is done in the sequel. We also discuss the measure $d\sigma_{n,m}$ in some detail.

2.11. *Notations* used in this reconstruction. $\bar{\Gamma} = SL_2(\mathbf{Z})$, Γ its full original in the universal covering group G_0 of $SL_2(\mathbf{R})$.

Let $r \in \mathbf{R}$; there is a character χ_r of Γ defined by

$$\chi_r(n(1)) = e^{\pi i r/6} \qquad \text{and} \qquad \chi_r\left(k\left(\frac{\pi}{2}\right)\right) = e^{\pi i r/2}$$

in the notation of [1, Sect. 2]. According to 4.4 of [1] the character χ_r corresponds to a multiplier system v_r of $\bar{\Gamma}$ with

$$v_r \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{\pi i r/6} \qquad \text{and} \qquad v_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e^{\pi i r/2}.$$

So v_r is the $2r$th power of the multiplier system of the eta function of Dedekind. It is known (see, e.g., [8, IX,1]) that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \bar{\Gamma}$ with $c \geqslant 1$,

$$v_r \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{-2\pi i r S(d,c) + \pi i r(a+d)/6c - \pi i r/2},$$

where $S(d, c)$ is the Dedekind sum.

We now take $r = 12n$, with $n \in \mathbf{R} \backslash \{0\}$, and define $\eta \in \{\pm 1\}$, $\tau \in [0, 1]$ by $r \equiv \eta\tau \bmod 2$. We put $\chi = \chi_{\eta r}$ in order to satisfy [1, 4.3.1].

We take $\alpha=(\infty, \eta n)$, $\beta=(\infty, \eta m)$, with $m \equiv n \bmod 1$; so $\alpha, \beta \in \Lambda^1$ (see [1, 7.2.1]). The corresponding Kloosterman sum (see [1, 8.4.4]) is

$$S(\alpha, \beta; c) = \sum_{d \bmod c}^{*} e^{2\pi i(\eta n a + \eta m d)/c + 2\pi i \eta r S(d,c) - \pi i \eta r(a+d)/6c + \pi i \eta r/2}$$

$$= e^{6\pi i n \eta} \sum_{d \bmod c}^{*} e^{2\pi i(12 n S(d,c) + (m-n)d/c)}. \tag{2.11.1}$$

2.12. *Application* of [1, 16.4.1]. Define $\varepsilon = \operatorname{sign}(nm)$. We fix $\psi \in \mathscr{T}_{\varepsilon, n}$ and apply [1, 16.4.1] with

$$\phi = \frac{4}{\pi} e^{-6\pi i \varepsilon |n|} \psi \tag{2.12.1}$$

so $f = b^{\pm}_{12|n|}$, and the right-hand side of [1, (16.4.1)] equals

$$2\pi e^{-6\pi i |n|} \delta_{n,m} |n| \int \psi(s)\, d\delta_{n,n}(s) + 2\pi e^{-6\pi i \varepsilon |n|} \sqrt{|nm|}\, K_{m-n}(n; f).$$

So in order to get Proposition 2.1 we take

$$\int \psi(s)\, d\sigma_{n,m} = \frac{1}{2\pi \sqrt{|nm|}} e^{6\pi i \varepsilon |n|} \cdot \text{(left-hand side of (16.4.1) in [1])}.$$

In [1, (16.4.1)] we see that $\int \psi(s)\, d\sigma_{n,m}(s)$ is the sum of three terms.

2.13. The first one is due to the continuous spectrum, only present if $n \in \mathbf{Z}$. The factor $t_{\varepsilon(\alpha), \varepsilon(\beta)}(s)^{-1}$ in [1, (16.4.1)] should be omitted. It is due to a mistake slipped in in [1, (15.6.3)]; this error propagates till [1, (16.4.20)], where I made the same mistake in the opposite direction. From [1, (9.5.12)] we obtain the following expression for this term:

$$\frac{8}{2\pi i} \int_0^{i\infty} \left|\frac{n}{m}\right|^s \psi(s) \frac{\sigma_{-2s}(n)\, \sigma_{2s}(m)}{\zeta(1+2s)\, \zeta(1-2s)}\, ds, \tag{2.13.1}$$

where ζ denotes the zeta function of Riemann and

$$\sigma_u(k) = \sum_{d|k} d^u$$

(d runs over the positive divisors of k).

2.14. The last term is due to holomorphic modular forms; it is only present if $mn > 0$, so if $\varepsilon = 1$. It describes a measure on $(-\frac{1}{2}, \infty)$ with support in $\{(b-1)/2 \colon b > 0,\ b \equiv 12|n| \bmod 2\}$. In [1, 16.4.5] we see that the mass of $d\sigma_{n,m}$ at $(b-1)/2$ may be described in the following way: Let $S_b(v_{12|n|})$

be the space of holomorphic modular cusp forms of weight b with multiplier system $v_{12|n|}$. Each $f \in S_b(v_{12|n|})$ has a Fourier expansion

$$f(z) = \sum_{k>0,\, k \equiv |n| \bmod 1} \rho_k(f)\, e^{2\pi i k z}. \tag{2.14.1}$$

Let $B(b, 12\,|n|)$ be an orthonormal basis of $S_b(v_{12|n|})$; we may choose it in such a way that all $\rho_k(f)$ are real for all f in this basis. Then the weight of $d\sigma_{n,m}$ at $(b-1)/2$ is for $nm > 0$ equal to

$$2\pi^{-2}(4\pi\sqrt{|nm|})^{1-b}\,\Gamma(b) \sum_{f \in B(b,12|n|)} \rho_{|n|}(f)\, \rho_{|m|}(f) \tag{2.14.2}$$

and equals 0 if $nm < 0$.

2.15. The middle term in [1, (16.4.1)] is due to square integrable real analytic modular forms of continuous series type. It is given by a measure with support in a discrete subset of $i(0, \infty) \cup [0, (1-\tau)/2)$. As for the modular case there are no exceptional eigenvalues (see [5, Proposition 2.1, p. 511] for the case $n \in \mathbf{Z}$, and [2, Theorem 2.15] for $n \notin \mathbf{Z}$); in our case the support is contained in $i(0, \infty)$. In [1, 16.4.4] a description of this measure is given in terms of Fourier coefficients of modular forms of weight τ and multiplier system $v_{12\eta n}$. In a similar way one arrives at the following description:

Choose $q \equiv 12n \bmod 2$. Let $A_q^0(s, v_{12n})$ be the space of real analytic modular cusp forms of weight q, eigenvalue $\frac{1}{4} - s^2$, and multiplier system v_{12n}. So $A_q^0(s, v_{12n})$ consists of the functions f on the upper half plane $\mathscr{H}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = v_{12n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{iq\arg(cz+d)} f(z)$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})$ with $-\pi < \arg(cz+d) \leqslant \pi$ and

$$f(z) = \sum_{k \equiv n \bmod 1,\, k \neq 0} \rho_k(f)\, W_{q\,\mathrm{sign}(k)/2,\, s}(4\pi\,|k|\,y)\, e^{2\pi i k x},$$

where $W_{\cdot,\cdot}$ denotes a Whittaker function (see [12, 1.7]).

$A_q^0(s, v_{12n}) \neq 0$ occurs only for $\frac{1}{4} - s^2$ in a discrete subset of $(\frac{1}{4}, \infty)$, and $\dim A_q^0(s, v_{12n}) < \infty$. Integration over $SL_2(\mathbf{Z})\backslash\mathscr{H}$ gives a scalar product. Let $f_1, f_2, \ldots$ be an orthonormal system with

(i) $f_k \in A_q^0(s, v_{12n})$, with $s_k \in i(0, \infty)$,

(ii) $\frac{1}{4} - {s_1}^2 \leqslant \frac{1}{4} - {s_2}^2 \leqslant \cdots$,

(iii) Each $A_q^0(s, v_{12n}) \neq 0$ is spanned by some of the f_k.

Then the continuous series part of $\int \psi(s)\, d\sigma_{n,m}$ is given by

$$\frac{8}{\pi}\sqrt{|nm|}\sum_{k=1}^{\infty} \Gamma\left(\frac{1}{2}+s_k+\frac{1}{2}q\operatorname{sign}(n)\right)$$
$$\times\Gamma\left(\frac{1}{2}-s_k+\frac{1}{2}q\operatorname{sign}(m)\right)\overline{\rho_n(f_k)}\,\rho_m(f_k)\,\psi(s_k).$$

2.16. We may summarize:
The integral $\int \psi(s)\, d\sigma_{n,m}(s)$ is given by the sum of the terms

$$\frac{8}{2\pi i}\int_0^{i\infty}\left|\frac{n}{m}\right|^s \psi(s)\frac{\sigma_{-2s}(n)\,\sigma_{2s}(m)}{\zeta(1+2s)\,\zeta(1-2s)}\,ds$$

only present if $n\in\mathbf{Z}$,

$$\frac{8}{\pi}\sqrt{|nm|}\sum_{k=1}^{\infty}\psi(s_k)\,\Gamma\left(\frac{1}{2}-s_k+\frac{1}{2}q\operatorname{sign}(m)\right)$$
$$\overline{\times\rho_m(f_k)\,\Gamma\left(\frac{1}{2}-s_k+\frac{1}{2}q\operatorname{sign}(n)\right)\rho_n(f_k)},$$

and

$$\sum_{b>0,\, b\equiv 12|n| \bmod 2}\psi\left(\frac{b-1}{2}\right)2\pi^{-2}\,(4\pi\sqrt{|nm|})^{1-b}$$
$$\times\Gamma(b)\sum_{f\in B(b,12|n|)}\rho_{|m|}(f)\,\rho_{|n|}(f)$$

only present if $mn>0$.

2.17. *Remark.* In most applications of the sum formula it is not necessary to use this complicated description of $d\sigma_{n,m}$. It often suffices to use the following facts, easily derived from 2.16, if one remarks that $v_{12|n|}=v_{12|m|}$ if $nm>0$.

2.18. PROPOSITION. *Let $n, m\in\mathbf{R}\backslash\{0\}$, $n\equiv m \bmod 1$.*

(i) *$d\sigma_{n,m}$ is a measure supported in*

$$i(0,\infty)\cup\left\{\frac{b-1}{2}: b>0,\ b\equiv 12\,|n| \bmod 2\right\}$$

if $nm>0$ and in $i(0,\infty)$ for $nm<0$.

(ii) *$d\sigma_{n,n}$ is a positive measure.*

(iii) *If* ψ *is integrable for* $d\sigma_{n,n}$ *and* $d\sigma_{m,m}$, *then it is integrable for* $d\sigma_{n,m}$ *and*

$$\left|\int \psi(s)\, d\sigma_{n,m}(s)\right| \leqslant \left\{\int |\psi(s)|\, d\sigma_{n,n}(s)\right\}^{1/2} \left\{\int |\psi(s)|\, d\sigma_{m,m}(s)\right\}^{1/2}.$$

3. Estimate of $d\sigma_{n,m}$

With Theorem 2 in [4] as an example we now prove a result on the distribution of the total mass of $d\sigma_{n,m}$. Our aim is not a large sieve inequality as in [4], but only an estimate for the measure $d\sigma_{n,m}$ itself. Furthermore, we have in our situation no analogon of Weil's estimate of the Kloosterman sum which is used in [4].

3.1. Proposition. *Let* $n \in \mathbf{R}\backslash\{0\}$.

$$\int_{|s| \leqslant T} d\sigma_{n,n}(s) \ll (1+|n|)\, T^2 \qquad \text{for} \quad T \to \infty.$$

3.2. Corollary. *Let* $n, m \in \mathbf{R}\backslash\{0\}$, $n \equiv m \bmod 1$.

$$\int_{|s| \leqslant T} |d\sigma_{n,m}(s)| \ll \sqrt{1+|n|}\, \sqrt{1+|m|}\, T^2 \qquad \text{for} \quad T \to \infty.$$

The corollary follows easily from the proposition if one uses Proposition 2.18(iii). The proof of the proposition is given in the following lemmas.

3.3. Lemma. *Let* $n \neq 0$ *and* $\frac{1}{2} < \sigma < 1$. *For* $v > 0$ *define*

$$\psi_v(s) = \begin{cases} e^{vs^2}(s/\sin \pi s)^2\, (\cos 2\pi s + \cos 12\pi n) & \text{for} \quad |\operatorname{Re} s| \leqslant \sigma \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\psi_v \in \mathscr{T}_{1,n}$ *and*

$$b_n^1 \psi_v(y) \ll_\sigma y^{2\sigma} v^{\sigma - 7/4} \qquad \text{for} \quad v \downarrow 0 \text{ if } y \leqslant 1,$$
$$b_n^1 \psi_v(y) \ll_\sigma v^{-2} \qquad \text{for} \quad v \downarrow 0 \text{ if } y \geqslant 1,$$

$$\int \psi_v(s)\, d\delta_{n,n}(s) \ll v^{-2} \qquad \text{for} \quad v \downarrow 0.$$

Proof. As $\psi_v((b-1)/2) = 0$ for all $b > 0$, $b \equiv 12\,|n| \bmod 2$, it is clear that $\psi_v \in \mathscr{T}_+(a, \sigma, n)$ for all $a > 2$. If we write down the integral representation of 2.7 with the path of integration moved to $\operatorname{Re} s = \sigma$, then

$$b_n^1 \psi_v(y) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = \sigma} \frac{16 e^{vs^2} \cos \pi(s + 6\,|n|)}{\sin^2 \pi s}\, s^3 J_{2s}(y)\, ds.$$

From the series representation of $J_{2s}(y)$ (see 2.7) follows

$$J_{2s}(y) \ll y^{2\operatorname{Re} s} \, |\Gamma(2s+1)|^{-1} \qquad \text{for} \quad y \leqslant 1. \tag{3.3.1}$$

Hence

$$\begin{aligned} b_n^1 \psi_v(y) &\ll e^{v\sigma^2} y^{2\sigma} \int_{-\infty}^{\infty} e^{-vt^2} (1+|t|)^{-2\sigma+5/2} \, dt \\ &\ll y^{2\sigma} v^{\sigma-7/4} \qquad \text{for} \quad v \downarrow 0, \; y \leqslant 1. \end{aligned}$$

We also have the integral representation

$$\begin{aligned} J_u(y) &= \frac{1}{\pi} \int_0^{\pi/2} \cos(u\theta - y \sin\theta) \, d\theta \\ &\quad + \frac{1}{\pi} \int_0^{\infty} e^{-tu} \sin\left(y \cosh t - \frac{1}{2} \pi u \right) dt \qquad \text{for} \quad \operatorname{Re} u > 0 \end{aligned} \tag{3.3.2}$$

(see [13, 6.2, formula (7)]). This implies

$$J_u(y) \ll e^{\pi |\operatorname{Im} u|/2} / \operatorname{Re} u \qquad \text{for} \quad \operatorname{Re} u > 0. \tag{3.3.3}$$

So for $y \geqslant 1$

$$\begin{aligned} b_n^1 \psi_v(y) &\ll \int_{-\infty}^{\infty} e^{-vt^2} (1+|t|^3) \, dt \\ &\ll v^{-2} \qquad \text{for} \quad v \downarrow 0. \end{aligned}$$

Finally

$$\begin{aligned} \int \psi_v(s) \, d\delta_{n,n}(s) &= \frac{-1}{2\pi i} \int_{\operatorname{Re} s = 0} \frac{1}{\pi} e^{vs^2} 8s^3 \cot \pi s \, ds \\ &\ll \int_0^{\infty} e^{-vt^2} t^3 \frac{1+e^{-2\pi t}}{1-e^{-2\pi t}} \, dt \\ &\ll \int_0^1 t^2 \, dt + \int_1^{\infty} e^{-vt^2} t^3 \, dt \\ &\ll v^{-2} \qquad \text{for} \quad v \downarrow 0. \end{aligned}$$

3.4. Lemma. *Let* $n \in \mathbf{R} \setminus \{0\}$.

$$\int_0^T d\sigma_{n,n}(it) \ll (1+|n|) T^2 \qquad \textit{for} \quad T \to \infty.$$

Proof. We use ψ_v as in Lemma 3.3.

$$K_0(12n, b_n^1\psi_v) \ll \sum_{c=1}^{\infty} |b_n^1\psi_v(4\pi\,|n|/c)|$$

$$\ll \sum_{1 \leqslant c < 4\pi|n|} v^{-2} + \sum_{c \geqslant 4\pi|n|} v^{\sigma-7/4}\left(\frac{4\pi\,|n|}{c}\right)^{2\sigma}$$

$$\ll \begin{cases} |n|^{2\sigma}\, v^{\sigma-7/4} & \text{if} \quad |n| < 1/4\pi \\ |n|\, v^{-2} & \text{if} \quad |n| \geqslant 1/4\pi. \end{cases}$$

The sum formula 2.1 now gives

$$\int \psi_v(s)\, d\sigma_{n,n}(s) \ll (1+|n|)\, v^{-2} \qquad \text{for} \quad v \downarrow 0.$$

Take $v = 2T^{-2}$ and remark that $\psi_v \geqslant 0$ on the support of $d\sigma_{n,n}$. So

$$\int_0^T t^2\, d\sigma_{n,n}(it) \ll e^2 \int_0^T t^2 e^{-vt^2}\, d\sigma_{n,n}(it) \ll \int_0^T \psi_v(it)\, d\sigma_{n,n}(it)$$

$$\ll \int \psi_v(s)\, d\sigma_{n,n}(s) \ll (1+|n|)\, T^4.$$

By partial summation the lemma follows.

3.5. Lemma. *Let* $\sigma > 1$, $a > 2$, $p > \sigma + 1$. *Let* $n \in \mathbf{R}\backslash\{0\}$, $2\sigma + 1 \not\equiv 12\,|n| \bmod 2$. *Put*

$$\psi(s) = \begin{cases} 0 & \textit{for} \quad |\mathrm{Re}\, s| > \sigma \\ (p^2 - s^2)^{-a/2} & \textit{for} \quad |\mathrm{Re}\, s| \leqslant \sigma \end{cases}$$

with $\arg(p^2 - s^2) = 0$ *for* $s \in i\mathbf{R}$. *Then*

(i) $\psi \in \mathscr{T}_{1,n}$,

(ii) $\int \psi(s)\, d\delta_{n,n}(s) \ll_{a,p} 1$,

(iii) $b_n^1\psi(y) \ll_{a,p,\varepsilon} \min(1, y^{1+\varepsilon})$ *for some* $\varepsilon > 0$.

Proof. The first assertion is clear. For the second one we easily get the estimate $\mathcal{O}(1)$ of $\int \psi(s)\, d\delta_{n,n}(s)$ for those values of n for which $1 + \cos 12\pi n$ stays away from zero. For $6n$ near $\frac{1}{2} \bmod 1$ we deform the path of integration in (2.5) away from zero. This gives an additional term in the sum, but a bound only depending on a and p may be obtained.

Consider now

$$b_n^1\psi(y)=\frac{1}{2\pi i}\int_{\mathrm{Re}\,s=\sigma_1} 8(p^2-s^2)^{-a/2} J_{2s}(y)\frac{s\,ds}{\cos\pi(s-6|n|)}$$
$$+\sum_{2\sigma_1+1<b\leqslant 2\sigma+1,\, b\equiv 12|n| \bmod 2} \frac{4}{\pi}(-1)^{(b-12|n|)/2}$$
$$\times (b-1)\,\psi\left(\frac{b-1}{2}\right) J_{b-1}(y)$$

with $\frac{1}{2}+\frac{1}{2}\varepsilon\leqslant\sigma_1\leqslant\sigma$ such that $\cos\pi(\sigma_1-6|n|)>\frac{1}{100}$. Use (3.3.3) to obtain $b_n^1\psi(y)\ll_{a,p}1$. For $y\leqslant 1$ use $J_u(y)\ll|\Gamma(u+1)|^{-1}y^{\mathrm{Re}\,u}$ (see (3.3.1)).

3.6. LEMMA. *Let* $n\in\mathbf{R}\backslash\{0\}$.

$$\int_{-1/2<s\leqslant 2} d\sigma_{n,n}(s)\ll 1+|n|.$$

Proof. Take ψ as in the previous lemma with $\sigma>2$. As $\psi\geqslant 0$ on the support of $d\sigma_{n,n}$ we get

$$\int_{-1/2<s\leqslant 2} d\sigma_{n,n}(s)\ll (p^2-\sigma^2)^{a/2}\int\psi(s)\,d\sigma_{n,n}(s)$$
$$\ll\left|\int\psi(s)\,d\delta_{n,n}(s)\right|+|K_0(12n,b_n^1\psi)|$$
$$\ll 1+\sum_{1\leqslant c<4\pi|n|}1+\sum_{c\geqslant 4\pi|n|}\left(\frac{4\pi|n|}{c}\right)^{1+\varepsilon}$$
$$\ll 1+\begin{cases}0+|n|^{1+\varepsilon} & \text{if } |n|\leqslant 1/4\pi\\ |n|+|n| & \text{if } |n|>1/4\pi\end{cases}$$
$$\ll 1+|n|.$$

3.7. LEMMA. *Let* $n\in\mathbf{R}\backslash\{0\}$, $b\equiv 12|n| \bmod 2$, $b>5$. *The mass of* $d\sigma_{n,n}$ *at* $(b-1)/2$ *is* $\mathcal{O}(|n|+b)$.

Proof. The sum formula states that this mass equals $K_0(12n,f)+2(b-1)/\pi^2$ with $f(y)=(4/\pi)(-1)^{(b-12|n|)/2}(b-1)J_{b-1}(y)$. From (3.3.1) and (3.3.3) follows

$$K_0(12n,f)\ll\sum_{c=1}^{\infty}\min\left(\left(\frac{4\pi|n|}{c}\right)^{b-1}\Gamma(b)^{-1},\frac{1}{b-1}\right)(b-1)$$
$$\ll 1+|n|.$$

3.8. LEMMA. *Let* $n \in \mathbf{R}\setminus\{0\}$.

$$\int_{2 \leqslant s \leqslant T} d\sigma_{n,n}(s) \ll |n|\, T + T^2 \qquad \text{for} \quad T \to \infty.$$

The *proof* is clear from the previous lemma.

4. ESTIMATES OF BESSEL TRANSFORMS

To use the sum formula in the opposite direction it is well to have estimates of $(b_n^{\pm})^{-1} f$ for $f \in C_c^{\infty}(0, \infty)$. Here I give more or less the results of [4, Lemma 7.1] in our notation, and slightly weaker, as I could not reproduce all the results in [4].

4.1. LEMMA. *Let* $\tau \in C_c^{\infty}(0, \infty)$, $0 \leqslant \tau \leqslant 1$, *let the support of* τ *be contained in the interval* $(1, 8)$, *and let*

$$\int_0^{\infty} |\tau'(y)|\, dy = 2, \qquad Y = \int_0^{\infty} |\tau''(y)|\, dy > 15.$$

For $X > 0$ *put*

$$f_X(y) = \tau(y/X)$$
$$\psi_{n,X}^{\pm} = (b_n^{\pm})^{-1} f_X \qquad \text{for} \quad n \neq 0.$$

(i) (a) *For* $|s| \leqslant \max(2, 4eX+1)$, $s \in \operatorname{supp}(d\sigma_{n,n})$, $s \notin (-\frac{1}{2}, 0)$:

$$\psi_{n,X}^{+}(s) \ll \frac{1 + |\log X|}{1 + X}.$$

(b) *For* $t \in \mathbf{R}$:

$$\psi_{n,X}^{-}(it) \ll \frac{1 + |\log X|}{1 + X}.$$

(ii) (a) *For* $\max(2, X+1) \leqslant |s| \leqslant \max(Y, 4eX+1)$, $s \in \operatorname{supp}(d\sigma_{n,n})$:

$$\psi_{n,X}^{+}(s) \ll |s|^{-3/2} + X\, |s|^{-2} \log |s|.$$

(b) *For* $t \in \mathbf{R}$, $|t| \geqslant 2$:

$$\psi_{n,X}^{-}(it) \ll |t|^{-3/2} + X\, |t|^{-2} \log |t|.$$

(iii) (a) *For* $|s| \geqslant \max(Y, 4eX+1)$, $s \in \operatorname{supp}(d\sigma_{n,n})$:

$$\psi^+_{n,X}(s) \ll Y\,|s|^{-5/2} + YX\,|s|^{-3}\log|s|.$$

(b) *For* $t \in \mathbf{R}, |t| \geqslant 2$:

$$\psi^-_{n,X}(it) \ll Y(|t|^{-5/2} + X^2\,|t|^{-7/2} + X^3\,|t|^{-4}).$$

(iv) (a) *For* $s \in (-\frac{1}{2}, 0)$, $s \in \operatorname{supp}(d\sigma_{n,n})$, $X \geqslant 1$:

$$\psi^+_{n,X}(s) \ll X^{2s}.$$

(b) *For* $s \in (-\frac{1}{2}, 0)$, $s \in \operatorname{supp}(d\sigma_{n,n})$, $X \leqslant 1$:

$$\psi^+_{n,X}(s) = \frac{\pi(-1)^{s-6|n|+1/2}\,\mathcal{M}\tau(2s)}{2^{2s+1}\Gamma(2s+1)}\,X^{2s} + \mathcal{O}(X^{2s+2})$$

with $\mathcal{M}\tau(u) = \int_0^\infty \tau(y)\,y^{u-1}\,dy$, *the Mellin transform.*

Proof. Throughout the proof we use Propositions 2.9 and 2.18. We denote $f = f_X$, $\psi^\pm = \psi^\pm_{n,X}$, and

$$jf(u) = \int_0^\infty f(y)\,J_u(y)\,\frac{dy}{y}$$

$$kf(u) = \int_0^\infty f(y)\,K_u(y)\,\frac{dy}{y}.$$

From the series for J_u (see (2.7.3)), we get for $X \leqslant 1$, $u > -1$

$$jf(u) = \Gamma(u+1)^{-1}\,2^{-u}X^u\mathcal{M}\tau(u) + \mathcal{O}(\Gamma(u+2)^{-1}\,X^{u+2});$$

in view of 2.9 this gives (iv)(b); we use the absence of exceptional eigenvalues of continuous series type.

For $\alpha \neq 0$ we have

$$\int_0^\infty e^{-\alpha y}f(y)\,\frac{dy}{y} = \frac{1}{\alpha}\int_0^\infty e^{-\alpha y}\,\frac{d}{dy}\,(f(y)/y)\,dy,$$

hence for $\operatorname{Re}\alpha \geqslant 0$, $\alpha \neq 0$:

$$\int_0^\infty e^{-\alpha y}f(y)\,\frac{dy}{y} \ll e^{-X\operatorname{Re}\alpha}\min(1, |\alpha|^{-1}\,X^{-1}).$$

If we use this in the integral representation (3.3.2) we find for $u > 0$:

$$jf(u) \ll \int_0^{\pi/2} \min\left(1, \frac{1}{X\sin\theta}\right)d\theta + \int_0^\infty e^{-ut}\min\left(1, \frac{1}{X\cosh t}\right)dt.$$

Now

$$\int_0^{\pi/2} \min\left(1, \frac{1}{X \sin \theta}\right) d\theta \ll 1 \qquad \text{if} \quad X \leqslant 1$$

and

$$\ll \arcsin X^{-1} + X^{-1} \log \tan \tfrac{1}{2}\theta \,|_{\theta = \arcsin X^{-1}}^{\pi/2} \ll X^{-1}(1 + \log X) \qquad \text{if} \quad X \geqslant 1.$$

For $X \geqslant 1$:

$$\int_0^{\infty} e^{-ut}(X \cosh t)^{-1}\, dt \ll X^{-1}$$

and for $X \leqslant 1$, $u > 0$, with $T = \log(X^{-1} + \sqrt{X^{-2} - 1})$:

$$\begin{aligned}
&\int_0^{\infty} e^{-ut} \min\left(1, \frac{1}{X \cosh t}\right) dt \\
&\quad = \int_0^T e^{-ut}\, dt + X^{-1} \int_T^{\infty} e^{-ut}(\cosh t)^{-1}\, dt \\
&\quad \ll T + X^{-1} \frac{e^{-(u+1)T}}{u+1} \\
&\quad \ll 1 + |\log X| + (u+1)^{-1} X^{-1}(X^{-1} + \sqrt{X^{-2} - 1})^{-u-1} \\
&\quad \ll 1 + |\log X|.
\end{aligned}$$

So for $u > 0$:

$$jf(u) \ll \frac{1 + |\log X|}{X + 1}.$$

This gives (i)(a) in the case $s \in (0, \infty)$.

From [13, 3.61, (1) and (2)] we see that

$$\begin{aligned}
&\frac{-\pi}{2 \sin 2\pi s} \{\cos \pi(s + 6\,|n|) J_{2s} - \cos \pi(s - 6\,|n|) J_{-2s}\} \\
&\quad = \frac{\pi i}{4} e^{\pi i(s - 6|n|)} H_{2s}^{(1)} - \frac{\pi i}{4} e^{-\pi i(s - 6|n|)} H_{2s}^{(2)},
\end{aligned}$$

and in [13, 6.21, (10), (11)] we find integral representations which for $t \in \mathbf{R}$ amount to

$$H_{2it}^{(1,2)}(y) = \pm \frac{1}{\pi i} e^{\pm \pi t} \int_{-\infty}^{\infty} e^{\pm iy \cosh \tau - 2it\tau}\, d\tau.$$

To be able to interchange the order of integration over y and τ we first move the path of integration to the contour given by $\mathbf{R} \to \mathbf{C}\colon x \mapsto \tau(x) = x \pm i\phi(x)$, where ϕ is a continuous function, $|\phi| \leqslant 1$, $\phi(x) = -1$ for

$x \leqslant -100$, and $\phi(x)=1$ for $x \geqslant 100$. This gives an integral which converges absolutely. Now integrate first over y and use

$$\left| \int_0^\infty f(y) e^{\mp iy\cosh \tau(x)} \frac{dy}{y} \right| \ll e^{-X \sin\phi(x) \cosh x}$$

to see that we can move back the path of integration. Hence for $t \in \mathbf{R}$:

$$\psi_{n,X}^{+}(it) \ll \int_{-\infty}^{\infty} \min\left(1, \frac{1}{X\cosh\tau}\right) d\tau$$
$$\ll \begin{cases} X^{-1} & \text{if } X \geqslant 1 \\ |\log X| + 1 & \text{if } X \leqslant 1. \end{cases}$$

This completes the proof of (i)(a).

We also use the integral representation

$$J_u(y) = \pi^{-1} \int_0^\pi \cos(u\theta - y\sin\theta)\, d\theta - \frac{\sin \pi u}{\pi} \int_0^\infty e^{-u\tau - y\sinh\tau}\, d\tau \qquad (4.1.1)$$

(see [13, 6.2, formula (4)]). We consider this for $-1 < u < 0$. The first term is $\mathcal{O}(1)$ and contributes $\mathcal{O}(1)$ to $\mathscr{J}f(u)$. The second term contributes for $X \geqslant 1$:

$$\ll |\sin \pi u| \int_0^\infty e^{-u\tau - X\sinh\tau} \min\left(1, \frac{1}{X\sinh\tau}\right) d\tau$$
$$\ll \int_0^T e^{-u\tau}\, d\tau + |\sin\pi u|\, X^{-1} \int_T^\infty e^{-u\tau - X\sinh\tau} \frac{d\tau}{\sinh\tau}$$
$$\text{with} \quad T = \log(X^{-1} + \sqrt{X^{-2}+1})$$
$$\ll Te^T + |\sin\pi u|\, X^{-1} \int_{1/X}^\infty (w + \sqrt{w^2+1})^{-u}\, e^{-Xw} \frac{dw}{w\sqrt{w^2+1}}$$
$$\ll X^{-1} + |\sin\pi u|\, X^{-1} \int_{1/X}^\infty w^{-u-2} \frac{1+\sqrt{1+w^{-2}}}{\sqrt{1+w^{-2}}}\, e^{-Xw}\, dw$$
$$\ll X^{-1} + |\sin\pi u|\, X^u \ll X^u.$$

This gives (iv)(a) (see 2.18).

Now use

$$K_u(y) = \frac{1}{2} \int_{-\infty}^\infty e^{-y\cosh\tau - u\tau}\, d\tau$$

(see [13, 6.22, (7)]); so for $t \in \mathbf{R}$

$$K_{2it}(y) = \int_0^\infty e^{-y\cosh\tau} \cos(2t\tau)\, d\tau.$$

For (i)(b) it suffices to consider $kf(2it)$ for $t \geqslant 0$.

$$kf(2it) = \frac{1}{2}\int_{-\infty}^{\infty} F(\cosh\tau)e^{2it\tau}\,d\tau \tag{4.1.2}$$

with $F(u) = \int_0^\infty f(y)\,e^{-uy}y^{-1}dy$. We have seen that $F(u) \ll e^{-X\operatorname{Re} u}$ for $\operatorname{Re} u > 0$. So for N large

$$\int_0^{\pi/2} F(\cosh(\pm N + ix))e^{2it(\pm N + ix)}\,dx$$

$$\ll \int_0^{\pi/2} e^{-X\cos x\cosh N}\,dx \ll \frac{1}{X\cosh N}.$$

This implies that we may move the line of integration in (4.1.2) to $\pi i/2 + \mathbf{R}$:

$$kf(2it) = \frac{1}{2}\int_{-\infty}^{\infty} F(i\sinh\tau)e^{2it\tau - \pi t}\,d\tau$$

$$\ll e^{-\pi t}\int_{-\infty}^{\infty} \min\left(1, \frac{1}{|\sinh\tau|\,X}\right)d\tau$$

$$\ll e^{-\pi t}\frac{1 + |\log X|}{1 + X}.$$

This gives (i)(b).

For $\operatorname{Re} u > 1$:

$$J_u(y) = \frac{1}{4\pi i}\int_{\operatorname{Re} s = -1}\left(\frac{1}{2}y\right)^{-s}\frac{\Gamma((u+s)/2)}{\Gamma(1 + (u-s)/2)}\,ds, \tag{4.1.3}$$

as may be shown by moving off the line of integration to the left. So for $u \geqslant 2$:

$$jf(u) = \frac{1}{4\pi i}\int_{\operatorname{Re} s = -1} 2^s\int_0^\infty f(y)\,y^{-1-s}\,dy\,\frac{\Gamma((u+s)/2)}{\Gamma(1 + (u-s)/2)}\,ds$$

$$= \frac{1}{4\pi i}\int_{\operatorname{Re} s = -1} 2^s s^{-1}\int_0^\infty f'(y)\,y^{-s}\,dy\,\frac{\Gamma((u+s)/2)}{\Gamma(1 + (u-s)/2)}\,ds$$

$$\ll \int_{-\infty}^{\infty} X\left|\frac{\Gamma((u+1+it)/2)}{\Gamma((u+1-it)/2)}\right|\frac{d\tau}{|-1+i\tau|\;|u-1+i\tau|\;|u+1-i\tau|}$$

$$\ll X\int_0^{u-1}\frac{1}{u^2-1}\frac{d\tau}{1+\tau} + X\int_{u-1}^{\infty}\frac{d\tau}{(\tau+1)^3}$$

$$\ll Xu^{-2}\log u.$$

This gives (ii)(a) for $s = (b-1)/2 \geqslant 2$.

If we move the line of integration in (4.1.3) to $-2-\operatorname{Re} u<\operatorname{Re} s<-\operatorname{Re} u$, we get an additional term:

$$J_u(y)=\frac{1}{4\pi i}\int_{\operatorname{Re} s=\sigma}\left(\frac{1}{2}y\right)^{-s}\frac{\Gamma((u+s)/2)}{\Gamma(1+(u-s)/2)}\,ds+\frac{(\frac{1}{2}y)^u}{\Gamma(u+1)}; \quad (4.1.4)$$

this is valid for $\sigma<0$, $-2-\operatorname{Re} u<\sigma<-\operatorname{Re} u$. We may take $\operatorname{Re} u=0$ and $\sigma=-1$. So for $t\geqslant 1$:

$$jf(2it)\ll\int_{-\infty}^{\infty}\left|\frac{\Gamma(i\tau+(1+i\tau)/2)}{\Gamma(i\tau+(1-i\tau)/2)}\right|\frac{X\,d\tau}{|-1+i\tau|\;|2it-1+i\tau|\;|2it+1-i\tau|}$$
$$+|\Gamma(2it+1)|^{-1}\,|t|^{-1}.$$

The latter term is $\mathcal{O}(e^{\pi|t|}\,|t|^{-3/2})$ and gives $\mathcal{O}(|s|^{-3/2})$ in (ii)(a) for $s\in i\mathbf{R}$. For the integral we get

$$\ll X\int_{-\infty}^{\infty}\frac{e^{(1/2)\pi(|t-\tau/2|-|t+\tau/2|)}}{(1+|\tau|)(1+|t+\frac{1}{2}\tau|)(1+|t-\frac{1}{2}\tau|)}\,d\tau$$

$$\ll X\int_{2t}^{\infty}\frac{e^{-\pi t}\,d\tau}{t(1+\tau)}+X\int_{0}^{2t}\frac{e^{-\pi\tau/2}\,d\tau}{(1+\tau)(1+t+\frac{1}{2}\tau)(1+t-\frac{1}{2}\tau)}$$
$$+X\int_{-2t}^{0}\frac{e^{-\pi\tau/2}\,d\tau}{(1-\tau)(1+t+\frac{1}{2}\tau)(1+t-\frac{1}{2}\tau)}$$
$$+X\int_{-\infty}^{-2t}\frac{e^{\pi\tau}\,d\tau}{(1-\tau)(1-t-\frac{1}{2}\tau)(1+t-\frac{1}{2}\tau)}.$$

The last term is the worst one; if we look at $-3t\leqslant\tau\leqslant-2t$ we see that we cannot do better than

$$\ll Xe^{\pi t}t^{-1}\int_{2t}^{\infty}\frac{d\tau}{(1+\tau)(2-2t+\tau)}\ll Xe^{\pi t}t^{-2}\log t.$$

The other terms are also $\mathcal{O}(Xe^{\pi t}t^{-2}\log t)$. This is sufficient to complete the proof of (ii)(a). The estimate stated in [4] for this case is a bit sharper than the one obtained here.

If we again use (4.1.4) and perform one more partial integration, we obtain from the latter term a contribution $\mathcal{O}(Yt^{-5/2}e^{\pi t})$ to $jf(2it)$ for $t\geqslant 2$. In the integral we get an additional factor $Y(1+|\tau|)^{-1}$. We obtain in the worst case

$$XY\int_{-\infty}^{-2t}\frac{e^{\pi t}\,d\tau}{(1-\tau)^2\,(1-t-\frac{1}{2}\tau)(1+t-\frac{1}{2}\tau)}$$
$$\ll XYe^{\pi t}t^{-2}\int_{2t}^{\infty}\frac{d\tau}{(1+\tau)(2-2t+\tau)}\ll XYe^{\pi t}t^{-3}\log t.$$

The other terms also satisfy this estimate. This gives (iii)(a) if $\operatorname{Re} s=0$.

For $u \geqslant 31$ we may move the line of integration in (4.1.3) to $\mathrm{Re}\, s = 1-u$. This gives

$$jf(u) = \frac{1}{4\pi i}\int_{\mathrm{Re}\, s = 1-u} \frac{2^s}{s(s-1)} \int_0^\infty f''(y)\, y^{1-s}\, dy \frac{\Gamma((u+s)/2)}{(1+(u-s)/2)}\, ds$$

$$\ll \int_{-\infty}^{\infty} \frac{2^{-u}}{|u-1+i\tau|^2}\, YX^{u-1} 8^u \left|\frac{\Gamma(\frac{1}{2}+\frac{1}{2}i\tau)}{\Gamma(\frac{1}{2}+u-\frac{1}{2}i\tau)}\right| d\tau$$

$$\ll Y(4X)^{u-1} \int_{-\infty}^{\infty} \frac{e^u}{|u-1+i\tau|^2} \left|\frac{1}{2}+u-\frac{1}{2}i\tau\right|^{-u} d\tau$$

$$\ll Y(4eX)^{u-1} \int_0^\infty |u-1+iu\tau|^{-2} \left|\frac{1}{2}+u-\frac{1}{2}iu\tau\right|^{-u} u\, d\tau$$

$$\ll Y(4eX)^{u-1}\, u^{-1-u} \int_0^\infty (1+\tau)^{-u-2}\, d\tau$$

$$\ll Y(4eX)^{u-1}\, u^{-2-u} \ll u^{-3} YX$$

under the condition $u = b-1 \geqslant 2\max(Y, 4eX+1)$. This completes the proof of (iii)(a).

For ψ^- we use

$$K_u(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\sigma} 2^{s-2} y^{-s}\, \Gamma\left(\frac{s+u}{2}\right) \times \Gamma\left(\frac{s-u}{2}\right) ds \qquad \text{for} \quad \sigma > |\mathrm{Re}\, u|; \tag{4.1.5}$$

by the Mellin transform this is equivalent to

$$\int_0^\infty K_u(y)\, y^s \frac{dy}{y} = 2^{s-2}\Gamma\left(\frac{s+u}{2}\right)\Gamma\left(\frac{s-u}{2}\right) \qquad \text{for} \quad \mathrm{Re}\, s > |\mathrm{Re}\, u|,$$

(see [13, 13.21, (8)]). By moving the line of integration we may deform (4.1.5) into

$$K_u(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=-1} 2^{s-1} y^{-s}\, \Gamma\left(\frac{u+s}{2}\right)\Gamma\left(\frac{s-u}{2}\right) ds + 2^u y^{-u}\Gamma(u) + 2^{-u} y^u \Gamma(-u) \qquad \text{for} \quad \mathrm{Re}\, u = 0,\ u \neq 0. \tag{4.1.6}$$

This gives for $t \geqslant 2$:

$$kf(2it) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=-1} 2^{s-1} s^{-1} \int_0^\infty f'(y)\, y^{-s}\, dy\, \Gamma\left(\frac{1}{2}s+it\right)\Gamma\left(\frac{1}{2}s-it\right) ds + \sum_{\pm} \pm 2^{\pm 2it}\Gamma(\pm 2it)\frac{1}{2it}\int_0^\infty f'(y)\, y^{\mp 2it}\, dy.$$

So

$$\psi^{-}(2it) \ll e^{\pi t}\left[X\int_{-\infty}^{\infty} |-1+it|^{-1}\left|\Gamma\left(-\frac{1}{2}+\frac{1}{2}i\tau+it\right)\right| \right.$$
$$\left. \times\left|\Gamma\left(-\frac{1}{2}+\frac{1}{2}i\tau-it\right)\right| d\tau + e^{-\pi t}t^{-3/2}\right].$$

The term with the integral satisfies

$$\ll e^{\pi t}X\int_{-\infty}^{\infty} \frac{e^{-(1/2)\pi(|t+\tau/2|+|t-\tau/2|)}\,d\tau}{(1+|\tau|)(1+|\tau+2t|)(1+|\tau-2t|)}$$
$$\ll e^{\pi t}X\left\{\int_0^{2t} \frac{e^{-\pi t}\,d\tau}{(1+\tau)(1+2t)(1+2t-\tau)}\right.$$
$$\left. +\int_{2t}^{\infty} \frac{e^{-\pi\tau}\,d\tau}{(1+\tau)(1+2t)(1+\tau-2t)}\right\}$$
$$\ll \frac{X}{t}\left(\frac{1}{2(1+t)}\log\frac{1+\tau}{1+2t-\tau}\bigg|_{\tau=0}^{2t} - \frac{1}{2t}\log\frac{1+\tau}{1+\tau-2t}\bigg|_{\tau=2t}^{\infty}\right)$$
$$\ll Xt^{-2}\log t.$$

This gives (ii)(b).

If we use one more partial integration, we get an additional factor $(1+|\tau|)^{-1}$ in the integral. If one looks at $-1\leqslant\tau\leqslant 1$, one sees that one cannot hope to obtain an additional factor t^{-1} in the estimate. To prove (iii)(b) we take $\operatorname{Re} s = -3$ in (4.1.5). This gives two more terms in (4.1.6):

$$-(\tfrac{1}{2}y)^{2-u}\,\Gamma(u-1) - (\tfrac{1}{2}y)^{2+u}\,\Gamma(-u-1).$$

For $u = 2it$, $t\geqslant 2$, all additional terms give contributions

$$\mathcal{O}(Yt^{-5/2}e^{-\pi t}), \qquad \text{resp. } \mathcal{O}(YX^2t^{-7/2}e^{-\pi t}).$$

For the integral we get

$$\ll \int_{-\infty}^{\infty} \frac{YX^3}{|3+i\tau|^2}\left|\Gamma\left(\frac{2it-3+i\tau}{2}\right)\Gamma\left(\frac{-2it-3+i\tau}{2}\right)\right| d\tau$$
$$\ll YX^3\int_0^{\infty} \frac{e^{-(1/2)\pi(t+\tau/2+|t-\tau/2|)}\,d\tau}{(1+\tau)^2\,(1+t+\frac{1}{2}\tau)^2\,(1+|t-\frac{1}{2}\tau|)^2}$$
$$\ll YX^3e^{-\pi t}t^{-4},$$

which completes the proof of (iii)(b).

4.2. LEMMA. *Let* $n, m \in \mathbf{R}\backslash\{0\}$, $n \equiv m \bmod 1$. *Take* $\psi^{\pm}_{n,X}$ *as in Lemma* 4.1.

(i) (a) *If* $nm > 0$ *and there exists* $b \in (0, 1)$ *such that* $b \equiv 12\,|n| \bmod 12$, *then for* $X \leqslant 1$:

$$\int \psi^{+}_{n,X}(s)\, d\sigma_{n,m}(s)$$
$$= \pi^{-1} \left(\frac{8\pi\sqrt{nm}}{X}\right)^{1-b} \frac{\mathscr{M}\tau(b-1)}{N(b)}\, p_{|n|-b/12}(b)\, p_{|m|-b/12}(b)$$
$$+ \mathcal{O}(\sqrt{1+|n|}\,\sqrt{1+|m|}\,(Y^{1/2} + |\log X|))$$

with

$$N(b) = \int_{\Gamma\backslash\mathscr{H}} |\eta(z)|^{4b}\, y^{b} \frac{dx\,dy}{y^2}$$

and p_k *the polynomial given by* $\prod_{k=1}^{\infty} (1-q)^{2b} = \sum_{k=0}^{\infty} p_k(b) q^k$ *for* $|q| < 1$.

(b) *If* $nm > 0$ *and* $|n| \notin (0, \frac{1}{12}) + \mathbf{Z}$, *then for* $X \leqslant 1$:

$$\int \psi^{+}_{n,X}(s)\, d\sigma_{n,m}(s) \ll \sqrt{1+|n|}\,\sqrt{1+|m|}\,(Y^{1/2} + |\log X|).$$

(c) *If* $nm > 0$, *then for* $X \geqslant 1$:

$$\int \psi^{+}_{n,X}(s)\, d\sigma_{n,m}(s)$$
$$\ll \sqrt{1+|n|}\,\sqrt{1+|m|} \begin{cases} (X(\log Y)^2 + Y^{1/2}) & \text{if } X \leqslant Y \\ X \log X & \text{if } X \geqslant Y. \end{cases}$$

(ii) *If* $nm < 0$, *then*

$$\int \psi^{-}_{n,X}(s)\, d\sigma_{n,m}(s)$$
$$\ll \sqrt{1+|n|}\,\sqrt{1+|m|} \begin{cases} (Y^{1/2} + |\log X|) & \text{if } X \leqslant 1 \\ (Y^{1/2} + X(\log Y)^2) & \text{if } 1 \leqslant X \leqslant Y^{1/2} \\ X(\log X)^2 & \text{if } X \geqslant Y^{1/2}. \end{cases}$$

4.3. *Remark* that the N in (i)(a) is continuous on $[0, \infty)$ and satisfies $N(0) = \pi/3$. The polynomials p_k satisfy $p_0 = 1$, $p_k(0) = 0$ for $k > 0$ and the degree of p_k is at most k.

4.4. *Proof of Lemma* 4.2. We first consider $\psi^- = \psi^-_{n,X}$. Remark that Lemma 4.1 gives for $t \in \mathbf{R}$:

$$\psi^-(it) \ll \begin{cases} (1+|\log X|)/(1+X) & \text{for } |t| \leqslant B \\ |t|^{-3/2} + X|t|^{-2}\log|t| & \text{for } B \leqslant |t| \leqslant C \\ Y|t|^{-5/2} + X^2 Y|t|^{-7/2} + X^3 Y|t|^{-4} & \text{for } |t| \geqslant C \end{cases}$$

with $B = \max(2, X)$ and $C = \max(X^2, Y)$. Put $\alpha = \sqrt{1+|n|}\sqrt{1+|m|}$; from Corollary 3.2 we get

$$\int_{|s| \leqslant B} \psi^-(s)\, d\sigma_{n,m}(s) \ll \alpha \frac{1+|\log X|}{1+X} B^2,$$

$$\int_{B \leqslant |s| \leqslant C} \psi^-(s)\, d\sigma_{n,m}(s) \ll \alpha(C^{1/2} + X(\log C)^2) \qquad \text{if} \quad C > B,$$

$$\int_{|s| \geqslant C} \psi^-(s)\, d\sigma_{n,m}(s) \ll \alpha Y(C^{-1/2} + X^2 C^{-3/2} + X^3 C^{-2}).$$

Hence

$$\int \psi^-(s)\, d\sigma_{n,m}(s) \ll \begin{cases} \alpha(Y^{1/2} + |\log X|) & \text{if } X \leqslant 1 \\ \alpha(Y^{1/2} + X(\log Y)^2) & \text{if } 1 \leqslant X \leqslant Y^{1/2} \\ \alpha X(\log X)^2 & \text{if } X \geqslant Y^{1/2}. \end{cases}$$

Now we consider $\psi^+ = \psi^+_{n,X}$. Let $s \in \operatorname{supp}(d\sigma_{n,m})$.

$$\psi^+(s) \ll \begin{cases} X^{2s} & \text{if } -\frac{1}{2} < s < 0,\ X \geqslant 1 \\ (1+|\log X|)/(1+X) & \text{if } |s| \leqslant L,\ s \notin (-\frac{1}{2}, 0) \\ |s|^{-3/2} + X|s|^{-2}\log|s| & \text{if } L \leqslant |s| \leqslant M \\ Y|s|^{-5/2} + YX|s|^{-3}\log|s| & \text{if } |s| \geqslant M \end{cases}$$

with $L = \max(2, 4eX+1)$, $M = \max(4eX+1, Y)$. Similarly as in the previous case we obtain

$$\int_{s \notin (-1/2, 0)} \psi^+(s)\, d\sigma_{n,m}(s)$$

$$\ll \sqrt{1+|n|}\sqrt{1+|m|} \begin{cases} (|\log X| + Y^{1/2}) & \text{for } X \leqslant 1 \\ (X(\log Y)^2 + Y^{1/2}) & \text{for } 1 \leqslant X \leqslant Y \\ X \log X & \text{for } X \geqslant Y. \end{cases}$$

For $X \geqslant 1$

$$\int_{-1/2}^{0} \psi^+(s)\, d\sigma_{n,m}(s) \ll \sqrt{1+|n|}\sqrt{1+|m|},$$

which may be absorbed into the estimate we already have.

From 2.18 we see that $(b-1)/2 \in (-\frac{1}{2}, 0) \cap \operatorname{supp}(d\sigma_{n,m})$ occurs only if $nm > 0$, $0 < b < 1$, and $12\,|n| \equiv b \bmod 2$. But in (2.14.2) we see that an additional condition is $S_b(v_{12|n|}) \neq 0$. This implies $12\,|n| \equiv b \bmod 12$ and $S_b(v_{12|n|}) = \mathbf{C} \cdot \eta^{2b}$, with η the Dedekind function

$$\eta(z) = e^{2\pi i z/24} \prod_{k \geq 1}^{\infty} (1 - e^{2\pi i k z}).$$

Define N and p_k as indicated in the lemma. From (2.14.2) we see that the weight of $d\sigma_{n,m}$ at $(b-1)/2$ is given by

$$\frac{2(4\pi\sqrt{|nm|})^{1-b}\,\Gamma(b)\,p_{|n|-b/12}(b)\,p_{|m|-b/12}(b)}{\pi^2 N(b)}. \tag{4.4.1}$$

So up to a term which may be absorbed into the error term we obtain from $\int_{-1/2}^{0} \psi^+(s)\,d\sigma_{n,m}(s)$ the contribution given in the lemma.

5. Distribution of Dedekind Sums

In this section Theorem 1.3 is proved.

5.1. *Notation.* Let $\kappa \in \mathbf{Z}$, τ be as in Lemma 4.1, and χ be a Schwartz function on $\mathbf{R}$. We define for $Z > 0$:

$$A(\kappa, \chi; Z) = \sum_{c=1}^{\infty} c^{-1}\tau(Z/c) \sum_{d \bmod c}^{*} e^{2\pi i \kappa d}\chi(12S(d, c)).$$

The dependence on τ does not show in the notation.

By investigating the behaviour of $A(\kappa, \chi; Z)$ for $Z \to \infty$ we get information on the distribution of the $(S(d, c), d/c)$.

5.2. Take $X = X(n) = 4\pi\sqrt{|n(n+\kappa)|}\,Z^{-1}$ for $n \in \mathbf{R}$. The sum defining $A(\kappa, \chi; Z)$ is finite for each Z, so by Fourier inversion:

$$A(\kappa, \chi; Z) = \int_{-\infty}^{\infty} \sum_{c=1}^{\infty} c^{-1}\tau\left(\frac{4\pi\sqrt{|n(n+\kappa)|}}{Xc}\right) \sum_{d \bmod c}^{*} e^{2\pi i \kappa d/c + 24\pi i n S(d,c)}\hat{\chi}(n)\,dn$$

$$= \int_{-\infty}^{\infty} K_\kappa(n, f_{X(n)})\,\hat{\chi}(n)\,dn.$$

By the sum formula 2.1 and Proposition 2.9(iv)

$$A(\kappa, \chi; Z) = \int_{n \in \mathbf{R},\, n \neq 0,\, -\kappa} \int \psi_{n,X}^{\varepsilon(n)}(s)\,d\sigma_{n,n+\kappa}(s)\,\hat{\chi}(n)\,dn \tag{5.2.1}$$

with $\varepsilon(n) = \operatorname{sign}(n(n+\kappa))$.

5.3. To estimate $A(\kappa, \chi; Z)$ we shall use Lemma 4.2. The dependence of the error terms on χ will be described by

$$N_l(\chi) = \int_{-\infty}^{\infty} |\chi^{(l)}(u)| \, du \qquad \text{for} \quad l \in \mathbf{Z},\ l \geq 0.$$

We assume $\chi \neq 0$, hence $N_l(\chi) > 0$ for each $l \geq 0$.

5.4. PROPOSITION. *For κ, χ, τ as above define*

$$g_\kappa(\chi; b) = \frac{1}{6\pi 2^b N(b)} \sum_{k=0}^{\infty} p_k(b)\, p_{k+|\kappa|}(b) \cdot \left(\hat{\chi}\left(\left(k + \frac{b}{12}\right) \operatorname{sign} \kappa \right) + \hat{\chi}\left(-\left(k + |\kappa| + \frac{b}{12}\right) \operatorname{sign} \kappa \right) \right)$$

with $\operatorname{sign} \kappa = 1$ *if* $\kappa = 0$. *Put*

$$A = \max(1, N_4(\chi)^{1/3} N_1(\chi)^{-1/3})$$
$$B = \min(1, N_1(\chi) N_0(\chi)^{-1}).$$

(i) $b \mapsto g_\kappa(\chi; b)$ *is continuous and bounded on* $[0, 1)$ *and*

$$g_\kappa(\chi; 0) = \begin{cases} \pi^{-2} \hat{\chi}(0) & \text{if } \kappa = 0 \\ 0 & \text{if } \kappa \neq 0. \end{cases}$$

(ii) *For* $Z \geq 4\pi(2 + |\kappa| + A)$:

$$A(\kappa, \chi; Z) = \int_0^1 Z^{1-b} g_\kappa(\chi; b)\, \mathscr{M}\tau(b-1)\, db + \mathcal{O}\left(\left(Y^{1/2} + \log \frac{Z}{4\pi(1 + |\kappa|)} \right) (1 + |\kappa|)(1 + \log(1 + |\kappa|))^2 \cdot N_1(\chi)(A + (\log B)^2) \right).$$

Proof. Let $b \in (0, 1)$, $k \in \mathbf{Z}$, $k \geq 0$. Take $n = k + b/12$ and $m = n + |\kappa|$. From Corollary 3.2 and (4.4.1) we see that

$$p_k(b)\, p_{k+|\kappa|}(b)\, N(b)^{-1} \ll (4\pi \sqrt{(k + b/12)(k + |\kappa| + b/12)})^{b-1} \cdot \frac{\sqrt{1 + k + b/12}\, \sqrt{1 + k + |\kappa| + b/12}}{\Gamma(b)}. \tag{5.4.1}$$

So for $k \geq 1$:

$$\frac{p_k(b)\, p_{k+|\kappa|}(b)}{N(b)} \ll \sqrt{k(k + |\kappa|)}.$$

This shows that the series defining $g_\kappa(\chi)$ converges absolutely, uniformly for $0 < b < 1$, with bounded sum. As all terms are continuous on $[0, 1)$

and as the terms with $k \geqslant 1$ are 0 at $b=0$ (see Lemma 4.2), we get the continuity on $[0, 1)$ and

$$g_\kappa(\chi; 0) = \frac{p_0(0)\, p_{|\kappa|}(0)}{6\pi N(0)}\, (\hat{\chi}(0) + \hat{\chi}(-\kappa)).$$

So (i) follows from Remark 4.3.

Now we apply Lemma 4.2 to estimate the integral in (5.2.1). We use $|\hat{\chi}(n)| \ll N_0(\chi)$ for $n \in \mathbb{R}$ and $|\hat{\chi}(n)| \ll N_l(\chi)\, |n|^{-l}$ for $n \neq 0$ and $l \geqslant 1$.

Let us first consider the integral over $n(n+\kappa) < 0$. This case is only present if $\kappa \neq 0$. Put $\varepsilon = \operatorname{sign} \kappa$. As $n(n+\kappa) = (n + \frac{1}{2}\kappa)^2 - \frac{1}{4}\kappa^2 < 0$, we have $|n(n+\kappa)| \leqslant \frac{1}{4}|\kappa|^2$, $|n| \leqslant |\kappa|$, and $X \leqslant 2\pi\, |\kappa|/Z \leqslant 1$. Further $(1+|n|)(1+|n+\kappa|) \ll (1+|\kappa|)^2$. So

$$\int_{n(n+\kappa)<0} \int \psi^-_{n,X}(s)\, d\sigma_{n,n+\kappa}(s)\, \hat{\chi}(n)\, dn$$

$$\ll \int_{0<|n| \leqslant |\kappa|/2,\, \varepsilon n<0} (1+|\kappa|) \left(Y^{1/2} + \log \frac{Z}{4\pi\sqrt{|n(n+\kappa)|}} \right) \cdot (|\hat{\chi}(n)| + |\hat{\chi}(-n-\kappa)|)\, dn.$$

We used the symmetry under $n \mapsto -\kappa - n$. The integral over $0 < |n| \leqslant C := \min(B, \frac{1}{2}|\kappa|)$ is estimated by

$$\ll (1+|\kappa|)\, N_0(\chi) \left(Y^{1/2} + \log \frac{Z}{4\pi\sqrt{|\kappa|/2}} + |\log C| \right) C$$

$$\ll (1+|\kappa|)\, N_1(\chi)\, \frac{C}{B} \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} + \log(1+|\kappa|) + |\log B| \right)$$

$$\ll \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} \right) (1+|\kappa|)(1+\log(1+|\kappa|)) \cdot N_1(\chi)(1+|\log B|).$$

For the region $C \leqslant |n| \leqslant D := \min(A, \frac{1}{2}|\kappa|)$ the integral, if present, is

$$\ll (1+|\kappa|) \left(Y^{1/2} + \log \frac{Z}{4\pi\sqrt{|\kappa|\, C/2}} \right) N_1(\chi) \int_C^D \frac{dx}{x}$$

$$\ll (1+|\kappa|) \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} + \log(1+|\kappa|) + |\log C| \right) \cdot N_1(\chi)(|\log B| + \log A)$$

$$\ll \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} \right) (1+|\kappa|)(1+\log(1+|\kappa|)) \cdot N_1(\chi)(\log A + |\log B|^2 + 1).$$

Only if $\frac{1}{2}|\kappa| > A$ we have to consider the integral over $A \leqslant |n| \leqslant \frac{1}{2}|\kappa|$:

$$\ll (1+|\kappa|)\left(Y^{1/2} + \log \frac{Z}{4\pi\sqrt{|\kappa| A/2}}\right) N_4(\chi) \int_A^{|\kappa|/2} x^{-4}\, dx$$

$$\ll (1+|\kappa|)\left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} + \log(1+|\kappa|) + \log A\right) N_4(\chi) A^{-3}$$

$$\ll \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)}\right)(1+|\kappa|)(1+\log(1+|\kappa|))\, N_1(\chi)(1+\log A).$$

Now we turn to the region $n(n+\kappa) > 0$. Lemma 4.2 gives several error terms to be used for $n(n+\kappa)$ in the regions $(0, (Z/4\pi)^2]$, $[(Z/4\pi)^2, (YZ/4\pi)^2]$, and $[(YZ/4\pi)^2, \infty)$, and an explicit term which may be present if $0 < n(n+\kappa) \leqslant (Z/4\pi)^2$.

We first estimate the integral of the error terms. We write $n = -\frac{1}{2}\kappa \pm x$, $x > \frac{1}{2}|\kappa|$. Then $n(n+\kappa) = x^2 - \frac{1}{4}\kappa^2$ and $\sqrt{1+|n|}\sqrt{1+|n+\kappa|} = \sqrt{(1+x)^2 - \frac{1}{4}\kappa^2}$.

The case $0 < n(n+\kappa) \leqslant (Z/4\pi)^2$ gives a contribution

$$\int_{|\kappa|/2}^{\sqrt{(Z/4\pi)^2 + |\kappa|^2/4}} \sqrt{(1+x)^2 - \frac{1}{4}\kappa^2}\left(Y^{1/2} + \log \frac{Z}{4\pi\sqrt{x^2-\kappa^2/4}}\right)$$

$$\cdot\left(\left|\hat{\chi}\left(\varepsilon\left(x - \frac{1}{2}|\kappa|\right)\right)\right| + \left|\hat{\chi}\left(-\varepsilon\left(x + \frac{1}{2}|\kappa|\right)\right)\right|\right) dx \quad \text{with} \quad \varepsilon = \operatorname{sign}(\kappa)$$

$$\ll \int_{|\kappa|/2}^{B+|\kappa|/2} (1+|\kappa|)\, N_0(\chi)$$

$$\cdot\left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} + 1 + \left|\log\left(x - \frac{1}{2}|\kappa|\right)\right| + \log(1+|\kappa|)\right) dx$$

$$+ \int_{B+|\kappa|/2}^{A+|\kappa|/2} (1+x)\left(Y^{1/2} + \log \frac{Z}{4\pi\sqrt{x^2-\kappa^2/4}}\right)$$

$$\cdot N_1(\chi)\left(x - \frac{1}{2}|\kappa|\right)^{-1} dx$$

$$+ \int_{A+|\kappa|/2}^{\sqrt{(Z/4\pi)^2+\kappa^2/4}} (1+x)\left(Y^{1/2} + \log \frac{Z}{4\pi\sqrt{x^2-\kappa^2/4}}\right)$$

$$\cdot N_4(\chi)\left(x - \frac{1}{2}|\kappa|\right)^{-4} dx.$$

The first term is

$$\ll \left(BY^{1/2} + B \log \frac{Z}{4\pi(1+|\kappa|)} + B \log(1+|\kappa|) + B\,|\log B| \right)$$
$$\cdot (1+|\kappa|)\, N_1(\chi) B^{-1}$$
$$\ll \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} \right) (1+|\kappa|)(1+\log(1+|\kappa|))$$
$$\cdot N_1(\chi)(|\log B| + 1);$$

the second term is

$$\ll \int_B^A \left(\frac{1+|\kappa|/2}{y} + 1 \right) \left(Y^{1/2} + \log \frac{Z}{4\pi} - \frac{1}{2} \log(y(y+|\kappa|)) \right) N_1(\chi)\, dy$$
$$\ll \left(\left(1 + \frac{1}{2}|\kappa| \right) \log \frac{A}{B} + A \right)$$
$$\cdot \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} + \log(1+|\kappa|) \right) N_1(\chi)$$
$$+ \left[\frac{1}{2} N_1(\chi) \int_B^1 \left(\frac{1+|\kappa|}{y} + 1 \right) (|\log B| + \log(1+|\kappa|))\, dy \right.$$
$$\left. + (\text{something} \leqslant 0) \right]$$
$$\ll \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} \right) (1+|\kappa|)(1+\log(1+|\kappa|))$$
$$\cdot N_1(\chi)(A + (\log B)^2);$$

and the third one is

$$\ll \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} + \log \frac{1+|\kappa|}{\sqrt{A(A+|\kappa|)}} \right) N_1(\chi) A^3$$
$$\cdot ((1+|\kappa|)A^{-3} + A^{-2})$$
$$\ll \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} + \log(1+|\kappa|) \right) N_1(\chi)(1+|\kappa|+A)$$
$$\ll \left(Y^{1/2} + \log \frac{Z}{4\pi(1+|\kappa|)} \right) (1+|\kappa|)(1+\log(1+|\kappa|))\, N_1(\chi) A.$$

The integral of the error terms over $n(n+\kappa) \geqslant (Z/4\pi)^2$ is estimated by

$$\int_{\sqrt{(Z/4\pi)^2+\kappa^2/4}}^{\sqrt{(YZ/4\pi)^2+\kappa^2/4}} (1+x)\left(Y^{1/2}+\frac{4\pi\sqrt{x^2-\kappa^2/4}}{Z}(\log Y)^2\right) \cdot N_4(\chi)\left(x-\frac{1}{2}|\kappa|\right)^{-4} dx$$

$$+\int_{\sqrt{(YZ/4\pi)^2+\kappa^2/4}}^{\infty} (1+x)\frac{4\pi\sqrt{x^2-\kappa^2/4}}{Z}\log\frac{4\pi\sqrt{x^2-\kappa^2/4}}{Z} \cdot N_4(\chi)\left(x-\frac{1}{2}|\kappa|\right)^{-4} dx$$

$$\ll N_1(\chi)\,A^3\Bigg[Y^{1/2}((1+|\kappa|)Z^{-3}+Z^{-2}) + Z^{-1}(\log Y)^2\,((1+|\kappa|)Z^{-2}+Z^{-1}) + \int_Y^\infty \left(1+\sqrt{\frac{\kappa^2}{4}+\left(\frac{yZ}{4\pi}\right)^2}\right) y\log y\frac{yZ^2\,dy}{(\sqrt{(yZ/4\pi)^2+\kappa^2/4}-|\kappa|/2)^5}\Bigg]$$

$$\ll N_1(\chi)\,A\left[(Y^{1/2}+(\log Y)^2)((1+|\kappa|)Z^{-1}+1)+\frac{1}{Z^2}\int_Y^\infty \frac{\log y\,dy}{y^2}\right]$$

$$\ll [Y^{1/2}+Z^{-2}Y^{-1}\log Y]\,N_1(\chi)A.$$

This shows that all error terms in Lemma 4.2 lead to contributions which may be absorbed into the error term of this proposition.

We are left with the following term:

$$\int_W (2Z)^{1-b}\frac{\mathscr{M}\tau(b-1)}{\pi N(b)}\,p_{|n|-b/12}(b)\,p_{|n+\kappa|-b/12}(b)\,\hat{\chi}(n)\,dn, \qquad (5.4.2)$$

where we integrate over $W=\{n\in\mathbf{R}\colon\ 0<n(n+\kappa)\leqslant(Z/4\pi)^2,\ |n|\in(0,\frac{1}{12})+\mathbf{Z}\}$, and where $b=b(n)$ is determined by $b(n)\equiv 12\,|n| \bmod 12$, $0<b(n)<1$.

Let $U=\{n\in\mathbf{R}\colon n(n+\kappa)\geqslant(Z/4\pi)^2,\ |n|\in(0,\frac{1}{12})+\mathbf{Z}\}$. If we integrate over U instead of over W, we get something that in view of (5.4.1) may be estimated by

$$\sum_{k\geqslant k_0}\int_0^1\left(\frac{2\pi\sqrt{k(k+|\kappa|)}}{Z}\right)^{b-1}\sqrt{1+k}\,\sqrt{1+k+|\kappa|}\,bk^{-4}\,db N_4(\chi)$$

$$\ll \sum_{k\geqslant k_0}\left(1+\log\frac{2\pi\sqrt{k(k+|\kappa|)}}{Z}\right)^{-1}k^{-3}N_4(\chi)$$

$$\ll k_0^{-2}A^3N_1(\chi)$$

with $k_0 = [\sqrt{(Z/4\pi)^2 + \kappa^2/4} - \frac{1}{2}|\kappa|]$, and $[\cdot]$ the integral part. So $k_0 \geqslant Z/8\pi - 1$, and the integral over U is $\mathcal{O}(AN_1(\chi))$. This means that we may replace the term in (5.4.2) by

$$\sum_{k=0}^{\infty} \frac{1}{12\pi} \int_0^1 (2Z)^{1-b} \frac{\mathcal{M}\tau(b-1)}{N(b)} p_k(b)\, p_{k+|\kappa|}(b)$$
$$\cdot (\hat{\chi}((k+b/12)\operatorname{sign}\kappa) + \hat{\chi}(-(k+|\kappa|+b/12)\operatorname{sign}\kappa))\, db$$
$$= \int_0^1 Z^{1-b} g_\kappa(\chi; b)\, \mathcal{M}\tau(b-1)\, db.$$

5.5. LEMMA. *For $\kappa \in \mathbf{Z}$, χ a Schwartz function on* $\mathbf{R}$, *put*

$$S_M(\kappa, \chi) = \sum_{M/2 < c \leqslant M} \frac{1}{c} \sum_{d \bmod c}^{*} \chi(12S(d,c)) e^{2\pi i \kappa d/c}.$$

Then

$$S_M(\kappa, \chi) = \int_0^{2/3} M^{1-b} g_\kappa(\chi; b) \frac{2^{b-1}-1}{b-1}\, db$$
$$+ \mathcal{O}_{\kappa,\chi}(M^{1/3}) \qquad \textit{for} \quad M \to \infty.$$

Remark. The constants in $\mathcal{O}$ may depend on κ and χ. From here on I do not keep this dependence explicit.

Proof. Take τ in Lemma 4.1 and Proposition 5.4 in such a way that

$$\tau(y) = \begin{cases} 1 & \text{for } 2 \leqslant y \leqslant 4 - 2\alpha \\ 0 & \text{for } y \leqslant 2 - \alpha \text{ and for } y \geqslant 4. \end{cases}$$

This may be arranged with $\alpha = \mathcal{O}(Y^{-1})$.

Put

$$T_M = \sum_{M < c < 2M/(2-\alpha)} \frac{1}{c} \sum_{d \bmod c}^{*} 1;$$

then

$$|S_M(\kappa, \chi) - A(\kappa, \chi; 2M)| \leqslant (T_M + T_{M/2}) \|\chi\|_\infty$$

with $\|\chi\|_\infty = \sup_{x \in \mathbf{R}} |\chi(x)|$. Trivially estimating T_M we obtain

$$S_M(\kappa, \chi) = A(\kappa, \chi; 2M) + \mathcal{O}_\chi(MY^{-1} + 1).$$

By Proposition 5.4

$$A(\kappa, \chi; 2M) = \int_0^1 (2M)^{1-b} g_\kappa(\chi; b)\, \mathcal{M}\tau(b-1)\, db$$
$$+ \mathcal{O}_{\kappa,\chi}(Y^{1/2} + \log M).$$

Now we take τ such that $Y = M^{2/3}$; so

$$S_M(\kappa, \chi) = \int_0^1 M^{1-b} 2^{1-b} g_\kappa(\chi; b)\, \mathscr{M}\tau(b-1)\, db + \mathcal{O}(M^{1/3} + \log M).$$

As $\mathscr{M}\tau(b-1) \ll 1$ for $0 \leqslant b \leqslant 1$ and $g_\kappa(\chi, \cdot)$ is bounded, we get

$$\int_{2/3}^1 M^{1-b} 2^{1-b} g_\kappa(\chi; b)\, \mathscr{M}\tau(b-1)\, db \ll M^{1/3}.$$

Let τ_0 be the characteristic function of $[2, 4]$, so $\mathscr{M}\tau_0(b-1) = (2^{2(b-1)} - 2^{b-1})/(b-1)$ and

$$2^{1-b} \mathscr{M}\tau(b-1) - \frac{2^{b-1}-1}{b-1} = 2^{1-b} \mathscr{M}(\tau - \tau_0)(b-1) \ll \alpha \ll Y^{-1}.$$

So

$$S_M(\kappa, \chi) = \int_0^{2/3} M^{1-b} g_\kappa(\chi; b)\, \frac{1-2^{b-1}}{1-b}\, db + \mathcal{O}(MY^{-1} + M^{1/3} + \log M).$$

5.6. **Proposition.** *Let χ be a Schwartz function on $\mathbf{R}$, $\kappa \in \mathbf{Z}$. Then*

$$\lim_{M \to \infty} \frac{\log M}{M} \sum_{c=1}^{M} \frac{1}{c} \sum_{d \bmod c}^{*} \chi(12S(d, c))\, e^{2\pi i \kappa d/c} = \begin{cases} \pi^{-2} \int_{-\infty}^{\infty} \chi(x)\, dx & \text{if } \kappa = 0 \\ 0 & \text{if } \kappa \neq 0. \end{cases}$$

Proof. The quantity to estimate equals

$$\frac{\log M}{M} \sum_{n=0}^{[{}^2\log M]} S_{M2^{-n}}(\kappa; \chi)$$

$$= \log M \int_0^{2/3} M^{-b} g_\kappa(\chi; b)\, \frac{1 - 2^{(b-1)([{}^2\log M]+1)}}{1-b}\, db + \mathcal{O}(M^{-2/3} \log M) \qquad \text{by Lemma 5.5}$$

$$= \log M \int_0^{2/3} M^{-b} f(b)\, db + \mathcal{O}(M^{-2/3} \log M)$$

with $f(b)=g_\kappa(\chi; b)/(1-b)$. This function f is continuous on $[0, \frac{2}{3}]$ and

$$f(0)=\begin{cases}\pi^{-2}\hat{\chi}(0) & \text{if } \kappa=0\\ 0 & \text{if } \kappa\neq 0.\end{cases}$$

We show that

$$\lim_{M\to\infty}\int_0^{2/3} M^{-b}\log Mf(b)\,db=f(0).$$

For given $\varepsilon>0$ take $\beta\in(0,\frac{2}{3})$ such that $|f(b)-f(0)|<\varepsilon$ for all $b\in[0,\beta)$, and take $M>\varepsilon^{-1/\beta}$; then

$$\begin{aligned}&\left|\log M\int_0^{2/3} M^{-b}f(b)\,db-f(0)\right|\\ &\leqslant \log M\int_0^{\beta} M^{-b}\,|f(b)-f(0)|\,db+M^{-\beta}\,|f(0)|\\ &\quad+\log M\int_\beta^{2/3} M^{-b}\,|f(b)|\,db\\ &\leqslant \varepsilon(1-M^{-\beta})+\varepsilon(|f(0)|+\|f\|_\infty).\end{aligned}$$

5.7. *Proof of Theorem* 1.3. Put $X=C_c(\mathbf{R}\times(\mathbf{R}/\mathbf{Z}))$; define the linear forms ν and ν_M on X by

$$\nu(g)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{\mathbf{R}/\mathbf{Z}} g(x, y)\,dy\,dx$$

$$\nu_M(g)=\frac{\log M}{M}\sum_{c=1}^{M}\frac{1}{c}\sum_{d \bmod c}^{*} g\left(12S(d, c), \frac{d}{c}\right).$$

We must prove that for all $g\in X$:

$$\lim_{m\to\infty}\nu_M(g)=\nu(g). \tag{5.7.1}$$

Remark that Proposition 5.6 gives (5.7.1) for

$$g=g_{\chi,\kappa}\colon (x, y)\mapsto \chi(x)e^{2\pi i\kappa y}$$

with χ a Schwartz function on $\mathbf{R}$ and $\kappa\in\mathbf{Z}$. So (5.7.1) is valid on the subspace Y of X spanned by the $g_{\chi,\kappa}$.

Let X_N be the subspace of X of functions g with $\operatorname{supp}(g)\subset(-N, N)\times(\mathbf{R}/\mathbf{Z})$. Put $Y_N=X_N\cap Y$. By the theorem of Stone–Weierstrass each $g\in X_N$

may be approximated in the supnorm on $[-N, N] \times (\mathbf{R}/\mathbf{Z})$ by elements of the form

$$(x, y) \mapsto \sum_{\kappa=-k}^{k} \phi_\kappa(x) e^{2\pi i \kappa y}$$

with $\phi_\kappa \in C^\infty([-N, N])$. We may find Schwartz functions χ_κ with support contained in $(-N-1, N+1)$ extending the ϕ_κ without making their supnorms larger. In this way we see that $g \in X_N$ may be approximated in the supnorm by elements of Y_{N+1}.

Fix a Schwartz function χ on $\mathbf{R}$ with $0 \leqslant \chi \leqslant 1$ and $\chi = 1$ on $[-N-1, N+1]$. For each $h \in X_{N+1}$ we have $h = hg_{\chi,0}$. So $|\nu_M(h)| \leqslant \|h\|_\infty \nu_M(g_{\chi,0})$. As (5.7.1) holds for $g = g_{\chi,0}$ we see that

$$|\nu_M(h)| \leqslant C_{N+1} \|h\|_\infty \qquad \text{for all} \quad M \geqslant 1 \text{ and all } h \in X_{N+1}.$$

So for given $g \in X_N$ choose $h \in Y_{N+1}$ with $\|g-h\|_\infty$ small and use

$$|\nu_M(g) - \nu(g)| \leqslant C_{N+1} \|g-h\|_\infty + |\nu_M(h) - \nu(h)| + \frac{2(N+1)}{\pi^2} \|h-g\|_\infty$$

to conclude that (5.7.1) holds for this g. As $X = \bigcup_{N \geqslant 1} X_N$ this completes the proof.

References

1. R. W. Bruggeman, Fourier coefficients of modular forms, *in* "Lecture Notes in Mathematics," Vol. 865, Springer-Verlag, Berlin, 1981.
2. R. W. Bruggeman, Modular forms of varying weight, III, *J. Reine Angew. Math.* **371** (1986), 144–190.
3. R. W. Bruggeman, Eisenstein series and the distribution of Dedekind sums, *Math. Z.* **202** (1989), 181–198.
4. J.-M. Deshouillers and H. Iwaniec, Kloosterman sums and Fourier coefficients of cusp forms, *Invent. Math.* **70** (1982), 219–288.
5. D. A. Hejhal, The Selberg trace formula for $PSL(2, \mathbf{R})$, Vol. 2, *in* "Lecture Notes in Mathematics," Vol. 1001, Springer-Verlag, Berlin, 1983.
6. D. Hickerson, Continued fractions and density results for Dedekind sums, *J. Reine Angew. Math.* **290** (1977), 113–116.
7. N. V. Kuznetsov, The conjecture of Petersson for forms of weight zero and the conjecture of Linnik: Sums of Kloosterman sums, *Mat. Sb.* **111** (1980), 334–383. [Russian]
8. S. Lang, Introduction to modular forms, *in* "Grundlehren der Mathematischen Wissenschaften," Vol. 222, Springer-Verlag, Berlin, 1976.
9. G. Myerson, Dedekind sums and uniform distribution, *J. Number Theory* **28** (1988), 233–239.
10. N. V. Proskurin, On general Kloosterman sums, preprint LOMI, R-3-80, Leningrad, 1980. [Russian]
11. H. Rademacher and E. Grosswald, "Dedekind Sums," Math. Assoc. Amer., 1972.
12. L. J. Slater, "Confluent Hypergeometric Functions," Cambridge, 1960.
13. G. N. Watson, "A Treatise on the Theory of Bessel Functions," 2nd ed., Cambridge Univ. Press, 1966.