Perturbation of zeros of the Selberg zeta-function for $\Gamma_0(4)$

Roelof Bruggeman, Markus Fraczek, Dieter Mayer

Abstract

We study the asymptotic behavior of zeros of the Selberg zeta-function for the congruence subgroup $\Gamma_0(4)$ as a function of a one-parameter family of characters tending to the trivial character. The motivation for the study comes from observations based on numerical computations. Some of the observed phenomena lead to precise theorems that we prove and compare with the original numerical results.

[1]

[2]

Introduction

This paper presents computational and theoretical results concerning zeros of the Selberg zeta-function. The thesis [Fr] shows that it is possible to use the transfer operator to compute in a precise way zeros of the Selberg zeta-function, and carries out computations for $\Gamma_0(4)$ for a one-parameter family of characters. The results show how zeros of the Selberg zeta-function follow curves in the complex plane

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parametrized by the character. In this paper we observe several phenomena in the behavior of the zeros as the character approaches the trivial character. Motivated by these observations we formulate a number of asymptotic results for these zeros, and prove these results with the spectral theory of automorphic forms. These asymptotic formulas predict certain aspects of the behavior of the zeros more precisely than we guessed from the data. We compare these predictions with the original data. In this way our paper forms an example of interaction between experimental and theoretical mathematics.

In [Se90] it is shown that for the group $\Gamma_0(4)$ and a specific one-parameter family of characters, the Selberg zeta-function not only has countably many zeros on the central line $\mathrm{Re}\,\beta = \frac{1}{2}$, but has also many zeros in the spectral plane situated on the left of the central line, the so-called resonances. Both types of zeros change when the character changes. As the character approaches the trivial character the resonances tend to points on the lines $\mathrm{Re}\,\beta = \frac{1}{2}$ or $\mathrm{Re}\,\beta = 0$, or to the non-trivial zeros of $\zeta(2\beta) = 0$, so presumably to points on the line $\mathrm{Re}\,\beta = \frac{1}{4}$. Many of these zeros have a real part tending to $-\infty$ as the parameter of the character approaches other specific values.

In this paper we focus on zeros on or near the central line $\mathrm{Re}\,\beta = \frac{1}{2}$, and consider their behavior as the character approaches the trivial character.

In Section 1 we describe observations in the results of the computations. We state the theoretical results, and compare predictions with the observations in the computational results. The approach of Fraczek is based on the use of a transfer operator, which makes it possible to consider eigenvalues and resonances in the same way. See [Fr, §7.4].

In Section 2 we give a short list of facts from the spectral theory of automorphic forms, and give the proofs of the statements in §1.

In Section 3 we recall the required results from spectral theory, applied to the group $\Gamma_0(4)$. Not all of the facts needed in §2 are readily available in the literature, some facts need additional arguments in the present situation. The spectral theory that we apply uses Maass forms with a bit of exponential growth at the cusps. In this way it goes beyond the classical spectral theory, which considers only Maass forms with at most polynomial growth. We close §3 with some further remarks on the method and on the interpretation of the results.

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1 Discussion of results

The congruence subgroup $\Gamma_0(4)$ consists of the elements $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ with $c \equiv 0 \bmod 4$. By $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we denote the image in $\mathrm{PGL}_2(\mathbb{R})$ of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$. The group $\Gamma_0(4)$ is free on the generators $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$. A family $\alpha \mapsto \chi_\alpha$ of characters parametrized by $\alpha \in \mathbb{C} \bmod \mathbb{Z}$ is determined by

$$\chi_\alpha\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = e^{2\pi i\alpha}, \quad \chi_\alpha\left(\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\right) = 1. \tag{1.1}$$

The character is unitary if $\alpha \in \mathbb{R} \bmod \mathbb{Z}$. This is the family of characters of $\Gamma_0(4)$ used in [Fr]. See especially §6.5. Up to conjugation and differences in parametrization, this is the family of characters considered in [Se90, §3], and in [PS92] and [PS94].

For a unitary character χ of a cofinite discrete group Γ the Selberg zeta-function $Z(\Gamma,\chi;\cdot)$ is a meromorphic function on $\mathbb{C}$ with both geometric and spectral relevance. As a reference we mention [He83, Chapter X, §2 and §5]. One may also consult [Fi], or [Ve90, Chapter 7].

The geometric significance is clear from the product representation

$$Z(\Gamma,\chi;\beta) = \prod_{k\geq 0}\prod_{\{\gamma\}}\left(1-\chi(\gamma)\,e^{-(\beta+k)\ell(\gamma)}\right) \qquad (\mathrm{Re}\,\beta > 1), \tag{1.2}$$

where k runs over integers and γ over representatives of primitive hyperbolic conjugacy classes. By $\ell(\gamma)$ is denoted the length of the associated closed geodesic. This geometric aspect is used in the investigations in [Fr]. By means of a transfer operator, Fraczek is able to compute zeros of the Selberg zeta function for $\Gamma_0(4)$ as a function of the character χ_α.

Via the Selberg trace formula, the zeros of function $Z(\Gamma,\chi;\cdot)$ are related to automorphic forms. This is the relation that we use in Sections 2 and 3 for our theoretical approach.

We denote by $Z(\alpha,\beta)$ the Selberg zeta-function $\beta \mapsto Z(\Gamma_0(4),\chi_\alpha;\beta)$ for $\alpha \in \mathbb{R}$. We consider its zeros in the region $\mathrm{Im}\,\beta > 0$.

For each value of $\alpha \in \mathbb{R}$ the zeros of $Z(\alpha,\cdot)$ form a discrete set. In Figure 1 we give the non-trivial zeros of $Z(\alpha,\cdot)$ in the region $[0,1]\times i[0,10]$ in the β-plane for the trivial character, $\alpha = 0$, [Fr, Table D.1], and the nearby value $\alpha = \frac{1}{10}$ (interpolation of data discussed in [Fr, §8.2]). In the unperturbed situation, $\alpha = 0$, the zeros to the left of the central line, the *resonances*, are known to occur at the zeros of $\zeta(2\beta)$, of which only one falls within the bounds in the figure. There are also zeros at points $\frac{\pi i\ell}{\log 2}$ with $\ell \in \mathbb{Z}$.

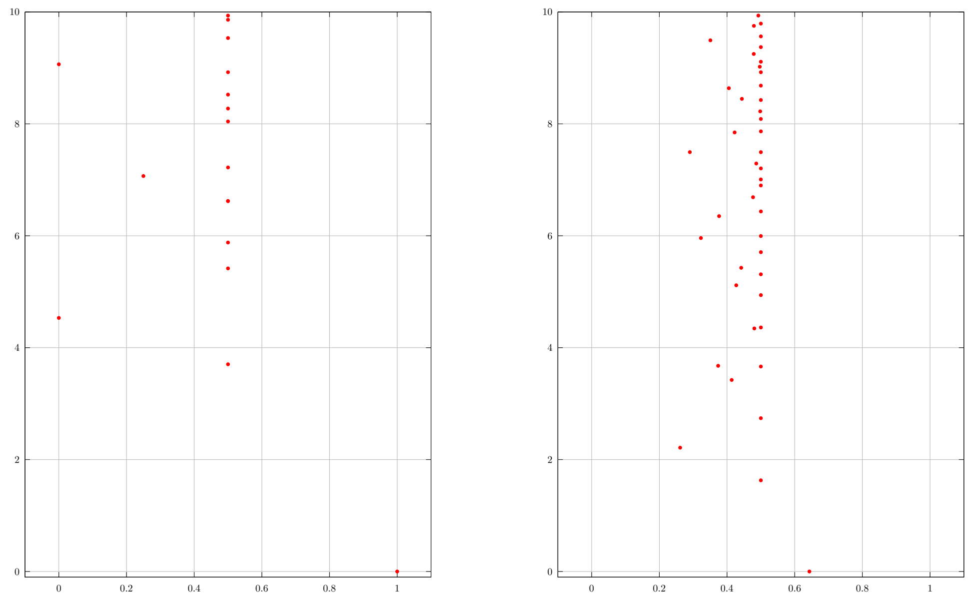


Figure 1: Zeros of the Selberg zeta-function $Z(\alpha, \cdot) = Z(\chi_\alpha, \cdot)$ in the region $[0, 1] \times i[0, 10]$ of the spectral plane. On the left for $\alpha = 0$, on the right for $\alpha = \frac{1}{10}$.

We call zeros β of $Z(\alpha,\cdot)=0$ with $\mathrm{Re}\,\beta=\frac{1}{2}$ *eigenvalues*, although we will see in §3.3 that $\beta-\beta^2$ qualifies better for that name. The lowest unperturbed eigenvalue is .5+3.70331 i. Perturbation to $\alpha=\frac{1}{10}$ gives a more complicated set of zeros, many of which are eigenvalues.

In [Fr, §8.2] it is explained how zeros are followed as a function of the parameter. They follow curves that either stay on the central line, or move to the left of the central line and touch the central line only at some points. Fraczek has prepared animations of the zeros of $\beta\mapsto Z(\alpha,\beta)$ as a function of α. See

`http://homepages.warwick.ac.uk/staff/M.Fraczek/character.html`

The computations for [Fr] were done with the author's own packages. See Appendix A "Project Morpheus" in *loc. cit.* The comparison of the theoretically obtained asymptotic formulas with the data has been carried out with [Pari] and [Sage]; for some of the pictures we used Mathematica.

1.1 Curves of eigenvalues

To exhibit curves of zeros of the Selberg zeta-function on the central line $\mathrm{Re}\,\beta=\frac{1}{2}$ we plot $\mathrm{Im}\,\beta$ as a function of α. The curves in Figure 2 were obtained in [Fr] by first determining for the arithmetical cases $\alpha\in\{\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2}\}$ all zeros in a region of the form $\frac{1}{2}+i[0,T]$. Here we display only those zeros which stay on the central line $\mathrm{Re}\,\beta=\frac{1}{2}$. The computations suggest that all these zeros go to $\beta=\frac{1}{2}$ as $\alpha\downarrow 0$, along curves that are almost vertical for small values of α. Our first result confirms this impression, and makes it more precise:

Theorem 1.1. *For each integer $k\geq 1$ there are $\zeta_k\in(0,1]$ and a real-analytic map $\tau_k:(0,\zeta_k)\to(0,\infty)$ such that $Z(\alpha,\frac{1}{2}+i\tau_k(\alpha))=0$ for all $\alpha\in(0,\zeta_k)$.*

For each $k\geq 1$

$$\tau_k(\alpha) \;=\; \frac{\pi k}{-\log(\pi^2\alpha/4)}+\mathrm{O}\Big(\frac{k^2}{(\log\alpha)^3}\Big) \qquad \text{as } \alpha\downarrow 0\,. \tag{1.3}$$

So the theory tells us that there are infinitely many curves of zeros going down as $\alpha\downarrow 0$, and that for each curve the quantity $-\pi^{-1}\,\mathrm{Im}\,\beta\,\log\frac{\pi^2\alpha}{4}$ tends to an integer. In Figure 3 we plot this quantity against α (on a logarithmic scale), and obtain the integers 1–19 as limit values. This nice agreement convinced us that our numerical and analytical results confirmed each other. The theorem does not state that all zeros of the Selberg zeta-function on the central line occur in these families. The spectral theory of automorphic forms allows the possibility that there are other families.

Figure 2 shows also a regular behavior near many parts on the central line. By theoretical means we obtain:

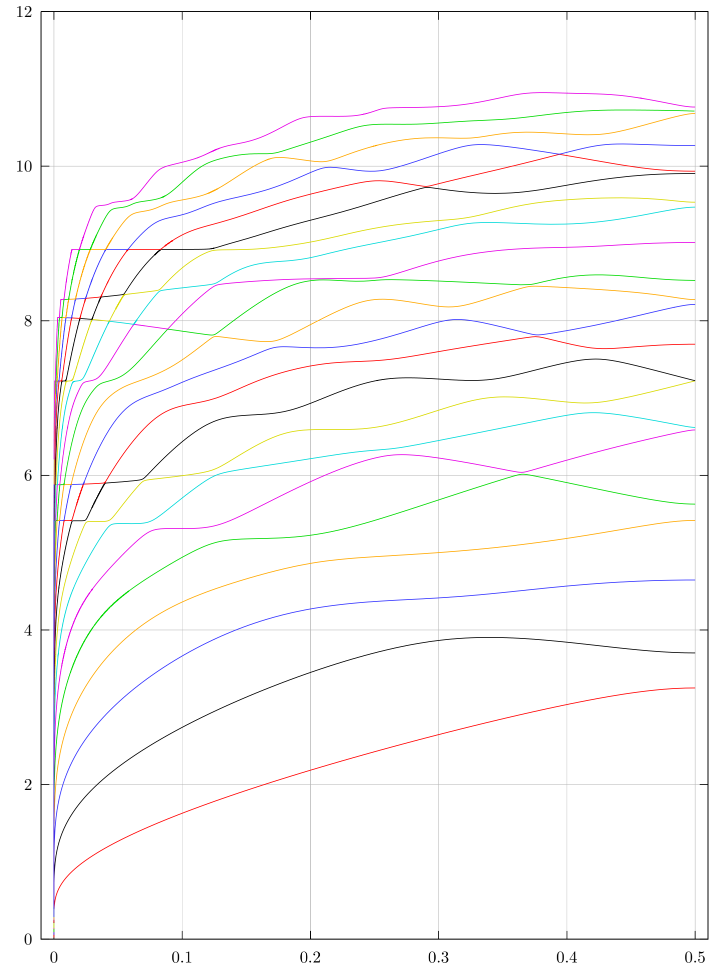


Figure 2: Zeros of $Z(\alpha, \beta)$ with $0 < \alpha \leq \frac{1}{2}, \beta \in \frac{1}{2} + i(0, 10)$. Horizontal: α; vertical: $\operatorname{Im} \beta$.

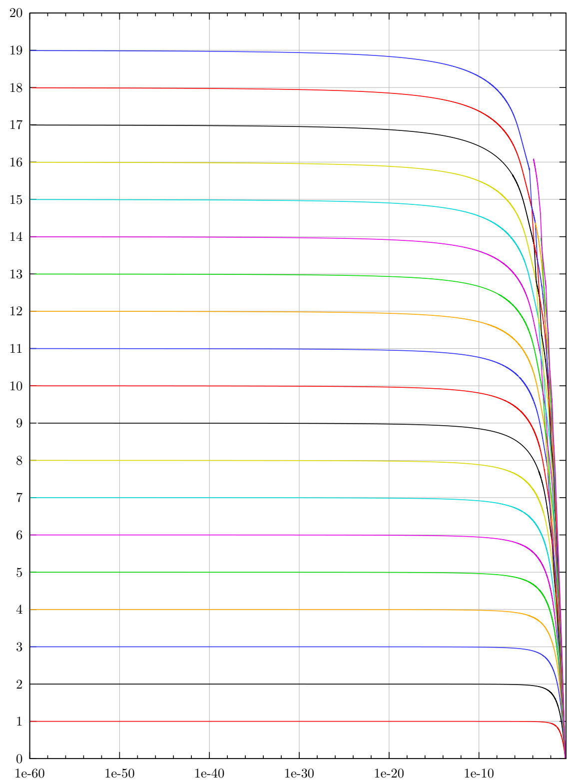


Figure 3: On the vertical axis the quantity $-\pi^{-1}\,\mathrm{Im}\,\beta\,\log\frac{\pi^2\alpha}{4}$ is given for those curves in the data set of [Fr] for which β goes to $\frac{1}{2}$ along the central line as $\alpha \downarrow 0$. The horizontal axis gives α on a logarithmic scale. So $\alpha \downarrow 0$ means going to the left in the graph.

Theorem 1.2. *Let $I \subset (0,\infty)$ be a bounded closed interval such that the interval $\frac{1}{2} + i\,I$ on the central line does not contain zeros of the unperturbed Selberg zeta-function $Z(0,\cdot)$.*

Let for $t \geq 0$

$$\varphi_-(\frac{1}{2}+it) \;=\; \arg\Bigl(\pi^{2it}\,\frac{2^{1+2it}-1}{2^{1-2it}-1}\,\frac{\zeta(-2it)}{\zeta(2it)}\Bigr), \tag{1.4}$$

with ζ the Riemann zeta-function, be the continuous choice of the argument that takes the value 0 for $t=0$.

For all sufficiently large $k \in \mathbb{Z}$ there is a function $a_k : I \to (0,1)$ inverting on I the function τ_k of the previous theorem: $\tau_k(a_k(t)) = t$ for $t \in I$. Uniformly for $t \in I$ we have

$$a_k(t) \;=\; \pi^{-1} e^{\varphi_-(1/2+it)/2t+\pi k_I/2t}\, e^{-\pi k/t}\Bigl(1+\mathrm{O}(e^{-\pi k/t})\Bigr) \qquad (k\to\infty)\,, \tag{1.5}$$

for some $k_I \in \mathbb{Z}$.

The theorem gives an assertion concerning the behavior of the zeros on the central line at given positive values t of $\operatorname{Im}\beta$, and describes the asymptotic behavior as the parameter k from the previous theorem tends to ∞. To compare this prediction with the data we determine by interpolation the value $a_k(t)$ for the curves used in Figure 3. The theorem predicts that

$$\log(\pi a_k(t)) + \frac{\pi k}{t} - \frac{\varphi_-(1/2+it)}{2t} \;\approx\; \frac{\pi k_I}{2t} + \mathrm{O}(e^{-\pi k/t})\,. \tag{1.6}$$

We used the data for the curves with $1 \leq k \leq 19$ to compute an approximation of the quantity on the left in (1.6). We consider this as a vector in $\mathbb{R}^{19}$, with coordinates parametrized by k, and project it orthogonally on the line spanned by $(1,1,\ldots,1)$ with respect to the scalar product $(x,y) = \sum_{k=1}^{19} k^{20}\, x_k\, y_k$, and thus obtain approximations of $\pi k_I/2t$, which are given in Table 1.

$t:$	1	2	3	4	5	6	7	8	9
	−2.000	−2.000	−2.000	−2.000	−2.000	2.000	2.000	4.015	10.17

Table 1: Approximation of k_I in (1.6).

Figure 4 illustrates the approximation of k_I for more values of t between 0.05 and 9.00. The intervals I in the theorem should not contain zeros of the unperturbed Selberg zeta-function. Actually, the proofs will tell us that not all unperturbed zeros are not allowed to occur in I, only those associated to Maass cusp forms that are odd

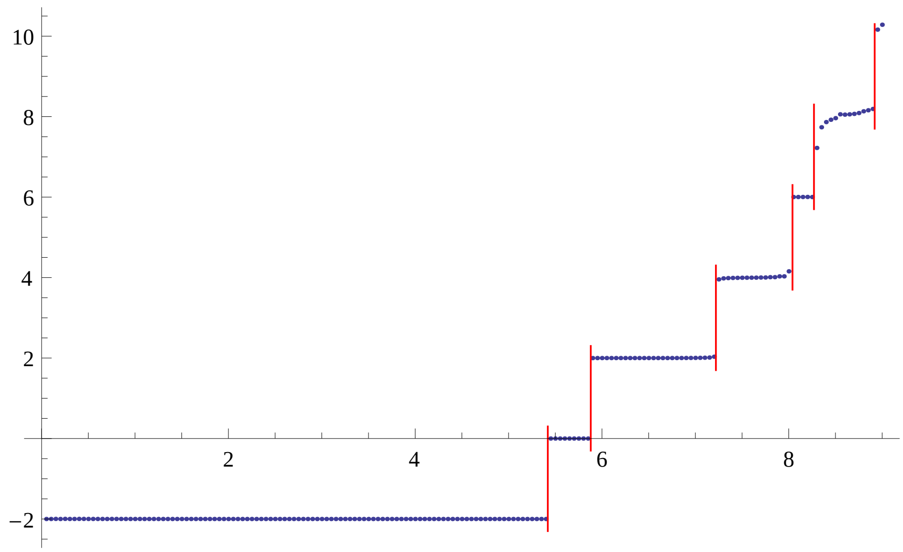


Figure 4: Approximation of k_I for $t \in (0, 9] \cap \frac{1}{20}\mathbb{Z}$. The vertical lines indicate the position of the unperturbed odd eigenvalues (from [Fr, Table D.1]).

for the involution induced by $z \mapsto \overline{z}/(2\overline{z} - 1)$. We have indicated the corresponding t-values by vertical lines in Figure 4.

The computational results suggest that k_I is even. We have no clear theoretical explanation of this observation.

1.1.1 Avoided crossings.

If one looks at the graphs of the functions τ_k in Figure 2 (ignoring the coloring) it seems that the graphs intersect each other. In the enlargement in Figure 5 most of these intersections turn out to be no intersections after all. This is the phenomenon of *avoided crossings* that is known to occur at other places as well; for instance in the computations in [Str]. In the computations for [Fr] care was taken to decrease the step length whenever curves of zeros approached each other. In all cases this indicated that the curves of zeros do not intersect each other. Theoretically, we know that no intersections occur for the zeros moving along the central line in the region indicated in Lemma 2.8.

In Remark 2.10 we will discuss that for some of the $t_0 > 0$ with $Z(0, \frac{1}{2} + it_0) = 0$ there may be a curve through t_0 in the (α, t)-plane such that $\tau_k'(\alpha)$ is relatively small for the value of α for which the graph of τ_k intersects the curve. We show this only under some simplifying assumptions formulated in Proposition 2.9.

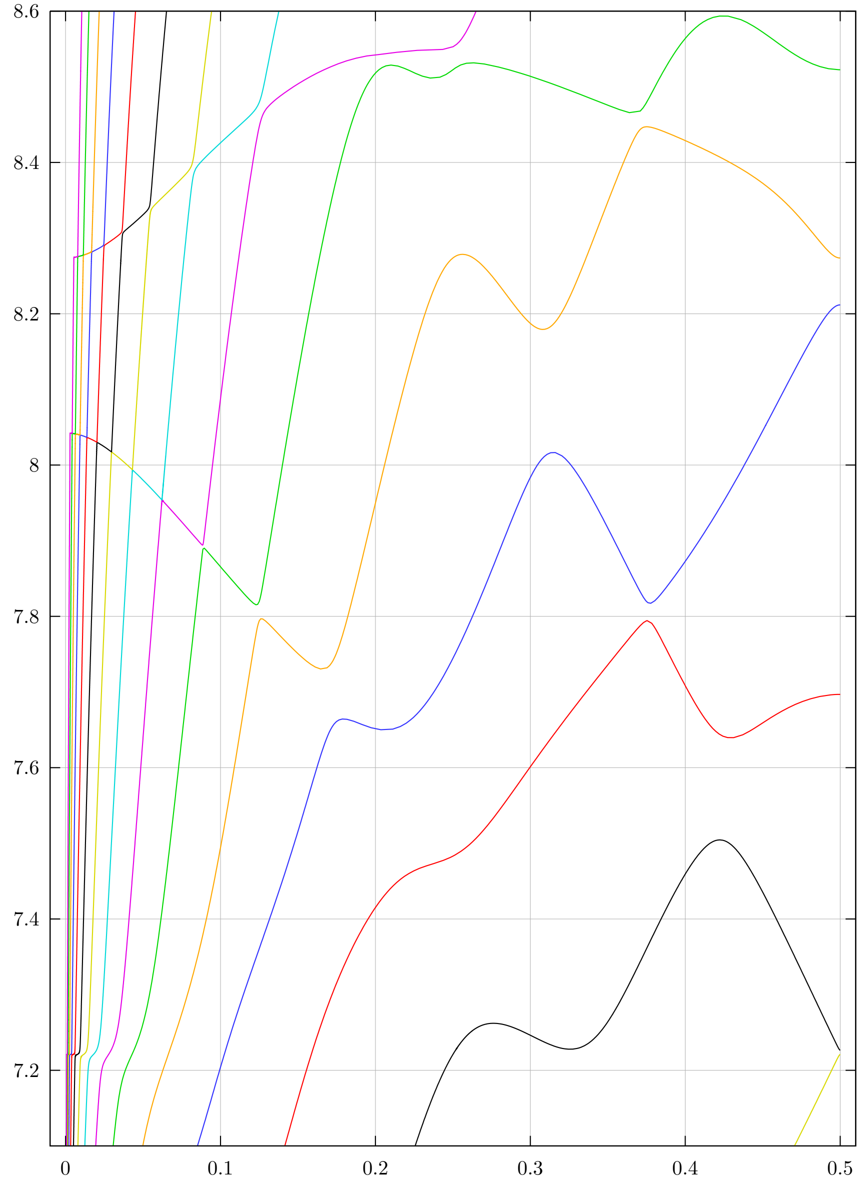


Figure 5: Enlargement of a subregion in Figure 2. Zeros of $Z(\alpha,\beta)$ with $0 < \alpha \le \frac{1}{2}$, $\beta \in \frac{1}{2} + i(7.1, 8.6)$.

1.2 Curves of resonances

The zeros of $Z(\alpha,\beta)$ with β to the left of the central line are more difficult to depict, since they form curves in the three-dimensional set of (α,β) with $\alpha \in (0,1)$ and $\beta \in \mathbb{C}$.

Figure 6 gives a three-dimensional picture. We see one curve in the horizontal plane, corresponding to $\mathrm{Im}\,\beta = 0$. In this paper we do not consider real zeros of the Selberg zeta-function. Many curves originate for $\alpha \approx 0$ from $\beta = \frac{1}{2}$ and move upwards in the direction of increasing values of $\mathrm{Im}\,\beta$. On the right we see also a few more curves that wriggle up starting from higher values of $\mathrm{Im}\,\beta$.

In Figure 7 we project the curves onto the β-plane. In this projection we cannot see the α-values along the curve. We see again the curves starting at $\beta = \frac{1}{2}$. Many of them seem to touch the central line at higher values of $\mathrm{Im}\,\beta$. The curves that start higher up are not well visible in this projection.

We can confirm certain aspects of these computational results by theoretical results. We start with the behavior of the resonances near $(0,\frac{1}{2})$.

Theorem 1.3. *There are $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that all (α,β) that satisfy $Z(\alpha,\beta) = 0$, $\alpha \in (0,\varepsilon_1]$, $\frac{1}{2} - \varepsilon_2 \le \mathrm{Re}\,\beta < \frac{1}{2}$, and $0 < \mathrm{Im}\,\beta \le \varepsilon_3$ occur on countably many curves*

$$t \mapsto (\alpha_k(t), \sigma_k(t) + it) \qquad (0 < t \le \varepsilon_3)\,,$$

parametrized by integers $k \ge 1$. The functions α_k and σ_k are real-analytic. The values of σ_k are in $[\frac{1}{2} - \varepsilon_2, \frac{1}{2})$. For each $k \ge 1$ the map α_k is strictly increasing and has an inverse t_k on some interval $(0,\zeta_k] \subset (0,\varepsilon_1]$. As $\alpha \downarrow 0$ we have

$$t_k(\alpha) \;=\; \frac{\pi k}{|\log \pi^2\alpha|} + \mathrm{O}\Big(\frac{1}{|\log \pi^2\alpha|^4}\Big)\,, \tag{1.7}$$

$$\sigma_k(t_k(\alpha)) \;=\; \frac{1}{2} - \frac{2(\pi k \log 2)^2}{|\log \pi^2\alpha|^3} + \mathrm{O}\Big(\frac{1}{|\log \pi^2\alpha|^4}\Big)\,. \tag{1.8}$$

The theorem confirms that there are many curves of resonances that approach the point $(0,\frac{1}{2})$ almost vertically as $\alpha \downarrow 0$. To check this, one may consider the three quantities

$$\begin{aligned} k_1(\alpha,\beta) \;&=\; \sqrt{(\frac{1}{2} - \mathrm{Re}\,\beta)\,|\log \pi^2\alpha|^3/2\pi^2(\log 2)^2}\,,\\ k_2(\alpha,\beta) \;&=\; \frac{1}{\pi}\,\big|\log(\pi^2\alpha)\big|\,\mathrm{Im}\,\beta\,,\\ k_3(\alpha,\beta) \;&=\; \frac{2(\log 2)^2\,(\mathrm{Im}\,\beta)^3}{\pi(\frac{1}{2} - \mathrm{Re}\,\beta)}\,, \end{aligned} \tag{1.9}$$

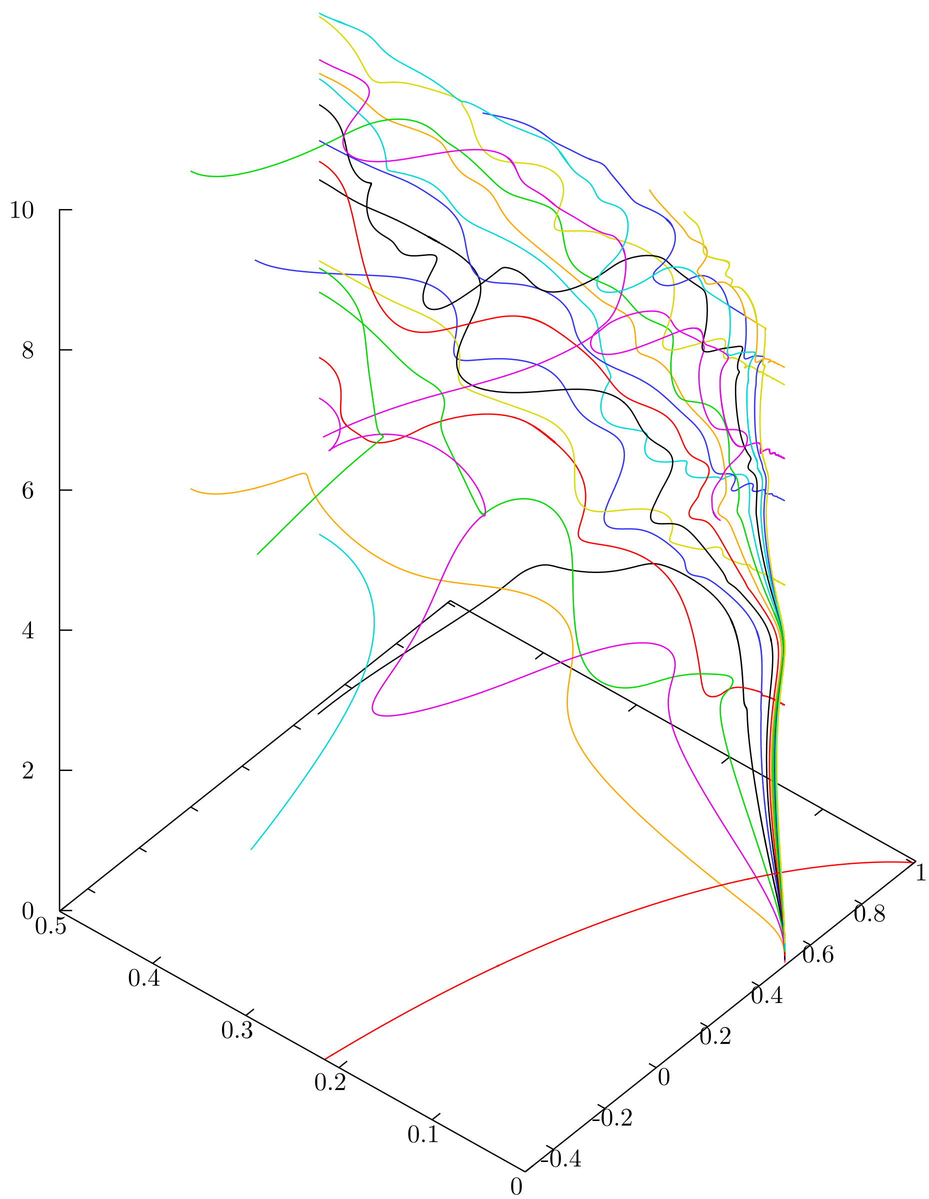


Figure 6: Zeros with $\mathrm{Re}\,\beta < \frac{1}{2}$, $\alpha > 0$, in a 3-dimensional graph. On the vertical axis $\mathrm{Im}\,\beta$ runs from 0 to 10. The 'horizontal' axis running to the left gives the coordinate $\alpha \in (0, \frac{1}{2})$, and the 'horizontal' axis to the right gives $\mathrm{Re}\,\beta \in (-\frac{1}{2}, 1)$.

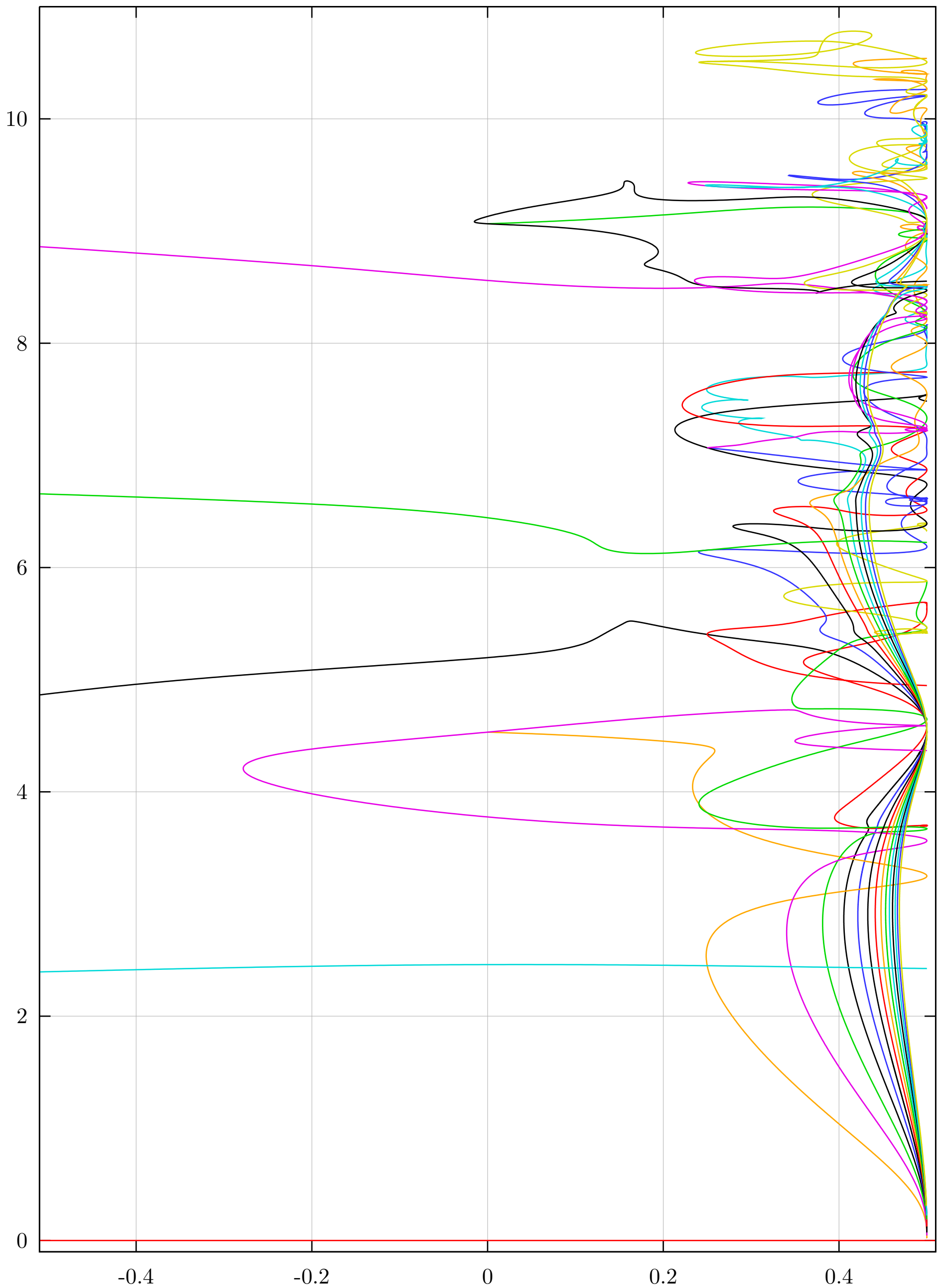


Figure 7: Curves of resonances in the complex plane. The vertical axis carries $\mathrm{Im}\,\beta$ and the horizontal one $\mathrm{Re}\,\beta$.

file name	$c_{0,1}$	$c_{0,2}$	$c_{0,3}$
data/n4-deform-S6.coff	1.00000095598	1.00000017452	0.999999581376
data/n4-deform-S8.coff	1.99992137802	2.00000702518	2.00002268168
data/n4-deform-S7.coff	2.99968426317	2.99999059702	3.00014897712
data/n4-deform-S1-1.coff	3.98471080167	3.99973024912	4.00794474117
data/n4-deform-S3.coff	4.97736802386	4.99941134657	5.01294455984
data/n4-deform-S16.coff	5.99987178401	5.99996232684	6.00002349805
data/n4-deform-S19.coff	6.9671815617	6.99744815359	7.03093500105
data/n4-deform-S23.coff	8.10285518358	7.99040140315	8.02661700448
data/n4-deform-S26.coff	9.04510381727	8.99230508629	9.04177665474
data/n4-deform-S11.coff	10.1363876464	9.9901573178	10.0254416847
data/n4-deform-S34.coff	11.4927014007	10.9860234485	10.9219590835
data/n4-deform-S43.coff	11.9413495934	11.9959052883	12.0661799032
data/n4-deform-S38.coff	13.3191668844	12.988532992	12.9785198908
data/n4-deform-S46.coff	14.0431747756	13.993197026	14.0160494173

Table 2: Least square approximation of the limits $c_{0,j}$ of the quantities k_j in (1.9). (The first column refers to the naming in [Fr] of the curves of zeros.)

which should each approximate the "real" k as $\alpha \downarrow 0$. A graphical approach does give less satisfactory results than those in Figure 3. The range of values of α for which we have data seems not to approach 0 sufficiently closely to draw definite conclusions concerning the limit behavior. In a non-graphical approach we approximate the limit value by finding the coefficients in a least square approximation

$$k_j(\alpha,\beta) \approx \sum_{\ell=0}^{4} \frac{c_{\ell,j}}{|\log\alpha|^\ell} \tag{1.10}$$

over the 500 data points with lowest values of α on each of the curves of resonances going to $(0,\frac{1}{2})$. The coefficient $c_{0,j}$ should be an approximation of the limit. The data are in Table 2. This gives a reasonable confirmation that (1.7) and (1.8) describe the asymptotic behavior of the data. We also experimented with direct least square approximation of the coefficients of the expansion of $t_k(\alpha)$ and $\sigma_k(t_k(\alpha))$ as a function of $\frac{1}{|\log\pi^2\alpha|}$. The results from the approximation of $\sigma_k(t_k(\alpha))$ were less convincing than those in Table 2.

The next result concerns curves higher up in the β-plane.

Theorem 1.4. *Let I be a bounded interval in $(0,\infty)$ such that $\{0\}\times\left(\frac{1}{2}+iI\right)$ does not contain zeros of the unperturbed Selberg zeta-function $Z(0,\cdot)$.*

There are countably many real-analytic curves of resonances of the form

$$t \mapsto (\alpha_k(t), \sigma_k(t)+it) \qquad \text{with } t\in I\,,$$

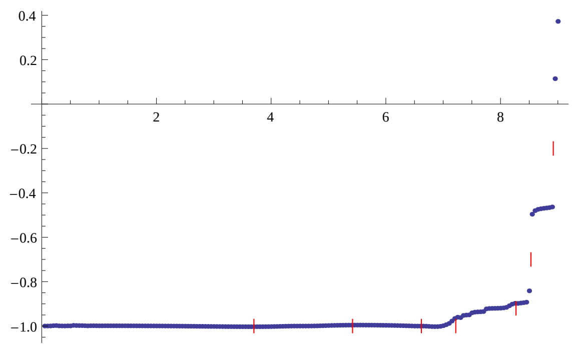


Figure 8: Approximation of k_I for $t \in (0,9] \cap \frac{1}{20}\mathbb{Z}$. The vertical lines indicate the position of unperturbed even eigenvalues (from [Fr, Table D.1]).

parametrized by integers $k \geq k_1$ for some integer k_1. Uniformly for $t \in I$ we have the relations

$$\alpha_k(t) = \frac{1}{\pi}\, e^{A(1/2+it)/2t}\, e^{-\pi k/t}\Bigl(1 + \mathrm{O}\bigl(\frac{1}{k}\bigr)\Bigr), \tag{1.11}$$

$$\frac{1}{2} - \sigma_k(t) = \frac{t}{2\pi k}\, M\bigl(\frac{1}{2} + it\bigr) + \mathrm{O}\Bigl(\frac{1}{k^2}\Bigr), \tag{1.12}$$

as $k \to \infty$, where $M(1/2+it)$ and $A(1/2+it)$ are the real and imaginary part of a continuous choice of

$$t \mapsto \log\Bigl(\pi^{2it}\,(2^{1+2it} - 1)\,\frac{\zeta(-2it)}{\zeta(2it)}\Bigr).$$

If we would use a standard choice of the argument the function A would have discontinuities. The parameter k is determined by the choice of the branch of the logarithm. The theory does not provide us, as far as we see, a way to relate the numbering of the branches for different intervals I.

We compare relation (1.11) with the data files in the same way as we used for Theorem 1.4. The theorem says that the relation holds for some choice A of the argument. We picked a continuous choice. Then we expect a factor $e^{\pi k_I/t}$ in (1.11) with k_I constant on intervals as indicated in the theorem. This leads to Figure 8. The function M in (1.12) has the simple form $M(1/2+it) = \log\bigl|2^{1+2it} - 1\bigr|$. Figure 9 gives this function and the approximation of it based on (1.12). Figures 8 and 9 show differences that we do not understand well.

In Figure 7 it seems that at $\beta \approx \frac{1}{2} + 4.5\, i$ many curves touch the central line. Moreover, relation (1.12) suggests that there are infinitely many curves that are

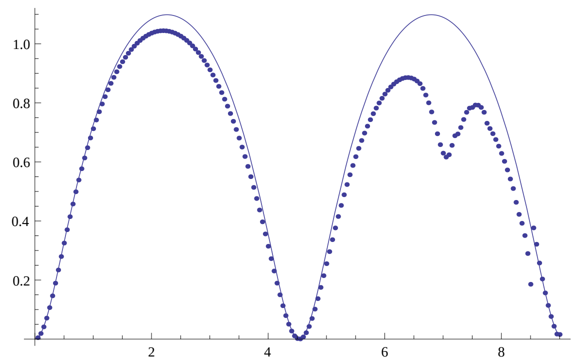


Figure 9: The function $t \mapsto \log\left|2^{1+2it}-1\right|$ and its approximation based on (1.12).

tangent to the central line at the points $\frac{1}{2}+\frac{\pi i}{\log 2}\ell$ with $\ell \in \mathbb{Z}$. In Figure 7 there seems to be a common touching to the central line at $\beta \approx \frac{1}{2}+9.0\,i$ as well. Figure 10 gives a closer few at the resonances near $\beta = \frac{1}{2}+\frac{\pi i}{\log 2}$ for curves computed in [Fr]. There is no common touching point, but a sequence of tangent points approaching $\frac{1}{2}+\frac{\pi i}{\log 2}$. See [Fr, Conclusions 8.2.32 and 8.2.33] for a further discussion. Concerning this phenomenon we have the following result:

Theorem 1.5. *Suppose that an interval I as in Theorem 1.4 contains in its interior a point $t_\ell := \frac{\pi\ell}{\log 2}$ with an integer $\ell \geq 1$. Then there is $k_2 \geq k_1$ such that for each $k \geq k_2$ the curve $t \mapsto \sigma_k(t)+it$ in Theorem 1.4 is tangent to the central line in a point $\frac{1}{2}+it_\ell+i\delta_k \in \frac{1}{2}+iI$, and the δ_k satisfy*

$$\delta_k \;=\; \frac{\eta_2}{\pi^2}e^{A(1/2+it_\ell)/t_\ell}\, e^{-2\pi k/t_\ell}\left(1+\mathrm{O}(k^{-1})\right), \tag{1.13}$$

for some $\eta_2 \in \mathbb{R}$. The function A is as in Theorem 1.4.

We do not get information concerning η_2 from the theory. [Fr, Table 8.8] gives approximated tangent points near $\frac{1}{2}+\frac{\pi i}{\log 2}$. In Figure 11 we give the corresponding approximations of $\log\eta_2$.

Remark 1.6. Theorems 1.1–1.5 have been motivated by part of the observations of Fraczek. In the next sections we present proofs that do not depend on the computations. The comparisons of the theoretically obtained asymptotic results with the computational data is in some cases convincing, and show in other cases discrepancies that we do not understand fully.

Remark 1.7. Figure 6 shows curves of resonances that do not approach $\beta = \frac{1}{2}$ as $\alpha \downarrow 0$. One of these curves is depicted in Figure 12, with an enlargement of the part

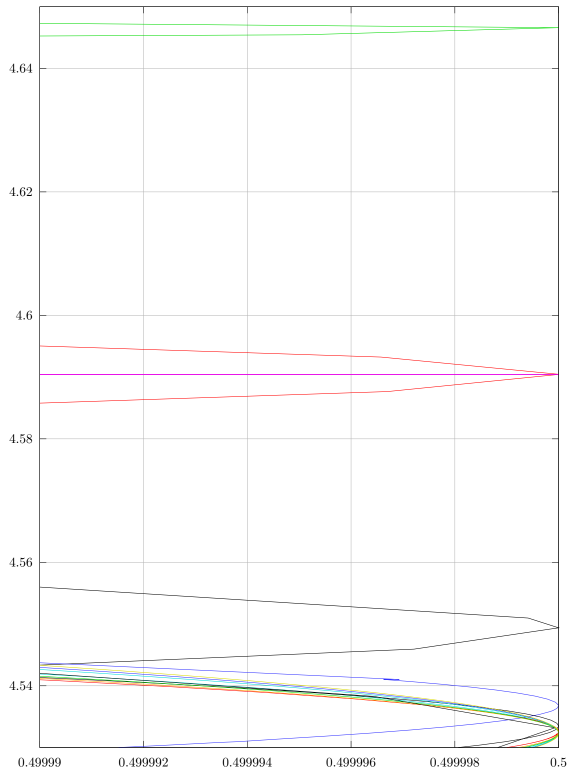


Figure 10: Enlargement of part of Figure 7 near $\beta = \frac{1}{2} + \frac{\pi i}{\log 2}$.

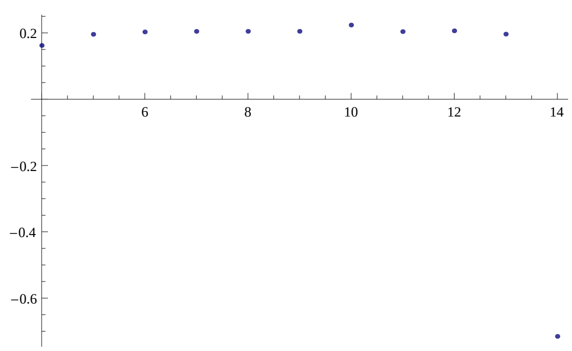


Figure 11: Approximation of η_2 in Theorem 1.5 for $\ell = 1$, based on [Fr, Table 8.8].

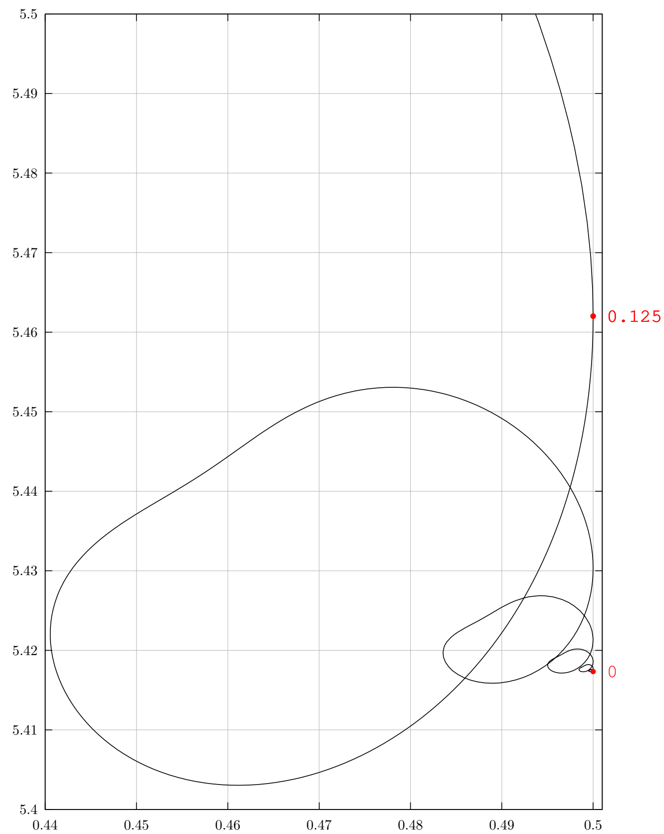


Figure 12: Curve of resonances starting at $\beta \approx \frac{1}{2} + i5.4173$. Values of α are given in red.

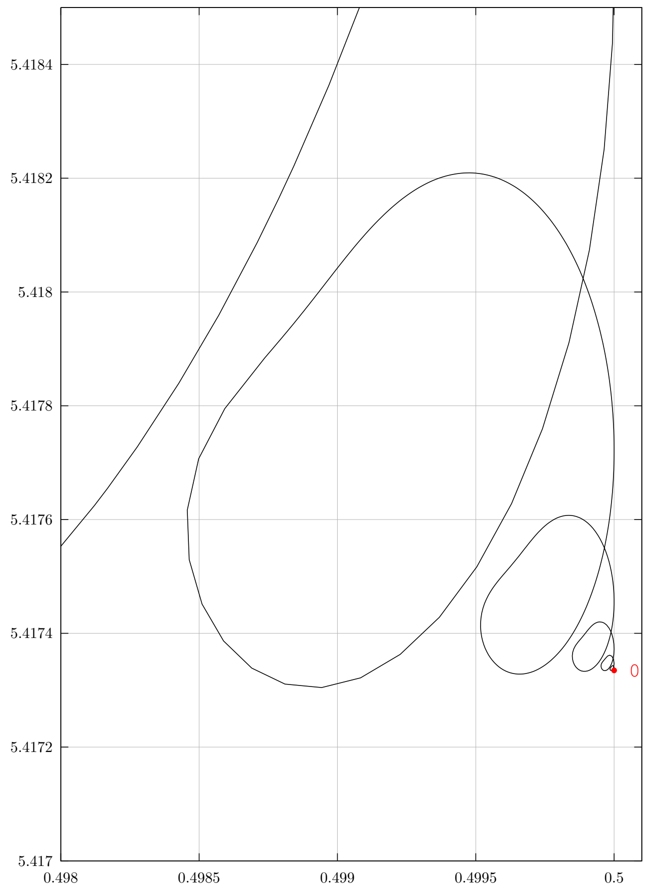


Figure 13: Enlargement of the initial part of the curve in Figure 12.

with small values of α in Figure 13. The suggestion is that this curve forms loops that repeatedly touch the central line.

We cannot prove that this type of behavior is bound to happen. In Proposition 2.20 we work under a number of assumptions, and then can prove some of the properties that can be seen in the data.

2 Proofs

In this section we prove the theoretical results stated in §1. The ingredients from the spectral theory of automorphic forms that we use are the scattering matrix and a generalization of it.

In §2.1–2.2 we summarize the facts we need. In §2.3–2.5 we prove the results in §1.

2.1 Facts from spectral theory

We will need a restricted list of facts from the spectral theory of automorphic forms. Table 3 gives a reference to a further discussion.

In the spectral theory of automorphic forms the *scattering matrix* plays an important role. For the unperturbed situation $\alpha = 0$ it is explicitly known:

$$\mathbf{C}_0(\beta) = \frac{1}{2^{2\beta}-1}\,\frac{\Lambda(2\beta-1)}{\Lambda(2\beta)}\begin{pmatrix} 2^{1-2\beta} & 1-2^{1-2\beta} & 1-2^{1-2\beta} \\ 1-2^{1-2\beta} & 2^{1-2\beta} & 1-2^{1-2\beta} \\ 1-2^{1-2\beta} & 1-2^{1-2\beta} & 2^{1-2\beta} \end{pmatrix}, \qquad (2.1)$$

where $\Lambda(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$ is the completed Riemann zeta-function.

Fact 2.1. *The zeros of* $Z(0,\cdot)$ *in the region* $\mathrm{Re}\,\beta < \frac{1}{2}$ *and* $\mathrm{Im}\,\beta > 0$ *are the values at which one of the matrix elements of the scattering matrix* $\mathbf{C}_0$ *has a singularity.*

The occurrence of $\zeta(2\beta)$ in the denominator of the matrix elements explains the zeros of $Z(s,\cdot)$ with $\mathrm{Re}\,\beta = \frac{1}{4}$. The factor $(2^{2\beta}-1)^{-1}$ produces zeros on the line $\mathrm{Re}\,\beta = 0$.

The scattering matrix in (2.1) can be embedded in a meromorphic family of matrices

$$\mathbf{C}(\alpha,\beta) = \begin{pmatrix} C_{0,0}(\alpha,\beta) & C_{0,\infty}(\alpha,\beta) & C_{0,\infty}(\alpha,\beta) \\ C_{0,\infty}(\alpha,\beta) & C_{\infty,\infty}(\alpha,\beta) & C_{\infty,-1/2}(\alpha,\beta) \\ C_{0,\infty}(\alpha,\beta) & C_{\infty,-1/2}(\alpha,\beta) & C_{\infty,\infty}(\alpha,\beta) \end{pmatrix} \qquad (2.2)$$

of 3×3-matrices on $U \times \mathbb{C}$, where U is a neighborhood of $(-1,1)$ in $\mathbb{C}$. We call it the *extended scattering matrix*. Its construction depends on functional analysis, and is far from explicit. We will use its properties in 2.3–2.6.

The columns and rows are indexed by 0, ∞ and $-\frac{1}{2}$ (representatives of the cuspidal orbits of $\Gamma_0(4)$). If all symmetries visible in the matrix in (2.1) would disappear under perturbation, there would be nine different matrix elements. However, some of the symmetries survive perturbation

Fact 2.2. *The extended scattering matrix satisfies the symmetries indicated by the coinciding matrix entries in* (2.2).

Fact 2.3. *The restriction* $\beta \mapsto \mathbf{C}(0,\beta)$ *exists and is the scattering matrix* $\beta \mapsto \mathbf{C}_0(\beta)$ *in* (2.1).

We note that a meromorphic function $(\alpha,\beta) \mapsto f(\alpha,\beta)$ on an open set of $\mathbb{C}^2$ may have a singularity at (α_0,β_0) that is not visible as a singularity of $\beta \mapsto f(\alpha_0,\beta)$.

(Consider for instance $f(\alpha,\beta) = \frac{\alpha-\beta}{\alpha+\beta}$ at $(0,0)$.) Such singularities are said to be of *indeterminate type*.

Fact 2.4. *Let* $\beta_0 \in \frac{1}{2} + i[0,\infty)$. *If the extended scattering matrix has a singularity at* $(0,\beta_0)$*, then* $Z(0,\beta_0) = 0$ *and* $\beta_0 \neq \frac{1}{2}$.

In (2.1) we see that such a singularity has necessarily indeterminate type.

There are functional equations:

Fact 2.5. *We have*

$$\begin{aligned} \mathbf{C}(-\alpha,\beta) &= \mathbf{C}(\alpha,\beta)^t\,, & \mathbf{C}(\alpha,1-\beta) &= \mathbf{C}(\alpha,\beta)^{-1}\,, \\ \overline{\mathbf{C}(\bar{\alpha},1-\bar{\beta})} &= (\mathbf{C}(\alpha,\beta)^t)^{-1}\,. \end{aligned} \tag{2.3}$$

as identities of meromorphic families of matrices on $U \times \mathbb{C}$.

For the perturbed situation there is also a scattering "matrix", with size 1×1. Unlike the scattering matrix for $\alpha = 0$, we have no explicit formula for it. However it can be expressed in the matrix elements of the extended scattering matrix.

Fact 2.6. *Let* X*,* C_+ *and* C_- *be the meromorphic functions on* $U_+ \times \mathbb{C}$*, with* $U_+ = \{\alpha \in U \;:\; \mathrm{Re}\,\alpha > 0\}$*, given by*

$$X(\alpha,\beta) \;=\; (\pi\alpha)^{2\beta-1}\,\Gamma(\frac{1}{2}-\beta)\,\Gamma(\beta-\frac{1}{2})^{-1}\,, \tag{2.4}$$

$$C_\pm(\alpha,\beta) \;=\; C_{\infty,\infty}(\alpha,\beta) \pm C_{\infty,-1/2}(\alpha,\beta)\,. \tag{2.5}$$

The meromorphic function

$$D_{0,0}(\alpha,\beta) \;=\; \frac{C_{0,0}(\alpha,\beta) - X(\alpha,\beta)\,(C_{0,0}(\alpha,\beta)\,C_+(\alpha,\beta) - 2\,C_{0,\infty}(\alpha,\beta)^2)}{1 - X(\alpha,\beta)\,C_+(\alpha,\beta)} \tag{2.6}$$

on $U_+ \times \mathbb{C}$*, has a meromorphic restriction* D_α *to the complex line* $\{\alpha\}\times\mathbb{C}$ *for each* $\alpha \in (0,1)$.

For $\alpha \in (0,1)$ *the zeros of the Selberg zeta-function* $Z(\alpha,\beta)$ *with* $\mathrm{Im}\,\beta > 0$ *satisfy* $\mathrm{Re}\,\beta \leq \frac{1}{2}$. *Those of these zeros that satisfy* $\mathrm{Re}\,\beta < \frac{1}{2}$ *are the values of* β *at which* $D_\alpha(\beta)$ *has a singularity.*

We note that the existence of the restriction to $\{\alpha\}\times\mathbb{C}$ is a non-trivial assertion. It says that the meromorphic function $D_{0,0}$ has no singularity along this complex line.

Fact 2.7. *If* C_- *is holomorphic at* $(\alpha,\beta) \in (0,1)\times\left(\frac{1}{2}+i\mathbb{R}\right)$ *and*

$$X(\alpha,\beta)\,C_-(\alpha,\beta) \;=\; 1\,,$$

then $Z(\alpha,\beta) = 0$.

(2.1)	[Hu84]	2.4	§3.5
2.1	§3.3	2.5	§3.1
2.2	(3.6) and §3.6	2.6	§3.2 and §3.3
2.3	§3.1	2.7	§3.4

Table 3: The places in §3 where a reference or a proof is given for the facts in §2.1.

2.2 The extended scattering matrix

The functional equations in 2.5 imply that $\mathbf{C}(\alpha,\beta)$ is a unitary matrix if $\alpha \in (-1,1)$ and $\beta \in \frac{1}{2} + i\mathbb{R}$. This implies that if $\alpha_0 \in (-1,1)$ then the matrix elements of $\mathbf{C}(\alpha_0,\beta)$ are bounded. So the restriction $\beta \mapsto \mathbf{C}(\alpha_0,\beta)$ cannot have singularities on the line $\{\alpha_0\} \times \left(\frac{1}{2} + i\mathbb{R}\right)$ as a function of the variable β. Nevertheless, the matrix elements can have singularities at (α_0,β_0) with $\mathrm{Re}\,\beta_0 = \frac{1}{2}$ as functions of the two complex variables (α,β).

The extended scattering matrix can be partly diagonalized:

$$\begin{aligned} \mathbf{U}\,\mathbf{C}\,\mathbf{U}^{-1} &= \begin{pmatrix} \mathbf{C}^{+} & 0 \\ 0 & C_{-} \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \\ \mathbf{C}^{+} &= \begin{pmatrix} C_{0,0} & \sqrt{2}\,C_{0,\infty} \\ \sqrt{2}\,C_{0,\infty} & C_{+} \end{pmatrix}. \end{aligned} \tag{2.7}$$

For $(\alpha,\beta) \in (-1,1) \times \left(\frac{1}{2} + i\mathbb{R}\right)$ the matrix $\mathbf{C}^{+}(\alpha,\beta)$ is unitary, and $\left|C_{-}(\alpha,\beta)\right| = 1$. The functional equations in 2.5 imply similar relations for $\mathbf{C}^{+}$. For $\alpha = 0$ we have

$$\mathbf{U}\,\mathbf{C}_0(0,\beta)\,\mathbf{U}^{-1} = \frac{\Lambda(2\beta-1)}{\Lambda 2\beta} \begin{pmatrix} \frac{2^{1-2\beta}}{2^{2\beta}-1} & \frac{\sqrt{2}(1-2^{1-2\beta})}{2^{2\beta}-1} & 0 \\ \frac{\sqrt{2}(1-2^{1-2\beta})}{2^{2\beta}-1} & \frac{1}{2^{2\beta}-1} & 0 \\ 0 & 0 & \frac{2^{2(1-\beta)}-1}{2^{2\beta}-1} \end{pmatrix}. \tag{2.8}$$

2.3 Curves of eigenvalues

For curves of eigenvalue s, *i.e.*, zeros of the Selberg zeta-function that stay on the central line, the fact 2.7 is important.

Lemma 2.8. *There is a simply connected set* $S_{-} \subset (0,1)\times(\frac{1}{2} + i(0,\infty))$ *in which* C_{-} *has no singularities. For each bounded interval* $I \subset (0,\infty)$ *there is* $\varepsilon_I \subset (0,1)$ *such that* $(0,\varepsilon_I)\times(\frac{1}{2} + iI) \subset S_{-}$.

Proof. If f and g are non-zero holomorphic functions on an open subset $U \subset \mathbb{C}^2$ without common factor in the ring of germs of holomorphic functions at $p \in U$ then their null sets intersect each other in an analytic subset of U that has dimension 0 near p. ([GR, Chap. 5, §2.4] implies that the null sets have dimension 1 at p. If their intersection would also have dimension 1 at p, then any prime component of this intersection in the Lasker-Noether decomposition [GR, p. 79] would be given by a common factor of f and g in the ring of germs at p.) So the quotient f/g can have singularities of indeterminate type only at a discrete set of points in U. Applying this to C_- the set of points to avoid is discrete in $[0, \frac{1}{2}] \times (\frac{1}{2} + i[0, \infty))$. Let $t_I = \max I$. We take ε_I equal to the minimum of the α for which (α, β) is one of the finitely many points of indeterminacy of C_- in $(0, \frac{1}{2}] \times (\frac{1}{2} + i[0, t_I])$. $\square$

We consider the equation $X C_- = 1$ in 2.7 in the set S_-. The function

$$Y_-(\alpha, \beta) \;=\; \frac{\Gamma(\beta - \frac{1}{2})}{\Gamma(\frac{1}{2} - \beta)} \frac{1}{C_-(\alpha, \beta)} \tag{2.9}$$

is holomorphic at all points of S_-, and has absolute value 1 at the points of S_-. The equation $X C_- = 1$ on S_- is equivalent to

$$(\pi\alpha)^{2\beta - 1} \;=\; Y_-(\alpha, \beta)\,.$$

We can choose a continuous argument $A_-(\alpha, \beta)$ of $Y_-(\alpha, \beta)$ on S_-, since this set is simply connected. The function $1/C_-$ may have singularities (of indeterminate type) at points of $\{0\} \times \left(\frac{1}{2} + i\mathbb{R}\right)$. If that occurs then the continuous extension of A_- to $\{0\} \times \left(\frac{1}{2} + i\mathbb{R}\right)$ minus the points where $1/C_-$ is singular does not have a constant difference with a continuous argument of

$$Y_-(0, \frac{1}{2} + it) \;=\; \pi^{2it} \, \frac{2^{1+2it} - 1}{2^{1-2it} - 1} \, \frac{\zeta(-2it)}{\zeta(2it)}\,.$$

See (2.1). From 2.4 we see that Y_- is holomorphic at $(0, \frac{1}{2})$. It has value 1 at $(0, \frac{1}{2})$. We normalize A_- such that its continuous extension has value 0 at $(0, \frac{1}{2})$. We find the Taylor approximation

$$A_-(\alpha, \frac{1}{2} + it) \;=\; 2t \log \frac{4}{\pi} + \mathrm{O}(t^2) + \mathrm{O}(\alpha) \qquad \text{as } (\alpha, t) \to (0, 0)\,. \tag{2.10}$$

With this preparation, we can reformulate the equation $X C_- = 1$ in S_- as

$$2t \, \log \pi\alpha \;=\; A_-(\alpha, \frac{1}{2} + it) - 2\pi k\,, \qquad (k \in \mathbb{Z})\,. \tag{2.11}$$

We have written $\beta = \frac{1}{2} + it$.

Proof of Theorem 1.1. The formulation in (2.11) shows that the solution set of $XC_- = 1$ in S_- is the disjoint union of components V_k, parametrized by the integer k in (2.11).

We take $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that $\Omega = (0, \varepsilon_1) \times \left(\frac{1}{2} + i(0, \varepsilon_2)\right) \subset S_-$ and such that $|A_-| < \pi$ on Ω. In the course of the proof we will impose a finite number of additional conditions on ε_1 and ε_2.

Equation (2.11) implies that for $(\alpha, \frac{1}{2} + it) \in V_k \cap \Omega$ we have

$$e^{-(C+\pi k)/t} \ \leq\ \pi\alpha \ \leq\ e^{(C-\pi k)/t}$$

for some $C \leq \frac{\pi}{2}$. On the basis of this first estimate we proceed more precisely, and obtain

$$2t \log \frac{\pi^2 \alpha}{4} \ =\ -2\pi k + \mathrm{O}(t^2) + \mathrm{O}(\alpha) \ =\ -2\pi k + \mathrm{O}(t^2)\,, \tag{2.12}$$

and conclude

$$\alpha \ =\ \frac{4}{\pi^2}\, e^{-\pi k/t}\,(1 + \mathrm{O}(t))\,. \tag{2.13}$$

If $k \leq 0$ this does not allow small values of α for $t \in (0, \varepsilon_2)$. Hence $k \geq 1$.

To show that $V_k \cap \Omega$ is the graph of a function, we apply the implicit function theorem. The set V_k is the level set $F(\alpha, t) = -2\pi k$ of the function $F(\alpha, t) = 2t \log \pi\alpha - A_-(\alpha, \frac{1}{2} + it)$, with derivatives

$$\frac{\partial F}{\partial \alpha} \ =\ \frac{2t}{\alpha} + \mathrm{O}(1)\,, \qquad \frac{\partial F}{\partial t} \ =\ 2 \log \frac{\pi^2 \alpha}{4} + \mathrm{O}(t) + \mathrm{O}(\alpha)\,.$$

So $\frac{\partial F}{\partial \alpha} > 0$ and $\frac{\partial F}{\partial t} < 0$ if we take ε_1 and ε_2 sufficiently small. So $V_k \cap \Omega$ is the graph of an injective function $\alpha \mapsto \frac{1}{2} + i\tau_k(\alpha)$ on $(0, \varepsilon_1)$. Since F is a real-analytic function, the analytic implicit function theorem shows that τ_k is real-analytic. (See, *e.g.*, [KrPa, Theorem 6.1.2].)

From (2.12) it follows that for $\alpha \in (0, \varepsilon_1)$, with ε_1 sufficiently small,

$$\tau_k(\alpha) \ =\ \mathrm{O}(k / \log(\pi^2 \alpha / 4))\,.$$

and then

$$\tau_k(\alpha) \ =\ \frac{-\pi k + \mathrm{O}(\tau_k(\alpha)^2)}{\log \frac{\pi^2 \alpha}{4}} \ =\ \frac{\pi k}{-\log \frac{\pi^2 \alpha}{4}} \left(1 + \mathrm{O}\left(k^2 / \log(\pi^2 \alpha / 4)^2\right)\right). \tag{2.14}$$

This gives (1.3). □

Proof of Theorem 1.2. Let $I \subset (0, \infty)$ be an interval as in the theorem. Lemma 2.8 provides us with ε_I such that $(0, \varepsilon_I) \times \left(\frac{1}{2} + iI\right) \subset S_-$. Since $\{0\} \times \left(\frac{1}{2} + iI\right)$ does not contain singularities of C_- the function A_- is continuous and hence bounded on $[0, \varepsilon_1] \times \left(\frac{1}{2} + iI\right)$ for $0 < \varepsilon_1 < \varepsilon_I$.

The graph τ_k is the level curve $F = -2\pi k$ of the function

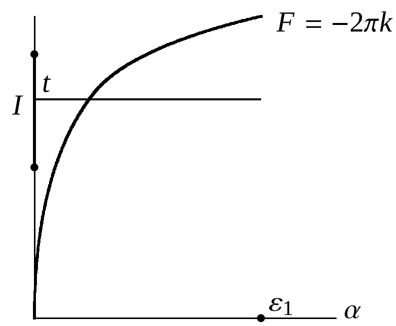
$$F(\alpha, t) \;=\; 2t\, \log \pi\alpha - A_-(\alpha, \frac{1}{2} + it)\,, \tag{2.15}$$

and hence the graphs of τ_k for different values of k do not intersect each other in S_-. The asymptotic relation (1.3) shows that $\tau_k(\alpha) < \tau_{k_1}(\alpha)$ if $k < k_1$ for sufficiently small α. Since the graphs have no intersections this relation is preserved throughout the intersection of the domains of τ_k and τ_{k_1}. The Selberg zeta-function is not the zero function, the eigenvalues $\tau_k(\varepsilon_1)$ form a discrete set, with only finitely elements under $\max I$. We take $k(\varepsilon_1)$ such that $\tau_k(\varepsilon_1) > \max I$ for all $k \geq k(\varepsilon_1)$.

Take $t \in I$ and $k \geq k(\varepsilon_1)$. The function F is equal to $-2\pi k$ along the graph of τ_k. We have $\lim_{\alpha \downarrow 0} F(\alpha, t) = -\infty$, and $F(\alpha, t)$ is larger than $-2\pi k$ under the graph of τ_k. In particular $F_k(\varepsilon_1, t) > -2\pi k$ since $\tau_k(\varepsilon_1) > \max I$. Differentiation gives

$$\frac{d}{d\alpha} F_k(\alpha, t) \;=\; \frac{2t}{\alpha} + \mathrm{O}(1)\,.$$

So the derivative of $\alpha \mapsto F_k(\alpha, t)$ is positive for $\alpha \in (0, \varepsilon_1]$ if we take ε_1 sufficiently small. So there is a unique $a_k(t) \in (0, \varepsilon_1]$ such that $\tau_k(a_k(t)) = t$. This function a_k inverts τ_k on I.



The estimate

$$2t\, \log \pi a_k(t) + 2\pi k \;=\; A_-(0, \frac{1}{2} + it) + \mathrm{O}(a_k(t))$$

is uniform for $t \in I$. It implies $\log a_k(t) = -\frac{\pi k}{t} + \mathrm{O}(1)$ uniform for $t \in I$ and $k \geq k(\varepsilon_1)$, and hence

$$\pi a_k(t) \;=\; \exp(-\pi k/t + A_-(0, 1/2 + it)/2t + \mathrm{O}(e^{-\pi k/t}))\,.$$

The function φ_- in the theorem is continuous on $[0, \infty)$. The argument $A_-(0, \frac{1}{2} + it)$ differs from it by $2\pi k_I$ for some integer depending on the interval I. (More precisely, depending on the component of I in $[0, \infty)$ minus the singularities of Y_-.) This gives (1.5). $\square$

Avoided crossing. For $(\alpha,\beta) = (\alpha, \frac{1}{2}+it)$ in the set S_- in Lemma 2.8 the gradient of F is

$$\nabla F(\alpha,t) \;=\; \begin{pmatrix} 2t/\alpha \\ 2\log \pi\alpha \end{pmatrix} - \nabla A_-(\alpha, 1/2+it)\,.$$

On the sets considered in Theorem 1.2 the gradient of $A_-(\alpha, \frac{1}{2}+it)$ is $\mathrm{O}(1)$. If $t = \tau_k(\alpha)$, then $\nabla F(\alpha,\beta)$ is proportional to $(\tau_k'(\alpha), -1)$, and hence

$$\tau_k'(\alpha) \;=\; \frac{2t\alpha^{-1} + \mathrm{O}(1)}{-2\,\log \pi\alpha + \mathrm{O}(1)} \;=\; \frac{t}{\alpha\,|\log \pi\alpha|}\left(1 + \mathrm{O}(1/\log \alpha|)\right).$$

This confirms that the graphs of the τ_k are steep for small α.

If we are near a singularity of C_- at $(0,\beta_0) \in \{0\} \times \left(\frac{1}{2} + i(0,\infty\right)$ this reasoning is no longer valid. In 2.4 we see that this can only occur if β_0 is an unperturbed eigenvalue. For $(\alpha,\beta) \in S_-$ we have $|C_-(\alpha,\beta)| = 1$. So if C_- has a zero or a pole at (α,β) near $(0,\beta_0)$ with α real, then $\mathrm{Re}\,\beta \neq \frac{1}{2}$.

It seems hard to analyze this precisely for a complicated singularity. Hence we work under simplifying assumptions.

Proposition 2.9. *Let* $\beta_0 = \frac{1}{2} + it_0$ *with* $t_0 > 0$. *We assume that on a neighborhood* Ω *of* $(0,\beta_0)$ *in* $\mathbb{C}^2$ *the matrix element* C_- *of the extended scattering matrix has the form*

$$C_-(\alpha,\beta) \;=\; \lambda(\alpha,\beta)\,\frac{\beta-\beta_0-n(\alpha)}{\beta-\beta_0-p(\alpha)}\,, \tag{2.16}$$

where n *and* p *are holomorphic functions on a neighborhood of* 0 *in* $\mathbb{C}$ *and* λ *is holomorphic on* Ω *without any zeros. We suppose furthermore that* $n(0) = p(0) = 0$, *and* $\mathrm{Re}\,p''(0) \neq 0$.

Then there are $\varepsilon_0 > 0$ *and* $k_0 \geq 1$ *such that for all* $k \geq k_0$ *there exists* $\alpha_k \in (0,\varepsilon_1]$ *such that* $\tau_k(\alpha_k) = t_0 + \mathrm{Im}\,p(\alpha_k)$, *and for these* α_k *we have*

$$\tau_k'(\alpha_k) \;=\; \mathrm{Im}\,p'(\alpha_k) - \frac{1}{2}t_0\,\mathrm{Re}\,p'(\alpha_k) + \mathrm{O}(\alpha_k^2)\,. \tag{2.17}$$

Remark 2.10. Suppose that $\beta_0 \in \frac{1}{2} + i(0,\infty)$ is an unperturbed eigenvalue, *i.e.*, $Z(0,\beta_0) = 0$. Then it might be associated to a singularity at $(0,\beta_0)$ of the extended scattering matrix, as in 2.4. This singularity might be visible as a singularity of the coefficient C_-. For that case, the assumptions in the proposition seem to describe the most general situation. If these assumptions are satisfied then there is the curve

$$K_{\beta_0} : \alpha \mapsto (\alpha, t_0 + \mathrm{Im}\,p(\alpha))$$

through $(0,t_0)$ such that the derivatives of the τ_k for all large k are relatively small at the points where the graph of τ_k crosses the curve K_{β_0}. (See also Remark 3.5 in §3.7.)

Proof of Proposition 2.9. The assumption that C_- has a singularity at β_0 implies that the functions p and n cannot be equal. From 2.5 and (2.5) it follows that $C_-(\alpha,\beta)$ is even in α. This evenness is inherited by the zero set and the set of singularities. Hence p and n are even functions. We also have $\overline{C_-(\bar{\alpha}, 1-\bar{\beta})} = C_-(\alpha,\beta)^{-1}$. This implies $n(\alpha) = -\overline{p(\bar{\alpha})}$. For real α we write $p_r(\alpha) = \mathrm{Re}\, p(\alpha)$ and $p_i(\alpha) = \mathrm{Im}\, p(\alpha)$. Hence we have for $\alpha \in (0, \varepsilon_1)$ and $t \approx t_0$:

$$Y_-(\alpha, \frac{1}{2} + it) \;=\; \lambda(\alpha, \frac{1}{2} + it)^{-1} \frac{\Gamma(it)\,(-p_r(\alpha) + i(t - t_0 - p_i(\alpha)))}{\Gamma(-it)\,(p_r(\alpha) + i(t - t_0 - p_i(\alpha)))}$$

So modulo $2\pi\mathbb{Z}$:

$$A_-(\alpha, \frac{1}{2} + it) \;\equiv\; 2\arg\big(p(\alpha) - i(t - t_0)\big) + \mathrm{O}(1)\,, \tag{2.18}$$

where the term indicated by $\mathrm{O}(1)$ has also bounded derivatives. So the gradient with respect to the variables α and t is

$$\nabla A_-(\alpha, \frac{1}{2} + it) \;=\; \mathrm{O}(1) + \mathrm{Im} \begin{pmatrix} \frac{2\,p'(\alpha)}{p(\alpha) - i(t-t_0)} \\ \frac{-2i}{p(\alpha) - i(t-t_0)} \end{pmatrix}. \tag{2.19}$$

The function $\alpha \mapsto F(\alpha, t_0 + p_i(\alpha))$, with F as in (2.15), tends to $-\infty$ as $\alpha \downarrow 0$, and has derivative

$$\begin{aligned} 2\,p_i'(\alpha)\,\log \pi\alpha + \frac{2\,(t_0 + p_i(\alpha))}{\alpha} - \mathrm{Im}\,\frac{2\,p'(\alpha)}{p_r(\alpha)} - \mathrm{Im}\,\frac{-2i}{p_r(\alpha)}\,p_i'(\alpha) + \mathrm{O}(1) \\ = \;\frac{2t_0 + 2p_i(\alpha)}{\alpha} + \frac{-2\,p_i'(\alpha) + 2p_i'(\alpha)}{p_r(\alpha)} + \mathrm{O}(1) \;=\; \frac{2t_0}{\alpha} + \mathrm{O}(1)\,, \end{aligned}$$

where we use that $p(\alpha) = \mathrm{O}(\alpha^2)$ and $p'(\alpha) = \mathrm{O}(\alpha)$ as $\alpha \downarrow 0$, since p is an even function vanishing at 0. So there is an interval $(0, \varepsilon_1]$ on which $\alpha \mapsto F(\alpha, t_0 + p_i(\alpha))$ is increasing. Hence for all sufficiently large integers k there are $\alpha_k \in (0, \varepsilon_1]$ such that $F(\alpha_k, t_0 + p_i(\alpha_k)) = -2\pi k$, and then $\tau_k(\alpha_k) = t_0 + p_i(\alpha_k)$.

We have $2\,\tau_k(\alpha_k)\,\log \pi\alpha_k \;=\; -2\pi k + \mathrm{O}(1)$, since the argument in (2.18) stays bounded in a neighborhood of $(0, \beta_0)$. So $\log \pi\alpha_k = \frac{-\pi k}{t_0} + \mathrm{O}(1)$ as $k \to \infty$, and hence $\alpha_k \downarrow 0$.

We have, again using $p(\alpha) = \mathrm{O}(\alpha^2)$ and $p'(\alpha) = \mathrm{O}(\alpha)$,

$$\begin{aligned} \nabla F(\alpha_k, \frac{1}{2} + i\tau_k(\alpha_k)) \;&=\; \begin{pmatrix} \frac{2\,\tau_k(\alpha_k)}{\alpha_k} - \mathrm{Im}\,\frac{2\,p'(\alpha_k)}{p_r(\alpha_k)} \\ 2\log \pi\alpha_k - \mathrm{Im}\,\frac{-2i}{p_r(\alpha_k)} \end{pmatrix} + \mathrm{O}(1) \\ &=\; \begin{pmatrix} \frac{2t_0}{\alpha_k} - \frac{2\,p_i'(\alpha_k)}{p_r(\alpha_k)} + \mathrm{O}(1) \\ \frac{2}{p_r(\alpha_k)} + \mathrm{O}(\log \alpha_k) \end{pmatrix}. \end{aligned}$$

Since the graph of τ_k is a level curve of F the gradient $\nabla F(\alpha_k, \frac{1}{2} + i\tau_k(\alpha_k))$ is orthogonal to $\begin{pmatrix} 1 \\ \tau_k'(\alpha_k) \end{pmatrix}$. We use $p(\alpha) = \frac{1}{2}\, p''(0)\,\alpha^2 + \mathrm{O}(\alpha^3)$ and $p'(\alpha) = p''(0)\,\alpha + \mathrm{O}(\alpha^2)$ as $\alpha \to 0$, and obtain:

$$\begin{aligned}
\tau_k'(\alpha_k) &= -\frac{2\,t_0/\alpha_k - 2\,p_i'(\alpha_k)/p_r'(\alpha_k) + \mathrm{O}(1)}{2/p_r(\alpha_k) + \mathrm{O}(\log \alpha_k)} \\
&= \frac{-2\,t_0\,\alpha_k + 4\,\alpha_k\,p_i''(0)/p_r''(0) + \mathrm{O}(\alpha_k^2)}{4/p_r''(0) + \mathrm{O}(\alpha_k^2 \log \alpha_k)} \\
&= \alpha_k(-\frac{1}{2}t_0\,p_r''(0) + p_i''(0)) + \mathrm{O}(\alpha_k^2) \;=\; p_i'(\alpha_k) - \frac{t_0}{2}p_r'(\alpha_k) + \mathrm{O}(\alpha_k^2)\,.
\end{aligned} \tag{2.20}$$

□

2.4 Curves of resonances originating at $(0, \frac{1}{2})$

To find resonances for $\alpha \in (0,1)$ we have to look for singularities of the scattering "matrix" $D_\alpha(\beta)$ in 2.6. If we work on a region where the extended scattering matrix has no singularities this means that we look for solutions of $X(\alpha,\beta)\,C_+(\alpha,\beta) = 1$ with the requirement that the resulting singularity of $D_{0,0}(\alpha,\beta)$ is not canceled by a zero of the numerator in (2.6).

Proposition 2.11. *Let* Ω *be a region in* $(0,1)\times\{\beta \in \mathbb{C} \;:\; \mathrm{Im}\,\beta > 0\}$ *that is invariant under* $(\alpha,\beta) \mapsto (\alpha, 1-\overline{\beta})$ *(reflection in the central line). Suppose that the matrix* $\mathbf{C}^+$ *in* (2.7) *is holomorphic on a neighborhood op* Ω *in* $\mathbb{C}^2$.

The denominator $M = 1 - X\,C_+$ *in the expression for* $D_{0,0}$ *in* (2.6) *vanishes at* $(\alpha, 1-\overline{\beta})$, *if and only if the numerator* $N = C_{0,0} - X\left(C_{0,0}\,C_+ - 2\,C_{0,\infty}^2\right)$ *vanishes at* (α,β).

Proof. The function $\Delta = C_{0,0}\,C_+ - 2\,C_{0,\infty}^2$ is the determinant of the matrix $\mathbf{C}^+$. It follows from 2.5 that $\overline{\mathbf{C}^+(\alpha, 1-\overline{\beta})} = (\mathbf{C}^+(\alpha,\beta)^t)^{-1}$ for $(\alpha,\beta) \in \Omega$. If Δ would have a zero at $(\alpha,\beta) \in \Omega$, this would contradict the holomorphy of $\mathbf{C}^+$ on Ω. Furthermore, $\overline{X(\alpha, 1-\overline{\beta})} = X(\alpha,\beta)^{-1}$.

We have

$$\overline{C_+(\alpha, 1-\overline{\beta})} \;=\; \text{coefficient at position } (2,2) \text{ of } (\mathbf{C}^+(\alpha,\beta)^t)^{-1} \;=\; \frac{C_{0,0}(\alpha,\beta)}{\Delta(\alpha,\beta)}$$

Since $\Delta(\alpha,\beta) \in \mathbb{C}^*$ we have equivalence of the following assertions:

$$\begin{aligned}
X(\alpha, 1-\overline{\beta})\,C_+(\alpha, 1-\overline{\beta}) &= 1\,, & \overline{C_+(\alpha, 1-\overline{\beta})} &= \overline{X(\alpha, 1-\overline{\beta})}^{-1}\,, \\
C_{0,0}(\alpha,\beta)/\Delta(\alpha,\beta) &= X(\alpha,\beta)\,, & C_{0,0}(\alpha,\beta) &= X(\alpha,\beta)\,\Delta(\alpha,\beta)\,.
\end{aligned}$$

$\square$

Remark 2.12. So zeros and singularities of $D_{0,0}(\alpha,\beta)$ are interchanged by the reflection in the central line. The meromorphic function $D_{0,0}$ is not the zero function, since its restriction to the complex lines $\{\alpha\}\times\mathbb{C}$ for $\alpha\in(0,1)$ are scattering "matrices", which are non-zero. So its sets of zeros and poles intersect each other only in a discrete set in $U_+\times\mathbb{C}$.

We now consider the equation $1 = X\,C_+$, in a region where $\mathbf{C}^+$ has no singularities. Analogously to (2.9), we put

$$Y_+(\alpha,\beta) \;=\; \frac{\Gamma(\beta-\frac{1}{2})}{\Gamma(\frac{1}{2}-\beta)}\,\frac{1}{C_+(\alpha,\beta)}\,. \tag{2.21}$$

This is a meromorphic function on $U\times\mathbb{C}$, and the equation $X\,C_+ = 1$ on $U_+\times\mathbb{C}$ is equivalent to

$$(\pi\alpha)^{2\beta-1} \;=\; Y_+(\alpha,\beta)\,. \tag{2.22}$$

A complication is that now we cannot restrict our consideration to a subset of $(0,1)\times\left(\frac{1}{2}+i(0,\infty)\right)$, but have to allow β to vary over a neighborhood of the central line. The presence of singularities of C_+ makes it impossible to choose a well defined argument globally.

Lemma 2.13. *Let $I\subset[0,\infty)$ be a bounded closed interval such that C_+ has no singularities at points $(0,\frac{1}{2}+it)$ with $t\in I$. There are $\varepsilon_1,\varepsilon_2>0$ such that the solution set of* (2.22) *in*

$$\Omega_I(\varepsilon_1,\varepsilon_2) \;=\; (0,\varepsilon_1]\times\Bigl([\frac{1}{2}-\varepsilon_2,\frac{1}{2}+\varepsilon_2]\times iI\Bigr) \tag{2.23}$$

consists of sets V_k parametrized by $k\in\mathbb{Z}$.

There exists $k_1\in\mathbb{Z}$ such that V_k is for all $k\geq k_1$ a real-analytic curve

$$t\mapsto(\alpha_k(t),\;\sigma_k(t)+it)\qquad(t\in I)\,.$$

Proof. Since C_+ is holomorphic at all points of the compact set $\{0\}\times\left(\frac{1}{2}+iI\right)$, it has the value given by the restriction to $\alpha=0$, which value we know explicitly from (2.1) and (2.5):

$$C_+(0,\frac{1}{2}+it) \;=\; \frac{\pi^{-2it}\,\Gamma(it)\,\zeta(2it)}{(2^{1+2it}-1)\,\Gamma(-it)\,\zeta(-2it)}\,.$$

So C_+ has also no zeros on $\{0\}\times(\frac{1}{2}+iI)$. We can choose $\varepsilon_1,\varepsilon_2>0$ such that C_+ also has no singularities or zeros with $\alpha\in(0,\varepsilon_1]$, $\left|\mathrm{Re}\,\beta-\frac{1}{2}\right|\leq\varepsilon_2$, and $\mathrm{Im}\,\beta\in I$.

We take real-analytic functions on $\overline{\Omega_I(\varepsilon_1, \varepsilon_2)}$

$$
\begin{aligned}
M_+(\alpha,\beta) &= \log\left|Y_+(\alpha,\beta)\right|, \\
A_+(\alpha,\beta) &= \arg Y_+(\alpha,\beta).
\end{aligned}
\tag{2.24}
$$

There is freedom in the choice of the argument. In this lemma we do not choose a normalization.

The solution set of (2.22) in $\Omega_I(\varepsilon_1, \varepsilon_2)$ is the disjoint union of components V_k given by

$$
\begin{aligned}
2t \log \pi\alpha &= A_+(\alpha, \sigma + it) - 2\pi k, \\
(2\sigma - 1) \log \pi\alpha &= M_+(\alpha, \sigma + it).
\end{aligned}
\tag{2.25}
$$

Here and in the sequel we write $\sigma = \mathrm{Re}\,\beta$ and $t = \mathrm{Im}\,\beta$. Changing the choice of A_+ causes a shift in the parameter k.

We want to use the fixed-point theorem to show that for each $t \in I$, $t > 0$, and each $k \geq k_1$ there is exactly one solution of (2.25). To do this, we write $\alpha(x) = e^{-1/x}/\pi$ and $\beta(y, t) = \frac{1}{2} - y + it$. Then $\alpha \in (0, \varepsilon_1]$ corresponds to $x \in (0, x_1]$ with $x_1 = -1/\log \pi\varepsilon_1$, and $|\mathrm{Re}\,\beta - \frac{1}{2}| \leq \varepsilon_2$ to $|y| \leq \varepsilon_2$. We take

$$
F_{t,k}(x, y) = \Big(\frac{2t}{2\pi k - A_+(\alpha(x), \beta(y, t))}, \frac{t\, M_+(\alpha(x), \beta(y, t))}{2\pi k - A_+(\alpha(x), \beta(y, t))}\Big).
\tag{2.26}
$$

By taking k_1 sufficiently large, we can make the denominators in (2.26) as large as we want, in particular non-zero. So $F_{t,k}$ is real-analytic on $(0, x_1] \times [-\varepsilon_2, \varepsilon_2]$. By defining $\alpha(0) = 0$ we extend $F_{t,k}$ to a C^∞-function on $[0, x_1] \times [-\varepsilon_2, \varepsilon_2]$.

Since C_+ has no zeros or poles in $\Omega_I(\varepsilon_1, \varepsilon_2)$ we have $M_+ = \mathrm{O}(1)$ and $A_+ = \mathrm{O}(1)$. So for all sufficiently large k we have

$$
F_{t,k}\Big([0, x_1] \times [-\varepsilon_1, \varepsilon_1]\Big) \subset (0, x_1) \times (-\varepsilon_2, \varepsilon_2).
\tag{2.27}
$$

To show that $F_{t,k}$ is contracting it suffices to bound the partial derivatives of the two components. For the first component we have $(2\pi k - A_+)^2$ in the denominator, which can be made large. In the numerator we have the derivatives

$$
\partial_x A_+ = \frac{d\alpha}{dx}\, \partial_\alpha A_+ \ll e^{-1/x}\, x^{-2}\, \alpha \ll 1, \qquad \partial_y A_+ = -\partial_\sigma A_+ = \mathrm{O}(1)
$$

The factor α is due to the fact that C_+, and hence A_+ is even in α. The factor t is bounded, since $t \in I$. For the other component we proceed similarly.

Controlling k, we can make all partial derivatives small. So $F_{t,k}$ is contracting for all $k \geq k_1$ with a suitable k_1. So for a given t there is exactly one point $(\alpha,\beta) \in \Omega_I(\varepsilon_,\varepsilon_2)$ satisfying (2.25). The fixed point is in the region where $F_{t,k}$ is real-analytic, jointly in its variables and in the parameter t. Hence the fixed point is a real-analytic function of t by the analytic implicit function theorem. (See, *e.g.* [KrPa, Theorem 6.1.2].) $\square$

In this lemma, we do not get information concerning the sets V_k with k below the bound k_1. If $0 \in I$ we can normalize the argument A_+ like we did in the previous subsection.

Lemma 2.14. *For sufficiently small* $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ *the solution set of* (2.22) *in the set*

$$\Omega(\varepsilon_1, \varepsilon_2, \varepsilon_3) \;=\; (0, \varepsilon_1] \times \Big([\frac{1}{2} - \varepsilon_2, \frac{1}{2} + \varepsilon_2] \times i(0, \varepsilon_3] \Big)$$

is equal to the union of real analytic curves

$$t \mapsto (\alpha_k(t), \sigma_k(t) + it) \qquad (t \in [0, \varepsilon_3])\,,$$

with $k \geq 1$.

Proof. If ε_1, ε_2 and ε_3 are sufficiently small, then C_+ has no singularities or zeros in the closure of $\Omega(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ in $\mathbb{R} \times \mathbb{C}$. So $\log Y_+$ can be defined holomorphically on a neighborhood of $(0, \frac{1}{2})$ in $\mathbb{C}^2$ that contains $\Omega(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. We choose the branch that has the following expansion at $(0, \frac{1}{2})$:

$$-2 \log \pi \, (\beta - \frac{1}{2}) - 4 \, (\log 2)^2 \, (\beta - \frac{1}{2})^2 + \mathrm{O}\Big((\beta - \frac{1}{2})^3 \Big) + \mathrm{O}(\alpha^2)\,. \tag{2.28}$$

For the behavior along $\{0\} \times \mathbb{C}$ we use (2.1). We also use that C_+ is even in α. (See 2.5 and (2.7).) So in this lemma we can work with

$$\begin{aligned} M_+(\alpha, \sigma + it) \;&=\; -2 \, \log \pi \, (\sigma - \frac{1}{2}) - 4 \, (\log 2)^2 \, (\sigma - \frac{1}{2})^2 + 4 \, (\log 2)^2 \, t^2 \\ &\qquad + \mathrm{O}\Big((\beta - \frac{1}{2})^3 \Big) + \mathrm{O}(\alpha^2)\,, \qquad (2.29) \\ A_+(\alpha, \sigma + it) \;&=\; -2t \, \log \pi - 8 \, (\log 2)^2 t \, (\sigma - \frac{1}{2}) + \mathrm{O}\Big((\beta - \frac{1}{2})^3 \Big) + \mathrm{O}(\alpha^2)\,. \end{aligned}$$

Now the parameter k in the previous lemma can be anchored to this choice of the argument.

For $t \in (0, \varepsilon_3]$ and $k \geq 1$ we define $F_{t,k}$ as in (2.26), and revisit the estimates in the proof of the previous lemma. We cannot use k to make the denominator large. By adapting the ε's we can make M_+ and A_+ as small as we want on $\Omega(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. (See the expansions in (2.29).) In particular, we arrange

$$|A_+| \;\leq\; 2\pi - 4 \quad \text{ and } \quad |M_+| \;\leq\; 1\,.$$

Then the denominator satisfies $D := 2\pi k - A_+ \geq 2\pi - A_+ > 2$, hence $2t/D < t \leq \varepsilon_3$, and $tM_+/D \leq \varepsilon_3/(2\pi - 4)$. Arranging $\varepsilon_3 < x_1 = -1/\log \pi \varepsilon_1$ and $\varepsilon_3 < (2\pi - 4)\varepsilon_2$, we get (2.27).

To get all partial derivatives of $F_{t,k}$ small, we have to work with the numerators of the derivatives, since we have lost control over the denominators, except for the lower bound 2. In the numerators we meet the following factors:

$$\begin{array}{lll} \frac{\partial A_+}{\partial x}\,, & t\,\frac{\partial M_+}{\partial x}\,, & t\,M_+\,\frac{\partial A_+}{\partial x}\,, \\ \frac{\partial A_+}{\partial y}\,, & t\,\frac{\partial M_+}{\partial y}\,, & t\,M_+\,\frac{\partial A_+}{\partial y}\,. \end{array}$$

We have $t = \mathrm{O}(\varepsilon_3)$ and $M_+ = \mathrm{O}(1)$, so we can concentrate on the derivatives of A_+ and M_+ with respect to x and y. Both derivatives $\frac{\partial A_+}{\partial \alpha}$ and $\frac{\partial M_+}{\partial \alpha}$ are $\mathrm{O}(\alpha)$ by (2.29), which is controlled by ε_1. Since $\frac{d\alpha}{dx} \ll x^{-2}e^{-1/x} = \mathrm{O}(1)$, all contributions in the first line can be made small by decreasing ε_1 and ε_2.

We have $\frac{\partial A_+}{\partial y} = -\frac{\partial A_+}{\partial \sigma} = \mathrm{O}(t) = \mathrm{O}(\varepsilon_3)$. Further, $\frac{\partial M_+}{\partial y} = -\frac{\partial M_+}{\partial \sigma} = \mathrm{O}(1)$. This derivative occurs only multiplied with $t = \mathrm{O}(\varepsilon_3)$. So adapting ε_1 and ε_2, and then ε_3, taking also into account the requirements $\varepsilon_3 < -1/\log \pi\varepsilon_1$ and $\varepsilon_3 < (2\pi - 4)\varepsilon_2$, we can arrange that all partial derivatives are very small on $\Omega(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

So $F_{t,k}$ is contracting. Its fixed point $(\alpha_k(t), \sigma_k(t))$ gives the sole point $(\alpha, \beta) \in V_k$ with $\mathrm{Im}\,\beta = t$. It depends on t in a real-analytic way.

Now let $k \le 0$. Suppose that there is a sequence $(\alpha_n, \beta_n) = (\alpha_n, \sigma_n + it_n) \in V_k$ that tends to $(0, \frac{1}{2})$. The expansions in (2.29) imply that $A_+(\alpha_n, \beta_n) = o(1)$ and $M_+(\alpha_n, \beta_n) = o(1)$. Then (2.25) implies that $2\,t_n \log \pi\alpha_n$ tends to $-2\pi k$. If we ensure that $\varepsilon_1 < \frac{1}{\pi}$, we have $\log \pi\alpha_n \le 0$, which shows that $k \le -1$ is impossible. Let $k = 0$. We have by (2.22) and (2.29)

$$\log \pi\alpha_n \;=\; -\log \pi + \mathrm{O}(\sigma_n - \frac{1}{2})\,,$$

in contradiction to $\log \alpha_n \to -\infty$. □

Proof of Theorem 1.3. We have to prove the invertibility of the α_k, the asymptotic behavior, possibly further decreasing the ε's. Then the inequality $\sigma_k < \frac{1}{2}$ follows from (1.8) (perhaps after adapting the ε's).

We consider one of the curves in Lemma 2.14. In the next computations we omit the index k. Differentiation of the relation (2.25) with respect to t leads to the system

$$\begin{pmatrix} \frac{2t}{\alpha} - \frac{\partial A_+}{\partial \alpha} & -\frac{\partial A_+}{\partial \sigma} \\ \frac{2\sigma - 1}{\alpha} - \frac{\partial M_+}{\partial \alpha} & 2\log \pi\alpha - \frac{\partial M_+}{\partial \sigma} \end{pmatrix} \begin{pmatrix} \dot\alpha \\ \dot\sigma \end{pmatrix} = \begin{pmatrix} -2\,\log \alpha + \frac{\partial A_+}{\partial t} \\ \frac{\partial M_+}{\partial t} \end{pmatrix}.$$

Here we consider A_+ and M_+ as functions of the three variables α, σ and t. By a dot we indicate differentiating with respect to t. The determinant of this system is

$$\Big(\frac{2t}{\alpha} + \mathrm{O}(\alpha)\Big)\Big(2\log \pi\alpha + \mathrm{O}(1)\Big) - \mathrm{O}(\alpha^{-1})\,\mathrm{O}(t) \;=\; \frac{4t\,\log \pi\alpha}{\alpha}\Big(1 + \mathrm{O}((\log \alpha)^{-1})\Big).$$

Adapting the ε's we arrange that this quantity is negative. Then we have

$$\begin{aligned} \dot{\alpha} &= \frac{\alpha}{4t\,\log \pi\alpha}\Big(1 + \mathrm{O}((\log\alpha)^{-1})\Big) \\ &\quad \cdot \Big(\Big(2\log\alpha + \mathrm{O}(1)\Big)\Big(-2\,\log\alpha + \mathrm{O}(1)\Big) + \mathrm{O}(t)\,\mathrm{O}(1)\Big) \\ &= \frac{\alpha\,|\log\pi\alpha|}{t}\,\Big(1 + \mathrm{O}((\log\alpha)^{-1})\Big)\,. \end{aligned}$$

So we can arrange that $\alpha_k'(t) = \dot{\alpha} > 0$. This shows that the real-analytic function $t \mapsto \alpha(t)$ on $[0, \varepsilon_3]$ has a real analytic inverse t_k on some interval $(\vartheta_k, \zeta_k] \subset (0, \varepsilon_1]$.

In the proof of Lemma 2.14 we have already arranged that $|A_+| < 2\pi - 4$. Hence $2t\,\log\pi\alpha < -4$ and $\log\pi\alpha < -\frac{2}{t}$. So $\alpha \downarrow 0$ as $t \downarrow 0$, which shows that $\vartheta_k = 0$ for all $k \geq 1$.

To derive the asymptotic expansions, we consider σ and t as functions of α along the curve parametrized by $k \geq 1$. We omit again the subscript k in the computations. We have along this curve

$$\begin{aligned} 2\,(\beta - \frac{1}{2})\,\log\pi\alpha &= -2\pi i k - 2\,\log\pi\,(\beta - \frac{1}{2}) - 4\,(\log 2)^2\,(\beta - \frac{1}{2})^2 \\ &\quad + \mathrm{O}\Big((\beta - \frac{1}{2})^3\Big) + \mathrm{O}(\alpha^2)\,. \end{aligned} \tag{2.30}$$

This implies that $\left(\beta - \frac{1}{2}\right)\log\pi\alpha = \mathrm{O}(1)$, hence

$$\beta - \frac{1}{2} \;=\; \mathrm{O}((\log\pi\alpha)^{-1}) \;=\; \mathrm{O}((\log\pi^2\alpha)^{-1})\,.$$

Next we get

$$\left(\beta - \frac{1}{2}\right)\log\pi^2\alpha = -\pi i k - 2\,(\log 2)^2\left(\beta - \frac{1}{2}\right)^2 + \mathrm{O}\left((\log\pi^2\alpha)^{-3}\right)\,.$$

We write $L = \log\pi^2\alpha$, which is a large negative quantity. We obtain

$$\begin{aligned} \beta - \frac{1}{2} &= \frac{-\pi i k}{L} - \frac{2\,(\log 2)^2\,(\beta - \frac{1}{2})^2}{L} + \mathrm{O}(L^{-4}) \\ &= -\frac{\pi i k}{L} - \frac{2\,(\log 2)^2}{L}\Big(-\frac{\pi^2 k^2}{L^2} + \mathrm{O}(L^{-3})\Big) \\ &= -\frac{\pi i k}{L} + \frac{2(\pi k\log 2)^2}{L^3} + \mathrm{O}(\ell^{-4})\,. \end{aligned}$$

Taking real and imaginary parts gives the asymptotic relations (1.7) and (1.8). $\square$

Proof of Theorem 1.4. Lemma 2.13 gives curves of this type in a region $\Omega_I(\varepsilon_1, \varepsilon_2)$ near $\{0\} \times (\frac{1}{2} + iI)$. In this region C_+ has no singularities or zeros, so $\log Y_+$ and its argument A_+ can be chosen in a continuous way. There seems no way to connect the branches of $\log Y_+$ more globally, so we may as well normalize A_+ by $A_+(0,\beta) = A(\beta)$ for $\beta \in \frac{1}{2} + iI$, where A and M are as in the theorem. We have on $\Omega_I(\varepsilon_1, \varepsilon_2)$

$$\begin{aligned} M_+(\alpha, \sigma + it) &= M(\frac{1}{2} + it) + \mathrm{O}(\sigma - \frac{1}{2}) + \mathrm{O}(\alpha^2)\,, \\ A_+(\alpha, \sigma + it) &= A(\frac{1}{2} + it) + \mathrm{O}(\sigma - \frac{1}{2}) + \mathrm{O}(\alpha^2)\,. \end{aligned} \tag{2.31}$$

The relations (2.25) hold for points $(\alpha, \sigma + it)$ on curves with number $k \geq k_1$(we omit the index k), and hence

$$\log \pi\alpha = \mathrm{O}(t^{-1})\,, \qquad \frac{1}{2} - \sigma = \mathrm{O}(|\log \pi\alpha|^{-1})\,.$$

Working more precisely, we get

$$2t\, \log \pi\alpha = -2\pi k + A(\frac{1}{2} + it) + \mathrm{O}(|\log \pi\alpha|^{-1})\,,$$

and hence $\log \pi\alpha \leq C\, k$ for some positive C. This gives

$$\log \pi\alpha = -\frac{\pi k}{t} + \frac{A(\frac{1}{2} + it)}{2t} + \mathrm{O}(k^{-1})\,,$$

which is (1.11). We also have

$$2(\sigma - \frac{1}{2})\Big(-\frac{\pi k}{t} + \mathrm{O}(1)\Big) = M(\frac{1}{2} + it) + \mathrm{O}(k^{-1})\,,$$

hence

$$\begin{aligned} (\frac{1}{2} - \sigma) &= \frac{M(\frac{1}{2} + it) + \mathrm{O}(k^{-1})}{2\pi k t^{-1} + \mathrm{O}(1)} = \frac{t}{2\pi k}\Big(M(\frac{1}{2} + it) + \mathrm{O}(k^{-1})\Big)\Big(1 + \mathrm{O}(t/k)\Big) \\ &= \frac{t\, M(\frac{1}{2} + it)}{2\pi k} + \mathrm{O}(k^{-2})\,, \end{aligned}$$

which is (1.12). □

Proof of Theorem 1.5. First we consider several statements equivalent to the statement that a curve as in Theorem 1.4 touches the central line at a point $t \in I$. Of course, this is equivalent to $\sigma_k(t) = \frac{1}{2}$. In (2.25) we see that it implies that $M_+(\alpha_k(t), \frac{1}{2} + it) = 0$. And since each curve with $k \geq k_1$ has t as a parameter, touching the central line in $\frac{1}{2} + it$ is equivalent to $M_+(\alpha_k(t), \frac{1}{2} + it) = 0$. By (2.24) and (2.21) this is equivalent to $\left|C_+(\alpha_k(t), \frac{1}{2}+it)\right| = 1$. At points with $\alpha \in (-1, 1)$ and $\mathrm{Re}\,\beta = \frac{1}{2}$ the matrix $\mathbf{C}^+(\alpha,\beta)$ in (2.7) is unitary. So the statement is also equivalent to $C_{0,\infty}(\alpha_k(t), \frac{1}{2} + it) = 0$.

In (2.1) we see that $\beta \mapsto C_{0,\infty}(0,\beta)$ has a simple zero at $\beta = \beta_\ell := \frac{1}{2} + it_\ell$. Since β_ℓ is not a zero of $Z(0,\cdot)$, we know from 2.4 that all matrix elements of $\mathbf{C}$ are holomorphic at $(0,\beta_\ell)$, in particular $C_{0,\infty}$ is holomorphic at $(0,\beta_\ell)$. So we have $C_{0,\infty}(\alpha,\beta) = \lambda(\alpha,\beta)\,P(\beta-\beta_\ell,\alpha)$ on a neighborhood of $(0,\beta_\ell)$, where P is a polynomial in $\beta-\beta_\ell$ with coefficients that are holomorphic in α on a neighborhood of 0 in $\mathbb{C}$, vanishing at 0, and where λ is holomorphic without zeros. (This follows from the Weierstrass preparation theorem. See, *e.g.*, [Ho, Corollary 6.1.2].) The restriction of $C_{0,\infty}$ to the complex line $\{0\}\times\mathbb{C}$ has a zero of order 1 at β_ℓ, hence $P(X,\alpha) = X - i\eta(\alpha)$, with η holomorphic on a neighborhood of 0 in $\mathbb{C}$ and $\eta(0) = 0$. Since $C_{0,\infty}(\alpha,\beta)$ is even in α, its zero set is also invariant under $\alpha \mapsto -\alpha$. Hence η is an even function. From 2.5 and (2.7) it follows that $(\alpha,\beta) \mapsto \overline{C_{0,\infty}(\bar\alpha, 1-\bar\beta)}$ has the same zero set as $C_{0,\infty}$. This implies $\overline{\eta(\bar\alpha)} = \eta(\alpha)$. So $\eta(\alpha) \in \mathbb{R}$ for real α. The power series expansion of η at 0 starts with $\eta(\alpha) = \eta_2\alpha^2 + \cdots$, with $\eta_2 \in \mathbb{R}$.

The asymptotic behavior of α_k in (1.11) shows that the curve $t \mapsto (\alpha_k(t), t)$ and the curve $\alpha \mapsto (\alpha, t_\ell + \eta(\alpha))$ intersect each other for all sufficiently large k. We call the intersection point$(a_k, t_\ell + \delta_k)$. So we have

$$a_k \;=\; \alpha_k(t_\ell + \delta_k)\,, \quad \delta_k \;=\; \eta(a_k)\,.$$

Furthermore, $C_{0,\infty}(a_k,\beta_\ell + i\delta_k) = 0$, hence the curve with number k touches the central line at $\beta_\ell + i\delta_k$.

Now we carry out estimates as $k \to \infty$. Theorem 1.4 gives $\alpha_k(t) \ll \exp(\mathrm{O}(1) - \pi k/t)$ uniformly on I. In particular, $a_k = \mathrm{O}(k^{-n})$ for each $n \geq 0$. In particular $a_k \downarrow 0$. Then $\delta_k = \eta(a_k)$ implies that $\delta_k = \mathrm{O}(k^{-2n})$ and also tends to zero.

We have

$$\begin{aligned}
\frac{A(\frac{1}{2} + it_\ell + i\delta_k) - 2\pi k}{2(t_\ell + \delta_k)} &= \frac{1}{2t_\ell}\Big(1 + \mathrm{O}(\delta_k/t_\ell)\Big)\Big(A(\frac{1}{2} + it_\ell) - 2\pi k + \mathrm{O}(\delta_k)\Big)\\
&= \Big(\frac{A(\frac{1}{2} + it_\ell) - 2\pi k}{2t_\ell} + \mathrm{O}(\delta_k)\Big)\Big(1 + \mathrm{O}(\delta_k)\Big)\\
&= \frac{A(\frac{1}{2} + it_\ell) - 2\pi k}{2t_\ell} + \mathrm{O}(k^{1-2n})\,.
\end{aligned}$$

Hence

$$
\begin{aligned}
a_k \;=\; \alpha_k(t_\ell+\delta_k) \;&=\; \frac{1}{\pi}\,\exp\Bigl(\frac{A(\frac12+it_\ell)-2\pi k}{2t_\ell}+\mathrm{O}(k^{1-2n})\Bigr)\Bigl(1+\mathrm{O}(k^{-1})\Bigr)\\
&=\; \frac{1}{\pi}\,e^{A(\frac12+it_\ell)/2t_\ell-\pi k/t_\ell}\Bigl(1+\mathrm{O}(k^{1-2n})\Bigr)\Bigl(1+\mathrm{O}(k^{-1})\Bigr)\\
&=\; \frac{1}{\pi}\,e^{A(\frac12+it_\ell)/2t_\ell-\pi k/t_\ell}\Bigl(1+\mathrm{O}(k^{-1})\Bigr).
\end{aligned}
$$

We obtain

$$
\delta_k \;=\; \eta(a_k) \;=\; \eta_2\,a_k^2+\mathrm{O}(a_k^4) \;=\; \frac{\eta_2}{\pi^2}\,e^{A\left(\frac12+it_\ell\right)/t_\ell-2\pi k/t_\ell}\Bigl(1+\mathrm{O}(k^{-1})+\mathrm{O}(k^{-4n})\Bigr).
$$

This gives (1.13).

We know that $\sigma_k(t)\le\frac12$ for all $t\in I$, by 2.6. So the points where the real-analytic curves touch the central line are tangent points. □

2.5 Curves of resonances originating higher up on the central line

We turn to a tentative explanation of curves of resonances like those in Figure 12 and 13. We cannot prove that these loops necessarily exist, and have to be content with a result that depends on a number of assumptions

Assumptions 2.15. (1) Let $\beta_0=\frac12+it_0$ with $t_0>0$. *We assume that the conjugated scattering matrix* $\mathbf{C}^+$ *in* (2.7) *has a singularity at* $(0,\beta_0)$. So β_0 is a zero of the unperturbed Selberg zeta-function $Z(0,\cdot)$ on the central line. (Not all such unperturbed eigenvalues need to be related to a singularity of $\mathbf{C}^+$.)
(2) The singularity of $\mathbf{C}^+$ at $(0,\beta_0)$ is as simple as possible, with a common denominator for all matrix elements. To make this precise, *we assume that there are holomorphic functions* p, $r_{0,0}$, $r_{0,\infty}$ *and* r_+ *on a neighborhood of* 0 *in* $\mathbb{C}$ *that all vanish at* 0*, such that on a neighborhood* Ω *of* $(0,\beta_0)$ *in* $\mathbb{C}^2$

$$
\begin{aligned}
C_{0,0}(\alpha,\beta) \;&=\; \gamma_{0,0}(\alpha,\beta)\,\frac{\beta-\beta_0-r_{0,0}(\alpha)}{\beta-\beta_0-p(\alpha)}\,,\\
C_{0,\infty}(\alpha,\beta) \;&=\; \gamma_{0,\infty}(\alpha,\beta)\,\frac{\beta-\beta_0-r_{0,\infty}(\alpha)}{\beta-\beta_0-p(\alpha)}\,,\\
C_{+}(\alpha,\beta) \;&=\; \gamma_{+}(\alpha,\beta)\,\frac{\beta-\beta_0-r_{+}(\alpha)}{\beta-\beta_0-p(\alpha)}\,,
\end{aligned}
\tag{2.32}
$$

where the γ*'s are holomorphic on* Ω *without zeros in* Ω.
(3) Since $\mathbf{C}^+(\alpha,\beta)$ is even in α, the sets of zeros of the matrix elements and the set of singularities are invariant under $\alpha\mapsto-\alpha$. So p and the r's are even functions.

We assume that already the first terms in their power series expansions are non-zero and all different: $p''(0)$, $r''_{0,0}(0)$, $r''_{0,\infty}(0)$ *and* $r''_{+}(0)$ *are four different non-zero complex numbers.*

(4) The restriction of $\mathbf{C}^{+}$ to the complex line $\{0\} \times \mathbb{C}$ is equal to

$$\begin{pmatrix} \gamma_{0,0}(0,\beta) & \sqrt{2}\,\gamma_{0,\infty}(0,\beta) \\ \sqrt{2}\,\gamma_{0,\infty}(0,\beta) & \gamma_{+}(0,\beta) \end{pmatrix}.$$

(See also (2.8).) *We assume that for $\beta = \beta_0$ all elements of this matrix are non-zero.*

Most of these assumptions mean that "nothing special happens", and hence seem not too unreasonable. Only the assumption that all matrix elements have the same set of singularities might be considered to be really restrictive.

Lemma 2.16. *Under the assumptions 2.15 the neighborhood Ω of $(0,\beta_0)$ can be chosen such that*

$$\Omega \cap \Big(\mathbb{R} \times (\frac{1}{2} + i\mathbb{R})\Big) \;=\; \{(0,\beta_0)\}\,.$$

Proof. Let $\alpha_1 \in \mathbb{R}$ and $\mathrm{Re}\,\beta_1 = \frac{1}{2}$, $(\alpha_1,\beta_1) \in \Omega$. The restriction $\beta \mapsto \mathbf{C}^{+}(\alpha_1,\beta)$ on $\frac{1}{2} + i\mathbb{R}$ is a family of unitary matrices, hence any singularity is of indeterminate type. Such singularities occur discretely, so taking Ω sufficiently small the sole possibility is $(\alpha_1,\beta_1) = (0,\beta_0)$. □

Lemma 2.17. *Under the assumptions 2.15 there is a neighborhood of* 0 *in* $\mathbb{C}$ *such that for all α in that neighborhood:*

$$r_{+}(\alpha) \;=\; -\overline{r_{0,0}(\bar{\alpha})}\,, \qquad r_{0,\infty}(\alpha) \;=\; -\overline{r_{0,\infty}(\bar{\alpha})}\,. \tag{2.33}$$

Proof. We have $\det \mathbf{C}^{+} = C_{0,0}\, C_{+} - 2\, C_{0,\infty}^{2}$. Hence

$$\begin{aligned} &(\beta - \beta_0 - p(\alpha))^2 \;\det \mathbf{C}^{+}(\alpha,\beta) \\ &\quad = \;\gamma_{0,0}(\alpha,\beta)\,\gamma_{+}(\alpha,\beta)\,(\beta-\beta_0 - r_{0,0}(\alpha))\,(\beta-\beta_0 - r_{+}(\alpha)) \\ &\qquad - 2\,\gamma_{0,\infty}(\alpha,\beta)^2\,(\beta-\beta_0 - r_{0,\infty}(\alpha))^2 \end{aligned}$$

is holomorphic on Ω, and its restriction to the complex line $\alpha = 0$ has a zero of at most order 2 at $\beta = \beta_0$. We use the Weierstrass preparation theorem to write it in the form $\delta(\alpha,\beta)\, Q(\beta-\beta_0,\alpha)$, with δ holomorphic without zeros on Ω and Q a polynomial in its first variable of degree at most 2 with coefficients that are holomorphic functions of α vanishing at $\alpha = 0$, and with highest coefficient equal to 1. So we have

$$\det \mathbf{C}^{+}(\alpha,\beta) \;=\; \delta(\alpha,\beta)\,\frac{Q(\beta-\beta_0,\alpha)}{P(\beta-\beta_0,\alpha)^2}\,,$$

where $P(T,\alpha) = T - p(\alpha)$. We define an involution $K \mapsto K^*$ in the space of polynomials in T with holomorphic coefficients in α by $K^*(T,\alpha) = (-1)^{\text{degree } K}\,\overline{K(-\overline{T},\overline{\alpha})}$. So $P^*(T,\alpha) = T + \overline{p(\overline{\alpha})}$. Lemma 2.16 implies that $P^* \neq P$.

The relation $\overline{\det \mathbf{C}^+(\overline{\alpha}, 1-\overline{\beta})} = \det \mathbf{C}^+(\alpha,\beta)^{-1}$, from 2.5 and (2.7), implies

$$\overline{\delta(\overline{\alpha},1-\overline{\beta})}\,\frac{(-1)^{\text{degree } Q}\,Q^*(\beta-\beta_0,\alpha)}{P^*(\beta-\beta_0,\alpha)^2} \;=\; \delta(\alpha,\beta)^{-1}\,\frac{P(\beta-\beta-0,\alpha)^2}{Q(\beta-\beta_0,\alpha)}\,,$$

and hence

$$\begin{aligned}(-1)^{\text{degree } Q}\,\overline{\delta(\overline{\alpha},1-\overline{\beta})}\,\delta(\alpha,\beta)\,Q^*(\beta-\beta_0,\alpha)\,Q(\beta-\beta_0,\alpha)&\\ = P^*(\beta-\beta_0,\alpha)^2\,P(\beta-\beta_0,\alpha)^2\,.&\end{aligned}$$

On the right is a fourth degree polynomial in $\beta-\beta_0$ with highest coefficient 1. This means that on the left we have also a polynomial of degree four, and that the highest coefficient is also equal to 1. So the product of the sign and the two δ's is equal to 1. (We note that not only $\delta(\alpha,\beta)$ but also $\overline{\delta(\overline{\alpha},1-\overline{\beta})}$ is non-zero for (α,β) sufficiently close to $(0,\beta_0)$.) Hence $Q^*\,Q = P^2\,(P^*)^2$, and since Q^* and Q have the same degree, this degree is equal to 2.

The polynomials P and P^* are irreducible, hence Q is equal to one of P^2, $P\,P^*$, and $(P^*)^2$. Hence

$$\det \mathbf{C}^+(\alpha,\beta) \;=\; \delta(\alpha,\beta)\Big(\frac{P^*(\beta-\beta_0,\alpha)}{P(\beta-\beta_0,\alpha)}\Big)^{\ell}\,,$$

with $\ell \in \{0,1,2\}$.

We also define $R_{0,0}(T,\alpha) = T - r_{0,0}(\alpha)$, so $R^*_{0,0}(T,\alpha) = T + \overline{r_{0,0}(\overline{\alpha})}$, and similarly for $r_{0,\infty}$ and r_+. Considering the relation $\overline{\mathbf{C}^+(\overline{\alpha},1-\overline{\beta})} = \mathbf{C}^+(\alpha,\beta)^{-1}$ itself we arrive at

$$R^*_{0,0}\,(P^*)^{\ell-1} \;=\; P^{\ell-1}\,R_+\,, \qquad R^*_{0,\infty}\,(P^*)^{\ell-1} \;=\; R_{0,\infty}\,P^{\ell-1}\,. \tag{2.34}$$

If $\ell = 0$ we find $R^*_{0,0}\,P = R_+\,P^*$. Assumption (3) implies that P and R_+ are different polynomials of the first degree with highest coefficient 1. So $P = P^*$, but we have already shown that that is impossible. So $\ell \in \{1,2\}$.

If $\ell = 2$ then $R^*_{0,0}\,P^* = P\,R_+$, and P divides $R^*_{0,0}$, and hence $P = R^*_{0,0}$, and then also $P^* = R_+$. We obtain $R_+ = (R^*_{0,0})^* = P^* = R_{0,0}$, in contradiction with assumption (3). Hence $\ell = 1$, and $R^*_{0,0} = R_+$, which gives the relation $\overline{r_{0,0}(\overline{\alpha})} = -r_+(\alpha)$. From (2.34) we now also get $R^*_{0,\infty} = R_{0,\infty}$, hence $r_{0,\infty}(\alpha) = -\overline{r_{0,\infty}(\overline{\alpha})}$. □

Lemma 2.18. *Under the assumptions 2.15 there are $\varepsilon > 0$ and a neighborhood U of β_0 in $\mathbb{C}$ such that for each $\alpha \in (0,\varepsilon]$ there is exactly one $\zeta(\alpha) \in U$ such that $X(\alpha,\zeta(\alpha))\,C_+(\alpha,\zeta(\alpha)) = 1$.*

We have $\lim_{\alpha\downarrow 0}\zeta(\alpha) = \beta_0$*, and as the point* $\zeta(\alpha)-\beta_0$ *moves to zero it passes the line segment between* $r_+(\alpha)$ *and* $p(\alpha)$ *infinitely often, circling around* $r_+(\alpha)$ *in negative direction or around* $p(\alpha)$ *in positive direction.*

Proof. We write $\beta = \beta_0 + z$. On Ω, the equation $X\,C_+ = 1$ becomes

$$(\pi\alpha)^{2it_0+2z}\,\tilde\gamma(\alpha,\beta_0+z)\,\frac{z-r_+(\alpha)}{z-p(\alpha)} \;=\; 1\,,$$

with $\tilde\gamma(\alpha,\beta) = \Gamma(\frac12-\beta)\,\Gamma(\beta-\frac12)\,\gamma_+(\alpha,\beta)$. We take $\varepsilon > 0$ and a simply connected, connected neighborhood U of β_0 such that $(0,\varepsilon]\times U \subset \Omega$. In the course of the proof we adapt ε and U. For sufficiently small $\varepsilon > 0$ the two points $r_+(\alpha)$ and $p(\alpha)$ are different points of U for all $\alpha\in(0,\varepsilon]$. The corresponding points $(\alpha,\beta_0+r_+(\alpha))$ and $(\alpha,\beta_0+p(\alpha))$ cannot be in the solution set of $X\,C_+ = 1$.

Taking a logarithm, we get the equation

$$2(it_0+z)\log\pi\alpha + \log\tilde\gamma(\alpha,\beta_0+z) + \log\frac{z-r_+(\alpha)}{z-p(\alpha)} \;\equiv\; 0 \bmod 2\pi i\mathbb{Z}\,,$$

where for $\log\tilde\gamma(\alpha,z)$ we use a continuous choice of the logarithm. The logarithm of the quotient $\frac{z-r_+}{z-p}$ is multivalued on $(0,\varepsilon]\times U$ and has branch points. We go over to the covering space by the parametrization

$$z \;=\; z(u) \;=\; \frac{e^u\,p(\alpha) - (\pi\alpha)^{2it_0}\,\tilde\gamma(0,\beta_0)\,r_+(\alpha)}{e^u - (\pi\alpha)^{2it_0}\,\tilde\gamma(0,\beta_0)}\,. \tag{2.35}$$

The variable u runs over a suitable subset of $\mathbb{C}$. The equation becomes

$$2\,z(u)\;\log\pi\alpha + \log\frac{\tilde\gamma(\alpha,\beta_0+z(u))}{\tilde\gamma(0,\beta_0)} + u \;=\; 0\,. \tag{2.36}$$

On the covering space the ambiguity modulo $2\pi i\mathbb{Z}$ is hidden in the choice of the variable u.

To make precise what is a suitable set in the u-plane, we use assumption (4). With $\beta\in\frac12+i\mathbb{R}$ all elements of the unitary matrix are non-zero, and hence have absolute value between 0 and 1. This implies that $0 < |\tilde\gamma(0,\beta)| < 1$, and we can take $\delta_- < 0$ such that $e^{\delta_-} > |\tilde\gamma(0,\beta_0)|$. We consider the region determined by $\delta_- \le \operatorname{Re} u \le \delta_+$ with some $\delta_+ > 0$. For these values of u the denominator of $z(u)$ satisfies

$$\left|e^u - (\pi\alpha)^{2it_0}\,\tilde\gamma(0,\beta_0)\right| \;\ge\; c_1 \;=\; c_1(\delta_-) \;>\; 0\,. \tag{2.37}$$

Hence we find

$$|z(u)| \;\le\; \frac{e^{\delta_+}\,|p(\alpha)| + |\tilde\gamma(0,\beta_0)|\,|r_+(\alpha)|}{c_1} \;\le\; c_2\,\alpha^2 \;=\; c_2(\delta_-,\delta_+)\,\alpha^2\,. \tag{2.38}$$

We have
$$z'(u) \;=\; \frac{(\pi\alpha)^{2it_0}\,\tilde\gamma(0,\beta_0)\,(r_+(\alpha)-p(\alpha))\,e^u}{\left(e^u-(\pi\alpha)^{2it_0}\,\tilde\gamma(0,\beta_0)\right)^2}\,, \tag{2.39}$$
which can be estimated in the following way:
$$\begin{aligned} z'(u) &\ll \frac{|\tilde\gamma(0,\beta_0)|\,\alpha^2\,e^{\delta_+}}{c_1^2}\,,\\ |z'(u)| &\le c_3\,\alpha^2 \;=\; c_3(\delta_-,\delta_+)\,\alpha^2\,. \end{aligned} \tag{2.40}$$

We consider on the region $\delta_- \le |\mathrm{Re}\,u| \le \delta_+$ the holomorphic function
$$F(u) \;=\; -2\,z(u)\,\log\pi\alpha - \log\frac{\tilde\gamma(\alpha,\beta_0+z(u))}{\tilde\gamma(0,\beta_0)}\,. \tag{2.41}$$
We have
$$\begin{aligned} \log\frac{\tilde\gamma(\alpha,\beta_0+z(u))}{\tilde\gamma(0,\beta_0)} &\ll \tilde\gamma(0,\beta_0)^{-1}\Big(\frac{\partial^2\tilde\gamma}{\partial\alpha^2}(0,\beta_0)\cdot\alpha^2 + \frac{\partial\tilde\gamma}{\partial\beta}(0,\beta_0)\cdot z(u)\Big)\\ &\ll \alpha^2 + |z(u)| \;\ll\; \alpha^2\,. \end{aligned}$$
(We have used that $\tilde\gamma$ is even in α.) We get $|F(u)| \;\le\; 2\,c_2\,\alpha^2\,|\log\pi\alpha| + \mathrm{O}(\alpha)$, and hence there is $c_4 = c_2(\delta_-,\delta_+)$ such that
$$\big|F(u)\big| \;\le\; c_4\,\alpha^2\,. \tag{2.42}$$
Taking ε such that $\varepsilon^2\,c_4 \in [\delta_-,\delta_+]$ we arrange that F maps the set
$$E \;=\; \{u\in\mathbb{C} \;:\; \delta_- \le \mathrm{Re}\,u \le \delta_+,\ -\varepsilon^2\,c_4 \le \mathrm{Im}\,u \le \varepsilon^2\,c_4\} \tag{2.43}$$
into itself.

The solutions of (2.36) in E are precisely the fixed points of F in E. The question is whether F is contracting on E.
$$\begin{aligned} F'(u) &= -2\,z'(u)\,\log\pi\alpha - \frac{\partial\tilde\gamma}{\partial\beta}(\alpha,\beta_0+z(u))\,z'(u)\\ &\ll \alpha^2\,\log\pi\alpha + \mathrm{O}(1)\,\alpha^2\,. \end{aligned}$$
Hence there is $c_5 = c_5(\delta_-,\delta_+)$ such that $|F'(u)| \le c_5\,\alpha^2\,|\log\pi\alpha|$ on E. We can adapt ε such that $c_5\,\alpha^2\,|\log\pi\alpha| \le c_6$ with some $c_6 \in (0,1)$. So F is contracting on E, and we find a unique fixed point $u(\alpha)\in E$. Projecting back we find a unique solution $\zeta(\alpha) = \beta_0 + z(u(\alpha))$ of the equation $X\,C_+ = 1$.

The denominator in (2.35) stays away from zero, by (2.37). Since $\operatorname{Re} u$ is bounded we have $z(\alpha) := z(u(\alpha)) = \mathrm{O}(\alpha^2)$. So $z(\alpha)$ tends to 0, and $\zeta(\alpha)$ tends to β_0. The relation

$$\frac{z(\alpha) - r_+(\alpha)}{z(\alpha) - p(\alpha)} \;=\; e^{u(\alpha)}\,(\pi\alpha)^{-2it_0}\,\tilde{\gamma}(0,\beta_0)^{-1} \tag{2.44}$$

shows that the argument of $\frac{z-r_+}{z-p}$ tends to ∞ as $\alpha \downarrow 0$. So $z(\alpha)$ crosses between $p(\alpha)$ and $r_+(\alpha)$ infinitely often, such that a continuous choice of the argument increases. This means that $z(\alpha)$ turns around $r_+(\alpha)$ in positive direction, or around $p(\alpha)$ in negative direction. □

Lemma 2.19. *Under the assumptions 2.15 there is a decreasing sequence* $(\alpha_k)_{k\geq k_0}$ *of positive numbers with limit zero such that for all* $\alpha \in (0, \alpha_{k_0})$

$$X(\alpha,\beta_0 + r_{0,\infty}(\alpha))\,C_+(\alpha,\beta_0 + r_{0,\infty}(\alpha)) \;=\; 1 \tag{2.45}$$

if and only if α *is one of the* α_k.

The α_k *satisfy*

$$\alpha_k \;=\; \frac{1}{\pi}\,e^{-(2\pi k+s_0)/2t_0}\left(1 + \mathrm{O}\big(k\,e^{-2\pi k/t_0}\big)\right), \tag{2.46}$$

for some real number s_0.

The value of $s_0 \bmod 2\pi\mathbb{Z}$ depends on the functions $r_{0,\infty}$, r_+ and p. We do not know it explicitly. The choice of s_0 in its class and the choice of the parameter k are related.

Proof. We consider the function

$$f(\alpha) \;=\; X(\alpha,\beta_0 + r_{0,\infty}(\alpha))\,C_+(\alpha,\beta_0 + r_{0,\infty}(\alpha))$$

on an interval $(0,\varepsilon_1]$ such that $(\alpha,\beta_0 + r_{0,\infty}(\alpha)) \in \Omega$. For small real values of α the values of $r_{0,\infty}(\alpha)$ are purely imaginary. Fact 2.5 and (2.7) imply that the matrix $\mathbf{C}^+(\alpha,\beta_0 + r_{0,\infty}(\alpha))$ is unitary. In (2.32) we see that $C_{0,\infty}(\alpha,\beta_0 + r_{0,\infty}(\alpha)) = 0$. So $\mathbf{C}^+(\alpha,\beta_0 + r_{0,\infty}(\alpha))$ is a unitary diagonal matrix. This implies that $|f(\alpha)| = 1$ for $\alpha \in (0,\varepsilon_1]$.

We make a continuous choice of $\alpha \mapsto s(\alpha)$ for $\alpha \in [0,\varepsilon_1)$ such that

$$e^{is(\alpha)} \;=\; \frac{\Gamma(\frac12 - \beta_0 - r_{0,\infty}(\alpha))}{\Gamma(-\frac12 + \beta_0 + r_{0,\infty}(\alpha))}\,\gamma_+(\alpha,\beta_0 + r_{0,\infty}(\alpha))\,\frac{r_{0,\infty}(\alpha) - r_+(\alpha)}{r_{0,\infty}(\alpha) - p(\alpha)}\,. \tag{2.47}$$

We note that s is an even function. The number s_0 in the statement of the lemma is equal to $s(0)$. Now

$$a(\alpha) \;=\; 2t_0\,\log\pi\alpha - 2i\,r_{0,\infty}(\alpha)\,\log\pi\alpha + s(\alpha) \qquad \text{for } \alpha \in (0,\varepsilon_1) \tag{2.48}$$

is a continuous choice of the argument of $f(\alpha)$. We have $a(\alpha) = 2t_0 \log \pi\alpha + \mathrm{O}(1)$ as $\alpha \downarrow 0$. The derivatives of the three term in (2.48) are

$$\frac{2t_0}{\alpha}, \quad \mathrm{O}(\alpha \log \pi\alpha), \quad \mathrm{O}(\alpha).$$

So for sufficiently small ε_1 the argument of $f(\alpha)$ is monotonously decreasing to $-\infty$ as $\alpha \downarrow 0$, and there is a sequence $(\alpha_k)_{k\geq k_0}$ of elements of $(0,\varepsilon)$ decreasing to zero such that for each $k \geq k_0$

$$2t_0 \log \pi\alpha_k - 2ir_{0,\infty}(\alpha_k) \log \pi\alpha_k + s(\alpha_k) = -2\pi k. \tag{2.49}$$

So the points $(\alpha_k, \beta_0 + r_{0,\infty}(\alpha_k))$ are solutions of $X C_+ = 1$, and the α's between two successive α_k do not satisfy this equation.

Since $\alpha_k = \mathrm{O}(1)$, we have directly $c_1 e^{-\pi k/t_0} \leq \alpha_k \leq c_2 e^{-\pi k/t_0}$ with positive c_1 and c_2. This gives

$$\log \pi\alpha_k = \frac{-2\pi k - s(\alpha_k)}{2t_0 + \mathrm{O}(\alpha_k^2)} = -\frac{2\pi k + s_0}{2t_0} + \mathrm{O}(k\, e^{-2\pi k/t_0}).$$

This gives (2.46). □

Proposition 2.20. *Under the assumptions 2.15 there is one curve $\alpha \mapsto (\alpha, \zeta(\alpha))$ on an interval $(0,\varepsilon_1)$ with limit $(0,\beta_0)$ for which $\mathrm{Z}(\alpha,\zeta(\alpha)) = 0$ for all $\alpha \in (0,\varepsilon_1)$.*

The curve touches the central line in $(\alpha_k, \zeta(\alpha_k))$ for a monotone sequence of α_k in $(0,\varepsilon_1]$ with limit 0. Hence the $\zeta(\alpha_k)$ are eigenvalues. The α_k satisfy the relation (2.46)*.*

As α runs through (α_{k+1}, α_k) the point $(\alpha, \zeta(\alpha))$ describes a curve in the region $\mathrm{Re}\,\beta < \frac{1}{2}$. The corresponding $\zeta(\alpha)$ are resonances, and $\zeta(\alpha_k) - \beta_0$ is proportional to α_k^2.

Proof. The $(\alpha, \zeta(\alpha))$ in Lemma 2.18 are solutions of $X C_+ = 1$. They satisfy $C_{0,\infty}(\alpha, \zeta(\alpha)) = 0$ precisely for the sequence (α_k) in Lemma 2.19. Proposition 2.11 shows that at these points the scattering ‘matrix’ $D_{0,0}$ has a singularity of indeterminate type. Hence the $\zeta(\alpha_k)$ are eigenvalues. For the other α, the function $D_{0,0}$ has value ∞ at $(\alpha, \zeta(\alpha))$, so $\mathrm{Re}\,\beta < \frac{1}{2}$ by 2.6, and $\zeta(\alpha)$ is a resonance. □

Remark 2.21. The computations reported in [Fr] provide us with six curves of resonances tending to a point on the central line with positive imaginary part. For two of these curves [Fr, Figure 8.26] suggests that indeed $\zeta(\alpha_k) - \beta_0$ is proportional to α_k^2.

3 Spectral theory of automorphic forms

We still have to indicate how the facts in §2.1 can be derived from published results. There is a vast literature on the spectral theory of Maass forms. A thorough discussion can be found in [Roe]; however the continuation of Eisenstein series was not yet fully known at that time. Of later literature we mention [Ve90], [Iw95], and [Bu98]. The material we need is also present in the treatment [He83, Chapters VI and VII] of the Selberg trace formula. For the extended scattering matrix we use results from analytic perturbation theory as discussed in [Br94].

3.1 Eisenstein series and scattering matrix

The group $\Gamma_0(4)$ has three cuspidal orbits, which we represent by 0, ∞ and $-\frac{1}{2}$. The corresponding Eisenstein series (for the trivial character $\chi_0 = 1$) are E_0^0, E_0^∞ and $E_0^{-1/2}$, each with Fourier expansions at all cusps of the form

$$E_0^\xi(\beta; g_\eta z) \;=\; \delta_{\xi,\eta}\, y^\beta + C_0(\eta,\xi;\beta)\, y^{1-\beta} + \cdots \tag{3.1}$$

By $\cdots$ we indicate the rapidly decreasing terms (as $\operatorname{Im} z \to \infty$) corresponding to the Fourier terms of non-zero order. We take $g_0 = \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$, $g_\infty = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $g_{-1/2} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, such that $g_\xi \infty = \xi$. The scattering matrix $\mathbf{C}_0(\beta)$ in (2.1) has $C_0(\eta,\xi;\beta)$ at position (η,ξ).

These Eisenstein series can be embedded in families of automorphic forms of a slightly more general type than is considered usually. For $-1 < \operatorname{Re}\alpha < 1$ and $\beta \in \mathbb{C}$ we use the space $A(\alpha,\beta)$ of functions f that satisfy

1. $f(\gamma z) = \chi_\alpha(\gamma)\, f(z)$ for all $\gamma \in \Gamma_0(4)$,
2. $-y^2\,(\partial_x^2 + \partial_y^2) f = \beta(1-\beta)\, f$,
3. for each cusp $\eta \in \{0, \infty, -\frac{1}{2}\}$ there is a Fourier expansion

   $$f(g_\eta z) \;=\; \sum_{n \equiv \kappa_\eta(\alpha) \bmod 1} (F_n^\eta f)(z)\,, \tag{3.2}$$

   where $F_n^\eta f(z)$ is the product of $e^{2\pi i n x} = e^{2\pi i n \operatorname{Re} z}$ and a function depending only on $y = \operatorname{Im} z$, and where $\kappa_0(\alpha) = 0$, $\kappa_\infty(\alpha) = \alpha$, and $\kappa_{-1/2} = -\alpha$.

   We *require* that the other Fourier terms $F_n^\eta f(z)$ with $n \neq \kappa_\eta(\alpha)$ are rapidly decreasing as $y \to \infty$.

We call $F_0^0 f$, $F_\alpha^\infty f$ and $F_{-\alpha}^{-1/2} f$ the *Fourier terms of order zero*. For $\xi \in \{0, \infty, \frac{1}{2}\}$ the Eisenstein series E_0^ξ can be embedded in a family $(\alpha,\beta) \mapsto E^\xi(\alpha,\beta)$ of automorphic forms that are meromorphic in the variable (α,β) for α in an open neighborhood of $(-1,1)$ in $\mathbb{C}$ and $\beta \in \mathbb{C}$. See [Br94, Theorem 10.2.1]. These families are uniquely determined by the form of their Fourier terms of order zero:

$$F^\eta_{\kappa_\eta(\alpha)} E^\xi(\alpha,\beta;z) \;=\; \delta_{\xi,\eta}\,\mu(\kappa_\eta(\alpha),\beta;z) + C_{\eta,\xi}(\alpha,\beta)\,\mu(\kappa_\eta(\alpha),1-\beta;z)\,, \tag{3.3}$$

with $\mu(n,\beta;z)$ a meromorphic extension of y^β of the form

$$\mu(n,\beta;z) \;=\; e^{2\pi i n z}\, y^\beta\, {}_1F_1(\beta;2\beta;4\pi n \operatorname{Im} z)\,. \tag{3.4}$$

The restriction $\beta \mapsto E^\xi(0,\beta)$ exists as a meromorphic family of automorphic forms and is equal to E_0^ξ. The scattering matrix $\mathbf{C}(\alpha,\beta)$ is formed by the $C_{\eta,\xi}(\alpha,\beta)$. Thus, we obtain fact 2.3.

With the methods of [Br94, 10.3.5] we obtain the functional equation $\mathbf{C}(\alpha,1-\beta)\mathbf{C}(\alpha,\beta) = \mathrm{I}$, as an equality between matrices of meromorphic functions. The Maass-Selberg relation, as discussed in, e.g., [Br94, Theorem 4.6.5], implies that $\mathbf{C}(-\alpha,\beta) = \mathbf{C}(\alpha,\beta)^t$ (transpose). An analysis of the effect of conjugation on the Fourier expansion gives the relation $\overline{\mathbf{C}(\bar\alpha,\bar\beta)} = \mathbf{C}(-\alpha,\beta)$. This leads to fact 2.5.

The character χ_α is invariant under conjugation by $j = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$. It is compatible with the involution $\iota : z \mapsto \bar z/(2\bar z - 1)$ on the upper half-plane $\mathbb{H}$:

$$\iota(\gamma z) = (j\gamma j^{-1})\,\iota z \qquad \text{for } \gamma \in \Gamma_0(4)\,. \tag{3.5}$$

Hence this involution induces an involution J, given by $(Jf)(z) = f(\iota z)$, in the space $A(\alpha,\beta)$, and leads to a decomposition into subspaces $A^\mp(\alpha,\beta)$ of odd and even automorphic forms. We have $JE^0 = E^0$ and $JE^\infty = E^{-1/2}$. This implies the equalities

$$\begin{aligned} C_{0,\infty} &= C_{0,-1/2}\,, & C_{\infty,0} &= C_{-1/2,0}\,, \\ C_{\infty,\infty} &= C_{-1/2,-1/2}\,, & C_{\infty,-1/2} &= C_{-1/2,\infty}\,, \end{aligned} \tag{3.6}$$

and leads to the partial diagonalization in §2.2.

3.2 Poincaré series

The basis $\{\mu(n,\beta;z),\, \mu(n,1-\beta;z)\}$ for the space of Fourier terms used in (3.3) has the drawback that, for general combinations of α and β, it consists of exponentially growing functions. For $\operatorname{Re} n \neq 0$ we may also use the basis consisting of $\mu(n,\beta;z)$ and the rapidly decreasing element

$$\omega(n,\beta;z) \;=\; 2\,(\varepsilon n)^{1/2}\, e^{2\pi i n x}\, \sqrt{y}\, K_{\beta-1/2}(2\pi\varepsilon n y)\,, \tag{3.7}$$

with $\varepsilon = \text{ Sign Re}\, n$. Of course, $\varepsilon n = |n|$ if n is real. The factor $2(\varepsilon n)^{1/2}$ allows n to be non-real complex, and keeps the notation consistent with [Br94, §4.2.8] (except that $\beta_{\text{here}} = s_{\text{there}} + \frac{1}{2}$).

Applying [Br94, Theorem 10.2.1] with this basis, we obtain meromorphic families $P^0(\alpha,\beta)$, $P^\infty(\alpha,\beta)$ and $P^{-1/2}(\alpha,\beta)$ of automorphic forms taking values in the spaces $A(\alpha,\beta)$, where α runs over a neighborhood of $(0,1)$ in $\mathbb{C}$ and $\beta \in \mathbb{C}$. For fixed $\alpha \in (0,1)$ the restriction $\beta \mapsto P^\xi(\alpha,\beta)$ exists, and is the meromorphic extension in β of a Poincaré series convergent for $\text{Re}\,\beta > 1$:

$$P^\xi_\alpha(\beta; z) \;=\; \sum_{\gamma\in\Gamma_\xi\backslash\Gamma} \chi_\alpha(\gamma)^{-1}\, \mu(\kappa_\xi(\alpha),\beta; g_\xi^{-1}\gamma z)\,. \tag{3.8}$$

For $\alpha \in (0,1)$ only the cusp 0 stays open, and $E^0_\alpha(\beta) := P^0_\alpha(\beta)$ is the corresponding Eisenstein series. Also the families P^ξ are characterized by their Fourier terms of order zero, which for $\eta = \infty$ and $\eta = -\frac{1}{2}$ have the following form:

$$\begin{aligned} F^\eta_{\kappa_\eta(\alpha)} P^\xi(\alpha,\beta; z) \;=\;& \delta_{\xi,\eta}\, \mu(\kappa_\eta(\alpha),\beta; z)\\ &+ D_{\eta,\xi}(\alpha,\beta)\, \omega(\kappa_\eta(\alpha),\beta; z)\,. \end{aligned} \tag{3.9}$$

With the method of [Br94, 10.3.4] the meromorphic functions $D_{\eta,\xi}(\alpha,\beta)$ can be expressed in terms of the $C_{\eta,\xi}(\alpha,\beta)$. The scattering matrix for the case $\alpha \in (0,1)$ is a 1×1-matrix, given by $\beta \mapsto D_{0,0}(\alpha,\beta)$. Its explicit expression in terms of the $C_{\eta,\xi}$ is given in (2.6) in fact 2.6.

3.3 Zeros of the Selberg zeta-function

The geometric description of the Selberg zeta-function $Z(\alpha,\beta) = Z(\Gamma_0(4),\chi_\alpha;\beta)$ in (1.2) is important for the computations in [Fr]. For this paper the relation of $Z(\alpha,\beta)$ to automorphic forms is important. We quote [He83, Theorem 5.3 in Chapter X, p. 498] as far as this concerns the region $\text{Im}\,\beta > 0$ for $\alpha \in \mathbb{R}$.

a) At points β on the central line $\frac{1}{2} + i(0,\infty)$ the function $Z(\alpha,\cdot)$ has a zero of order equal to the dimension of the space $\mathsf{Maass}^0(\alpha,\beta) \subset A(\alpha,\beta)$ of cusp forms. This space consists of the automorphic forms for the character χ_α that have rapid decay at all cusps.

   These values of β are called *eigenvalues* in §1 and §2, although the cumbersome description 'spectral parameters of cusp forms' would be more correct.

   For the group $\Gamma_0(4)$ it is known that if $\mathsf{Maass}^0(0,\beta) \neq \{0\}$ then $\beta \in \frac{1}{2} + i\mathbb{R}$, $\beta \neq \frac{1}{2}$. The theorem of [Hu85, p. 250] gives this for $\Gamma^0(4)$, which is conjugate to $\Gamma_0(4)$.

b) At points with $\mathrm{Re}\,\beta < \frac{1}{2}$ and $\mathrm{Im}\,\beta > 0$ the function $Z(\alpha, \cdot)$ has a zero of the same order as the singularity of the determinant of the scattering matrix at β. These are the *resonances*.

For $\alpha \in (0, 1)$ the scattering "matrix" is the quantity $D_\alpha(\beta)$ in 2.6, hence this gives the last statement in fact 2.6. (Actually, Hejhal speaks of a zero of the determinant of the scattering matrix at $1 - \bar{\beta}$. This is equivalent to what we need by the functional equations.)

There are no zeros with $\mathrm{Re}\,\beta > \frac{1}{2}$ and $\mathrm{Im}\,\beta > 0$.

3.4 Cusp forms and singularities of Poincaré series

A Poincaré series $P^\xi_\alpha(\beta)$ with $\alpha \in (0, 1)$ has a singularity at $\beta_0 \in \frac{1}{2} + i(0, \infty)$ if and only if there exists a cusp form $f \in \mathsf{Maass}^0(\alpha, \beta_0)$ for which the Fourier term of order $n_\xi(\alpha)$ at the cusp ξ is non-zero. See, *e.g.*, [Br94, Proposition 11.3.9].

The Fourier term of order α at ∞ of the family of Poincaré series

$$\frac{1}{\sqrt{2}} \left(P^\infty_\alpha(\cdot) - P^{-1/2}_\alpha(\cdot) \right)$$

contains the factor $C_-/(1 - XC_-)$, in the notation of (2.4) and (2.5). If for $\alpha \in (0, 1)$, $\beta \in \frac{1}{2} + i\mathbb{R}$ we have $X(\alpha, \beta)\, C_-(\alpha, \beta) = 1$ and C_- is holomorphic at (α, β), then this Poincaré series has a singularity at β, and hence there is a cusp form with this spectral parameter. This cusp form arises as a residue of this odd Poincaré series, and hence it is an odd cusp form. This implies fact 2.7.

3.5 Fact 2.4

Suppose that the extended scattering matrix $\mathbf{C}$ has a singularity at $(0, \beta_0)$ with $\mathrm{Re}\,\beta_0 = \frac{1}{2}$. From [Br94, Proposition 10.2.14] we conclude that this happens if and only if the vector $(E^0, E^\infty, E^{-1/2})$ has a singularity at $(0, \beta_0)$. If $\beta_0 \neq \frac{1}{2}$ we apply [Br94, Proposition 12.4.2, i)] to conclude that $\beta_0(1 - \beta_0)$ is an eigenvalue, and hence $Z(0, \beta_0) = 0$.

We still have to consider the possibility that $\beta_0 = \frac{1}{2}$. The space $A(0, \frac{1}{2})$ is spanned by the derivatives $\partial_\beta E^\xi(0, \beta)\big|_{\beta=1/2}$ for $\xi \in \{0, \infty, -1/2\}$. Near $(0, \frac{1}{2})$ the basis $\mu(\alpha, \beta)$, $\mu(\alpha, 1 - \beta)$ for the Fourier terms of order zero is unsuitable. We can use the basis $\lambda(\alpha, \beta)$, $\mu(\alpha, 1 - \beta)$ in [Br94, Lemma 7.6.14 i)]. The derivative of E^ξ has at the cusp ξ a Fourier term in which $\lambda(1, \frac{1}{2})$ occurs with non-zero factor. If we suppose that the vector $\mathsf{Eis} = (E^0, E^\infty, E^{-1/2})$ has a singularity at $(0, \frac{1}{2})$ then we can follow its principal part along a local curve through $(0, \frac{1}{2})$ contained in the singular set of Eis. See the discussion in [Br94, §12.1]. This gives a non-zero meromorphic family of automorphic forms along a one-dimensional curve. The principal part of this family at $\alpha = 0$ gives a non-zero vector V in $A(0, \frac{1}{2})^3$.

A closer examination of the Fourier expansions shows that the terms of order zero can be expressed through $\mu(0, \frac{1}{2})$. So V is not a linear combination of the derivatives of the E^ξ. Hence it is a cusp form. However for $(\alpha, \beta) = (0, \frac{1}{2})$ the space of cusp forms is zero as shown by [Hu84]. So we get the contradictory conclusion that V is the zero vector in $A(0, \frac{1}{2})^3$.

3.6 Fact 2.2

We have already obtained the equalities in (3.6). We still need to establish the equality $C_{0,\infty} = C_{\infty,0}$. From 2.5 we have $C_{\infty,0}(\alpha,\beta) = C_{0,\infty}(-\alpha,\beta)$. So it suffices to show that $C_{0,\infty}(\alpha,\beta)$ is even in α.

The formulas in [Br94, §5.2] give for $\alpha \in (0,1)$ and $\mathrm{Re}\,\beta > 1$

$$D_{0,\infty}(\alpha,\beta) \;=\; \frac{\pi^\beta\,\alpha^{\beta-1}}{\Gamma(\beta)}\,\Phi_{0,\infty}(\alpha,\beta)\,, \tag{3.10}$$

with an absolutely convergent series

$$\Phi_{0,\infty}(\alpha,\beta) \;=\; \sum_{c>0} c^{-2\beta} \sum_{d \bmod c} \chi_\alpha(\gamma_g)^{-1}\, e^{2\pi i\alpha a/c}\,, \tag{3.11}$$

in which the variables run over $g = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ such that $\gamma_g = \left[\begin{smallmatrix} -2b & a/2 \\ -2d & c/2\end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] g_0^{-1} \in \Gamma$. In particular, c runs over the positive numbers satisfying $c \equiv 2 \bmod 4$ and d over the even numbers modulo c satisfying $(2d, c/2) = 1$.

Right multiplication of g by $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right]$ does not change $\frac{a}{c}$, and multiplies γ_g by $\left[\begin{smallmatrix} 1 & 0 \\ -4 & 1\end{smallmatrix}\right]$ on the right; hence it leaves $\chi_\alpha(\gamma_g)$ invariant as well. So the terms are functions of $d \bmod c$. Left multiplication of g with $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right]$ gives a factor $e^{-2\pi i\alpha}$ in $\chi_\alpha(\gamma_g)^{-1}$. This is compensated by $e^{2\pi i\alpha a/c}$ since a is changed to $a+c$. So the terms in the sum are well defined.

We have $\chi_\alpha(\gamma) = e^{i\alpha\Omega(\gamma)}$, where $\Omega : \Gamma_0(4) \to \mathbb{Z}$ is the group homomorphism determined by $\Omega\left(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right]\right) = 1$ and $\Omega\left(\left[\begin{smallmatrix} 1 & 0 \\ -4 & 1\end{smallmatrix}\right]\right) = 0$. The value of $\Omega\left(\left[\begin{smallmatrix} r & s \\ -4p & q\end{smallmatrix}\right]\right)$ is determined by the upper row of the matrix, and we have

$$\begin{aligned} \Omega(1,0) &= 0\,, & \Omega(-r,-s) &= \Omega(r,s)\,,\\ \Omega(r,s) &= \Omega(r-4s,s)\,, & \Omega(r,s) &= \Omega(r,s-r)+1\,, \end{aligned} \tag{3.12}$$

where $r, s \in \mathbb{Z}$, $r \equiv 1 \bmod 2$ and $(r,s) = 1$. With induction we obtain

$$\left|\Omega(r,s)\right| \;\le\; |s|\,, \quad \text{and} \quad \Omega(r,-s) \;=\; -\Omega(r,s)\,. \tag{3.13}$$

The inner sum in (3.11) for a given positive $c \equiv 2 \bmod 4$ has the form

$$\sum_{d \bmod c,\, d\equiv 0 \bmod 2,\, (2d,c/2)=1} e^{-2\pi i\alpha\,\Omega(-2b,a/2)}\, e^{2\pi i a/c}\,.$$

Conjugation with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has the effect $(a,b,c,d) \mapsto (-a,b,c,-d)$. Under this transformation the set of d in the sum is unchanged, while we have $\Omega(-2b,a/2) \mapsto \Omega(-2b,-a/2) = -\Omega(-2b,a/2)$ and $\frac{a}{c} \mapsto -\frac{a}{c}$. Hence

$$\Phi_{0,\infty}(-\alpha,\beta) \;=\; \Phi_{0,\infty}(\alpha,\beta) \tag{3.14}$$

for $\alpha \in \mathbb{R}$ and $\mathrm{Re}\,\beta > 1$.

For other $\xi, \eta \in \{0, \infty, -\frac{1}{2}\}$ we can carry out similar computations. (See [Br94, §4.2].) In all cases we have for $\alpha \in (0,1)$, $\mathrm{Re}\,\beta > 1$

$$D_{\eta,\xi}(\alpha,\beta) \;=\; \frac{\pi^{\beta}\alpha^{\beta-1}}{\Gamma(\beta)}\,\Phi_{\eta,\xi}(\alpha,\beta)\,,$$

with $\Phi_{\eta,\xi}(\alpha,\beta)$ holomorphic in β on the region $\mathrm{Re}\,\beta > 1$ and continuous in $\alpha \in \mathbb{R}$ (not necessarily even). It is given by a more complicated series than that for $\Phi_{0,\infty}$, since the factor $c^{-2\beta}$ is replaced by a more complicated expression involving Bessel functions.

The relation between the families E^{ξ} in §3.1 and the families P^{ξ} in §3.2 implies

$$C_{0,\infty}(\alpha,\beta) \;=\; \bigl(1 - X(\alpha,\beta)C_{+}(\alpha,\beta)\bigr)\, D_{0,\infty}(\alpha,\beta)\,, \tag{3.15}$$

with X and C_{+} as in (2.4) and (2.5). Furthermore, for $\alpha \in (0,1)$ and $\mathrm{Re}\,\beta > 1$

$$C_{+}(\alpha,\beta) \;=\; \frac{\pi^{1/2}\Gamma(\beta-\frac{1}{2})\,\Phi_{+}(\alpha,\beta)}{\Gamma(\beta) + \pi^{1/2}(\pi\alpha)^{2\beta-1}\Gamma(\frac{1}{2}-\beta)\,\Phi_{+}(\alpha,\beta)}\,, \tag{3.16}$$

where Φ_{+} is a linear combination of functions $\Phi_{\eta,\xi}$.

For a given integer $k \geq 2$ we conclude for $\mathrm{Re}\,\beta > \frac{k+1}{2}$

$$C_{+}(\alpha,\beta) \;=\; \pi^{1/2}\Gamma(\beta-1/2)\,\Gamma(\beta)^{-1}\,\Phi_{+}(\alpha,\beta) + \mathrm{O}(\alpha^{k}) \;=\; \mathrm{O}(1)\,,$$

as $\alpha \downarrow 0$. Since we have also $X(\alpha,\beta) = \mathrm{O}(\alpha^{k})$, we obtain

$$C_{0,\infty}(\alpha,\beta) \;=\; \bigl(1 + \mathrm{O}(\alpha^{k})\bigr)\,\pi^{1/2}\,\Gamma(\beta-1/2)\Gamma(\beta)^{-1}\,\Phi_{0,\infty}(\alpha,\beta)\,.$$

Now take $\beta \in \mathbb{C}$ with $\mathrm{Re}\,\beta > \frac{k+1}{2}$ such that $\mathbf{C}$ is holomorphic at $(0,\beta)$. The holomorphic function $\alpha \mapsto C_{0,\infty}(\alpha,\beta)$ has a Taylor expansion at $\alpha = 0$ of any order. So the continuous function $\alpha \mapsto \Phi_{0,\infty}(\alpha,\beta)$ on $\mathbb{R}$ has a Taylor expansion of order $k-1$. Since it is an even function the terms of odd order less than k vanish. So $\alpha \mapsto C_{0,\infty}(\alpha,\beta)$ has only even terms in its Taylor expansion at $\alpha = 0$ of order $k-1$. This holds for a dense set of β with $\mathrm{Re}\,\beta > \frac{k+1}{2}$, hence for odd $n < k$ we have

$$\partial_\alpha^n C_{0,\infty}(\alpha,\beta)\Big|_{\alpha=0} \;=\; 0\,, \tag{3.17}$$

as an identity of meromorphic functions of $\beta \in \mathbb{C}$. Since we can take k arbitrarily large the conclusion is that $C_{0,\infty}(\alpha,\beta)$ is an even function of α.

3.7 Concluding remarks

In the preceding subsections we have explained those parts of the theory of automorphic forms that we needed to discuss some observations in the computational results in [Fr].

Remark 3.1. An essential point enabling us to get some hold on the behavior of the zeros of the Selberg zeta-function is the fact that the Poincaré series for the perturbed character can be expressed in the Eisenstein families E^ξ, which leads to (2.6).

In all cases in §2 we have an equation where $(\pi\alpha)^{2\beta-1}$ or $(\pi^2\alpha)^{2\beta-1}$ is equal to some meromorphic function on a neighborhood of $\{0\}\times\mathbb{C}$ in $\mathbb{C}^2$. This causes a proportionality relation between α and $e^{-\pi k/\mathrm{Im}\,\beta}$ in many cases.

Remark 3.2. The full result of 2.2 is not needed for the proofs. We could have managed with the estimate $\mathrm{O}(\alpha^2)$ for the matrix elements of the extended scattering matrix.

Remark 3.3. All zeros of the Selberg zeta-function with the spectral parameter on the central line that are visible in the computations in [Fr] are related to properties of the extended scattering matrix. The spectral theory of automorphic forms allows the existence of cusp forms $f \in \mathsf{Maass}^0(\alpha,\beta)$ with $\alpha \in (0,1)$ for which the Fourier terms of order zero at all cusps vanish. The presence of such cusp forms has not been detected in the computations.

Remark 3.4. In Theorem 1.1 we have stated that the functions τ_k are defined on an interval $(0,\zeta_k) \subset (0,1)$. Actually, one can prove, that the families of cusp forms associated with the eigenvalue $\alpha \mapsto \frac{1}{4} + \tau_k(\alpha)^2$ are real-analytic on $(0,1)$. They belong to a so-called Kato basis. Compare [PS94, §2].

Remark 3.5. All automorphic forms for $\Gamma_0(4)$ with the family of characters $\alpha \mapsto \chi_\alpha$ split completely in an even and an odd part. The zeros of the Selberg zeta-function are related to eigenfunctions of a transfer operator, to which also a parity is associated. In [FM] it is shown that this parity corresponds to the parity of automorphic forms. It turns out that zeros of the Selberg zeta-function in §1.1 are odd, and those in §1.2 even.

Remark 3.6. All odd cusp forms observed in the computations occur in families on an interval contained in $(0,1)$ and have no real-analytic extension across $\alpha = 0$. Such an extension would be allowed by the theory, and would give at $\alpha = 0$ an unperturbed odd cusp form. All odd unperturbed cusp forms inferred from the computations do not occur in such families, but make their influence noticeable by the phenomenon of avoided crossing.

Remark 3.7. All inferred even cusp forms for a nontrivial character, $\alpha \in (0,1)$, occur discretely as cusp forms. Their parameters (α,β) occur on curves of resonances, where they touch the central line.

The limit point $(0,\beta_0)$ for a curve of resonances as $\alpha \downarrow 0$ is equal to $(0,\frac{1}{2})$ for countably many curves. All inferred unperturbed cusp forms with parameters $(0,\beta_0) \neq (0,\frac{1}{2})$ are approached by a curve of resonances. Such a curve describes infinitely many loops, giving rise to a sequence $(\alpha_k,\beta_k) \to (0,\beta_0)$ of parameters of even perturbed cusp forms.

Remark 3.8. The considerations in this paper concern a special situation, namely the cofinite discrete subgroup $\Gamma_0(4)$ of $\mathrm{PSL}_2(\mathbb{R})$ and the 1-parameter family of characters $\alpha \mapsto \chi_\alpha$. We have tried to make use of all special properties of this specific situation that we could obtain. It remains to be investigated how much of the results of this paper are valid more generally. Computations in [Fr] indicate that for $\Gamma_0(8)$ similar phenomena occur.

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