11d. Parametrization

See §3.4, Proposition 3.17

In[ ]:=
```
Clear[S1, S2]
S1[{j_, nu_}] := (1 / 2) {3 nu - j, j + nu} // Simplify
S2[{j_, nu_}] := (1 / 2) {-3 nu - j, nu - j} // Simplify
```

Invariance

In[ ]:=
```
Clear[j, nu]
ld2[S1[{j, nu}]] == ld2[j, nu] // Simplify
ld2[S2[{j, nu}]] == ld2[j, nu] // Simplify
ld3[S1[{j, nu}]] == ld3[j, nu] // Simplify
ld3[S2[{j, nu}]] == ld3[j, nu] // Simplify
```

Out[ ]= True

Out[ ]= True

Out[ ]= True

Out[ ]= True

Eigenvalue computation

In[ ]:=
```
Clear[j, nu, h, p, th]
```

$$th = \frac{1}{8}\left(nu^2 - (2+j)^2\right)$$

```
eqn = {ld2[j, nu] == (h^2/3 + p^2 + 2 h + 2 p) + 4 (p + 2) (p + 1)^(-1) th,
   ld3[j, nu] == (h - 3 p + 6) ((h / 9) (h + 3 p + 12) + 2 (p + 2) (p + 1)^(-1) th)}
Solve[eqn, {j, nu}] // Simplify
Dimensions[%]
```

Out[ ]= $\frac{1}{8}\left(-(2+j)^2 + nu^2\right)$

Out[ ]= $\left\{-4 + \frac{j^2}{3} + nu^2 == 2\,h + \frac{h^2}{3} + 2\,p + p^2 + \frac{\left(-(2+j)^2 + nu^2\right)(2+p)}{2 \times (1+p)},\right.$
$\left.(3+j) \times \left(-\frac{1}{9}\,(-6+j)^2 + nu^2\right) == (6 + h - 3\,p)\left(\frac{\left(-(2+j)^2 + nu^2\right)(2+p)}{4 \times (1+p)} + \frac{1}{9}\,h\,(12 + h + 3\,p)\right)\right\}$

Out[ ]= $\left\{\left\{j \to \frac{1}{2}\,(h - 3\,p),\ nu \to -\frac{1}{2}\,\sqrt{80 + h^2 + 24\,p - 7\,p^2 + 2\,h\,(12 + 5\,p)}\right\},\right.$
$\left\{j \to \frac{1}{2}\,(h - 3\,p),\ nu \to \frac{1}{2}\,\sqrt{80 + h^2 + 24\,p - 7\,p^2 + 2\,h\,(12 + 5\,p)}\right\},\ \left\{j \to -\frac{1}{8 \times (3 + 2\,p)}\left((18 + h)\,p - 3\,p^2 +\right.\right.$

$$3 \times \left(12 + \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)}\right)\Big),$$

$$\text{nu} \to -\frac{1}{4\,\sqrt{2}\,\sqrt{p}}\,i\,\sqrt{\Bigg(-\frac{1}{(3 + 2\,p)^2}\,p\,\Big(144 + 528\,p + 772\,p^2 + 504\,p^3 + 121\,p^4 + h^2\left(40 + 52\,p + 17\,p^2\right) +}$$
$$12 \times \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} +$$
$$30\,p\,\sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} +$$
$$15\,p^2\,\sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} -$$
$$h\,\Big(48 + 216\,p + 228\,p^2 + 70\,p^3 + 8 \times \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 +\right.}$$
$$\left.h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right) + 5\,p\,\sqrt{\left(144 + 336\,p +\right.}$$
$$\left.284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)\Big)\Big)\Bigg)\Bigg\},$$

$$\Big\{j \to -\frac{1}{8 \times (3 + 2\,p)}\Big((18 + h)\,p - 3\,p^2 + 3 \times \Big(12 + \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 +\right.}$$
$$\left.h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)\Big)\Big),$$

$$\text{nu} \to \frac{1}{4\,\sqrt{2}\,\sqrt{p}}\,i\,\sqrt{\Bigg(-\frac{1}{(3 + 2\,p)^2}\,p\,\Big(144 + 528\,p + 772\,p^2 + 504\,p^3 + 121\,p^4 + h^2\left(40 + 52\,p + 17\,p^2\right) +}$$
$$12 \times \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} +$$
$$30\,p\,\sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} +$$
$$15\,p^2\,\sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} -$$
$$h\,\Big(48 + 216\,p + 228\,p^2 + 70\,p^3 + 8 \times \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 +\right.}$$
$$\left.h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right) + 5\,p\,\sqrt{\left(144 + 336\,p +\right.}$$
$$\left.284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)\Big)\Big)\Bigg)\Bigg\},$$

$$\Big\{j \to \frac{1}{8 \times (3 + 2\,p)}\Big(-((18 + h)\,p) + 3\,p^2 + 3 \times \Big(-12 + \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 +\right.}$$
$$\left.17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)\Big)\Big),$$

$$\text{nu} \to -\frac{1}{4\,\sqrt{2}\,\sqrt{p}}\,i\,\sqrt{\Bigg(-\frac{1}{(3 + 2\,p)^2}\,p\,\Big(144 + 528\,p + 772\,p^2 + 504\,p^3 + 121\,p^4 + h^2\left(40 + 52\,p + 17\,p^2\right) -}$$
$$12 \times \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} -$$
$$30\,p\,\sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} -$$
$$15\,p^2\,\sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)} +$$
$$h\,\Big(-48 - 216\,p - 228\,p^2 - 70\,p^3 + 8 \times \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 + 17\,p^4 +\right.}$$
$$\left.h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right) + 5\,p\,\sqrt{\left(144 + 336\,p +\right.}$$
$$\left.284\,p^2 + 108\,p^3 + 17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)\Big)\Big)\Bigg)\Bigg\},$$

$$\Big\{j \to \frac{1}{8 \times (3 + 2\,p)}\Big(-((18 + h)\,p) + 3\,p^2 + 3 \times \Big(-12 + \sqrt{\left(144 + 336\,p + 284\,p^2 + 108\,p^3 +\right.}$$
$$\left.17\,p^4 + h^2\,(4 + 3\,p)^2 + 2\,h\left(48 + 84\,p + 42\,p^2 + 5\,p^3\right)\right)\Big)\Big),$$

$$nu \to \frac{1}{4\sqrt{2}\sqrt{p}} i \sqrt{\left(-\frac{1}{(3+2p)^2} p \left(144 + 528 p + 772 p^2 + 504 p^3 + 121 p^4 + h^2 (40 + 52 p + 17 p^2) - 12 \times \sqrt{(144 + 336 p + 284 p^2 + 108 p^3 + 17 p^4 + h^2 (4+3p)^2 + 2 h (48 + 84 p + 42 p^2 + 5 p^3))} - 30 p \sqrt{(144 + 336 p + 284 p^2 + 108 p^3 + 17 p^4 + h^2 (4+3p)^2 + 2 h (48 + 84 p + 42 p^2 + 5 p^3))} - 15 p^2 \sqrt{(144 + 336 p + 284 p^2 + 108 p^3 + 17 p^4 + h^2 (4+3p)^2 + 2 h (48 + 84 p + 42 p^2 + 5 p^3))} + h \left(-48 - 216 p - 228 p^2 - 70 p^3 + 8 \times \sqrt{(144 + 336 p + 284 p^2 + 108 p^3 + 17 p^4 + h^2 (4+3p)^2 + 2 h (48 + 84 p + 42 p^2 + 5 p^3))} + 5 p \sqrt{(144 + 336 p + 284 p^2 + 108 p^3 + 17 p^4 + h^2 (4+3p)^2 + 2 h (48 + 84 p + 42 p^2 + 5 p^3))}\right)\right)\right)}\}\}$$

Out[ ]= {6, 2}

Mathematica does not find more than 6 solutions.

Freedom in the parametrization

In[ ]:=
```
Clear[nu, nu1, j, j1]
{ld2[j, nu] == ld2[j1, nu1], ld3[j, nu] == ld3[j1, nu1]}
Solve[%, {j1, nu1}]
```

Out[ ]= $\left\{-4 + \frac{j^2}{3} + nu^2 == -4 + \frac{j1^2}{3} + nu1^2,\ (3 + j) \times \left(-\frac{1}{9} (-6 + j)^2 + nu^2\right) == (3 + j1) \times \left(-\frac{1}{9} (-6 + j1)^2 + nu1^2\right)\right\}$

Out[ ]= $\{\{j1 \to j,\ nu1 \to -nu\},\ \{j1 \to j,\ nu1 \to nu\},$
$\left\{j1 \to \frac{1}{2}(-j - 3\,nu),\ nu1 \to \frac{j - nu}{2}\right\},\ \left\{j1 \to \frac{1}{2}(-j - 3\,nu),\ nu1 \to \frac{1}{2}(-j + nu)\right\},$
$\left\{j1 \to \frac{1}{2}(-j + 3\,nu),\ nu1 \to \frac{1}{2}(-j - nu)\right\},\ \left\{j1 \to \frac{1}{2}(-j + 3\,nu),\ nu1 \to \frac{j + nu}{2}\right\}\}$

The Weyl group permutes these solutions