11f. Eigenfunction equations, abelian eigenfunction module

In Lemma 3.20 we use a non-trivial character of N

In[ ]:=
```
Clear[p, r, h, g, j, nu]
F = chbt g[r, t] × Phi[h, p, r, p]
(* the sum over r is not explicitly given *)
```

Out[ ]= chbt g[r, t] × Phi[h, p, r, p]

The following computation takes time

In[ ]:=
```
eieq = {eR[CasZ, F, subab] - ld2[j, nu] F, eR[Dt3Z, F, subab] - ld3[j, nu] F} // Simplify
```

Out[ ]=
$$\Big\{-\mathrm{chbt}\left(-4+\frac{j^2}{3}+nu^2\right) g[r,t]\times \mathrm{Phi}[h,p,r,p] + \frac{1}{12}\mathrm{chbt}\, g[r,t]\big(24\, i\, \mathrm{betac}\,\pi\,(2+p-r)\,t\,\mathrm{Phi}[h,p,-2+r,p] + ((h-3r)^2-48\pi^2 t^2 \mathrm{Abs}[\mathrm{beta}]^2)\,\mathrm{Phi}[h,p,r,p] - 24\, i\,\mathrm{beta}\,\pi\,(2+p+r)\,t\,\mathrm{Phi}[h,p,2+r,p]\big) + \mathrm{chbt}\, t\,\mathrm{Phi}[h,p,r,p]\left(-3 g^{(0,1)}[r,t]+t\, g^{(0,2)}[r,t]\right),$$
$$-\mathrm{chbt}\,(3+j)\times\left(-\frac{1}{9}(-6+j)^2+nu^2\right) g[r,t]\times\mathrm{Phi}[h,p,r,p] + \frac{1}{72} i\,\mathrm{chbt}\,\big(g[r,t]\big(36\,\mathrm{betac}\,\pi\,(h-3r)(-2-p+r)\,t\,\mathrm{Phi}[h,p,-2+r,p] + i\,(h^3-9h^2(2+r) - 27r(-16+6r+r^2)+9h(-16+12r+3r^2)+144\pi^2(6+h+3r)\,t^2\mathrm{Abs}[\mathrm{beta}]^2)\,\mathrm{Phi}[h,p,r,p] + 36\,\mathrm{beta}\,\pi\,(2+p+r)(h-3\times(8+r))\,t\,\mathrm{Phi}[h,p,2+r,p]\big) + 36\,t\big(6\,\mathrm{betac}\,\pi\,(2+p-r)\,t\,\mathrm{Phi}[h,p,-2+r,p]\, g^{(0,1)}[r,t] + 6\,\mathrm{beta}\,\pi\,(2+p+r)\,t\,\mathrm{Phi}[h,p,2+r,p]\, g^{(0,1)}[r,t] + i\,(6+h-3r)\,\mathrm{Phi}[h,p,r,p]\left(3 g^{(0,1)}[r,t]-t\, g^{(0,2)}[r,t]\right)\big)\big)\Big\}$$

In the various terms we carry out a substitution of the implicit summation variable r to get the same eigenfunction Phi[h,p,r,p] in all terms.
Then we consider the term for one value of r, and divide out the common factors.

In[ ]:=
```
eieq[[1]] // compr
ei2 = % (chbt Phi[h, p, r, p])^(-1) // Simplify
```

Out[ ]=
$$\frac{1}{12}\mathrm{chbt}\,\mathrm{Phi}[h,p,r,p]\big(-24\, i\,\mathrm{beta}\,\pi\,(p+r)\,t\, g[-2+r,t] + (48+h^2-4j^2-12nu^2-6hr+9r^2-48\pi^2t^2\mathrm{Abs}[\mathrm{beta}]^2)\, g[r,t] + 12\,t\,(2\, i\,\mathrm{betac}\,\pi\,(p-r)\, g[2+r,t]-3 g^{(0,1)}[r,t]+t\, g^{(0,2)}[r,t])\big)$$

Out[ ]=
$$-2\, i\,\mathrm{beta}\,\pi\,(p+r)\,t\, g[-2+r,t] + \frac{1}{12}\times(48+h^2-4j^2-12nu^2-6hr+9r^2-48\pi^2t^2\mathrm{Abs}[\mathrm{beta}]^2)\, g[r,t] + t\,(2\, i\,\mathrm{betac}\,\pi\,(p-r)\, g[2+r,t]-3 g^{(0,1)}[r,t]+t\, g^{(0,2)}[r,t])$$

In[∘]:=
```
eieq[[2]] // compr
ei3 = % (chbt Phi[h, p, r, p])^(-1) // Simplify
```

Out[∘]=
$$\frac{1}{72}\,\text{chbt Phi}[h,p,r,p]\,\Big(36\,i\,\text{beta}\,\pi\,(p+r)\,(h-3\times(6+r))\,t\,g[-2+r,t]-\big(-864+h^3+72\,j^2-8\,j^3+216\,\text{nu}^2+72\,j\,\text{nu}^2+432\,r-162\,r^2-27\,r^3-9\,h^2\,(2+r)+9\,h\,(-16+12\,r+3\,r^2)+144\,\pi^2\,(6+h+3\,r)\,t^2\,\text{Abs[beta]}^2\big)\,g[r,t]+36\,i\,t\,\big(\text{betac}\,\pi\,(-p+r)\,(h-3\times(2+r))\,g[2+r,t]+6\,\text{beta}\,\pi\,(p+r)\,t\,g^{(0,1)}[-2+r,t]+18\,i\,g^{(0,1)}[r,t]+3\,i\,h\,g^{(0,1)}[r,t]-9\,i\,r\,g^{(0,1)}[r,t]+6\,\text{betac}\,p\,\pi\,t\,g^{(0,1)}[2+r,t]-6\,\text{betac}\,\pi\,r\,t\,g^{(0,1)}[2+r,t]-6\,i\,t\,g^{(0,2)}[r,t]-i\,h\,t\,g^{(0,2)}[r,t]+3\,i\,r\,t\,g^{(0,2)}[r,t]\big)\Big)$$

Out[∘]=
$$\frac{1}{72}\times\Big(36\,i\,\text{beta}\,\pi\,(p+r)\,(h-3\times(6+r))\,t\,g[-2+r,t]-\big(-864+h^3+72\,j^2-8\,j^3+216\,\text{nu}^2+72\,j\,\text{nu}^2+432\,r-162\,r^2-27\,r^3-9\,h^2\,(2+r)+9\,h\,(-16+12\,r+3\,r^2)+144\,\pi^2\,(6+h+3\,r)\,t^2\,\text{Abs[beta]}^2\big)\,g[r,t]+36\,i\,t\,\big(\text{betac}\,\pi\,(-p+r)\,(h-3\times(2+r))\,g[2+r,t]+6\,\text{beta}\,\pi\,(p+r)\,t\,g^{(0,1)}[-2+r,t]+18\,i\,g^{(0,1)}[r,t]+3\,i\,h\,g^{(0,1)}[r,t]-9\,i\,r\,g^{(0,1)}[r,t]+6\,\text{betac}\,p\,\pi\,t\,g^{(0,1)}[2+r,t]-6\,\text{betac}\,\pi\,r\,t\,g^{(0,1)}[2+r,t]-6\,i\,t\,g^{(0,2)}[r,t]-i\,h\,t\,g^{(0,2)}[r,t]+3\,i\,r\,t\,g^{(0,2)}[r,t]\big)\Big)$$

In[∘]:=
```
Coefficient[{ei2, ei3}, g^(0,2)[r, t]]
```

Out[∘]=
$$\Big\{t^2,\ \frac{1}{72}\times(216\,t^2+36\,h\,t^2-108\,r\,t^2)\Big\}$$

In[∘]:=
```
(216 t^2 + 36 h t^2 - 108 r t^2) / 72 // Expand
```

Out[∘]=
$$3\,t^2+\frac{h\,t^2}{2}-\frac{3\,r\,t^2}{2}$$

A simplification is possible replacing the contribution of Δ_3 by a linear combination of both contributions.

In[∘]:=
```
2 ei3 - (6 + h - 3 r) ei2 // FullSimplify
ei3a = % /. beta Conjugate[beta] → Abs[beta]^2 // Simplify
```

Out[∘]=
$$3\,i\,\text{beta}\,\pi\,(-2+h-3\,r)\,(p+r)\,t\,g[-2+r,t]-\frac{1}{9}\,\big((h-2\,j-3\,r)\,(h+j+3\,\text{nu}-3\,r)\,(h+j-3\,(\text{nu}+r))+216\,\text{beta}\,\pi^2\,r\,t^2\,\text{Conjugate[beta]}\big)\,g[r,t]+3\,i\,\pi\,t\,\big(\text{betac}\,(2+h-3\,r)\,(-p+r)\,g[2+r,t]+2\,t\,(\text{beta}\,(p+r)\,g^{(0,1)}[-2+r,t]+\text{betac}\,(p-r)\,g^{(0,1)}[2+r,t])\big)$$

Out[∘]=
$$3\,i\,\text{beta}\,\pi\,(-2+h-3\,r)\,(p+r)\,t\,g[-2+r,t]-\frac{1}{9}\,\big((h-2\,j-3\,r)\,(h+j+3\,\text{nu}-3\,r)\,(h+j-3\,(\text{nu}+r))+216\,\pi^2\,r\,t^2\,\text{Abs[beta]}^2\big)\,g[r,t]+3\,i\,\pi\,t\,\big(\text{betac}\,(2+h-3\,r)\,(-p+r)\,g[2+r,t]+2\,t\,(\text{beta}\,(p+r)\,g^{(0,1)}[-2+r,t]+\text{betac}\,(p-r)\,g^{(0,1)}[2+r,t])\big)$$

In[ ]:=
```
Coefficient[-9 ei3a, g[r, t]]
Coefficient[-9 ei3a, g[r + 2, t]] // Simplify
Coefficient[-9 ei3a, g^(0,1)[r + 2, t]] // Simplify
Coefficient[-9 ei3a, g[r - 2, t]] // Simplify
Coefficient[-9 ei3a, g^(0,1)[r - 2, t]] // Simplify
```

Out[ ]= $(h - 2 j - 3 r) (h + j + 3 nu - 3 r) (h + j - 3 (nu + r)) + 216 \pi^2 r t^2 \text{Abs}[beta]^2$

Out[ ]= $27 i\, betac\, \pi (2 + h - 3 r) (p - r) t$

Out[ ]= $-54 i\, betac\, \pi (p - r) t^2$

Out[ ]= $-27 i\, beta\, \pi (-2 + h - 3 r) (p + r) t$

Out[ ]= $-54 i\, beta\, \pi (p + r) t^2$

The factor - 9 gives agreement with **efeqt** (N-trivial case) for **beta**=0.

In[ ]:=
```
{ei2, -9 ei3a} /. beta → 0 /. betac → 0
% == efeqt[h, p, r, g] // Simplify
```

Out[ ]= $\left\{\frac{1}{12} \times (48 + h^2 - 4 j^2 - 12 nu^2 - 6 h r + 9 r^2) g[r, t] + t \left(-3 g^{(0,1)}[r, t] + t g^{(0,2)}[r, t]\right),\right.$
$\left.(h - 2 j - 3 r) (h + j + 3 nu - 3 r) (h + j - 3 (nu + r)) g[r, t]\right\}$

Out[ ]= True

Check of routine in 11h.

In[ ]:=
```
efeqa[h, p, r, g, beta] == {ei2, -9 ei3a} /. betac → Conjugate[beta] // Simplify
```

Out[ ]= True