## 12c. Non-abelian eigenfunction modules

See §3.4.2.3

Eigenfunction equation, separately for $\varepsilon$=1 and -1

In[ ]:=
```
Clear[f, nu, j, ell, m, h]
{deqp, relp} = efeqn[h, 0, 0, f, ell, m, 1] /. p → 0 // Simplify
{deqm, relm} = efeqn[h, 0, 0, f, ell, m, -1] /. p → 0 // Simplify
```

Out[ ]=
$$\left\{\frac{1}{12}\left(h^2 - 24\,\text{ell}\,h\,\pi\,t^2 - 4\left(j^2 + 3\times\left(-4 + \text{nu}^2 + 4\,\text{ell}\,\pi\,t^2 + 4\,\text{ell}^2\,\pi^2\,t^4\right)\right) - 96\,m\,\pi\,t^2\,\text{Abs[ell]}\right) f[0, t] + t\left(-3\,f^{(0,1)}[0, t] + t\,f^{(0,2)}[0, t]\right),\ -\frac{1}{9}(h - 2\,j)\left(h^2 + 2\,h\,j + j^2 - 9\,\text{nu}^2\right) f[0, t]\right\}$$

Out[ ]=
$$\left\{\frac{1}{12}\left(h^2 - 24\,\text{ell}\,h\,\pi\,t^2 - 4\left(j^2 + 3\times\left(-4 + \text{nu}^2 + 4\,\text{ell}\,\pi\,t^2 + 4\,\text{ell}^2\,\pi^2\,t^4\right)\right) - 96\times(1 + m)\,\pi\,t^2\,\text{Abs[ell]}\right) f[0, t] + t\left(-3\,f^{(0,1)}[0, t] + t\,f^{(0,2)}[0, t]\right),\ -\frac{1}{9}(h - 2\,j)\left(h^2 + 2\,h\,j + j^2 - 9\,\text{nu}^2\right) f[0, t]\right\}$$

In[ ]:=
```
relp // Factor
relm // Factor
```

Out[ ]=
$$-\frac{1}{9}(h - 2\,j)(h + j - 3\,\text{nu})(h + j + 3\,\text{nu})\,f[0, t]$$

Out[ ]=
$$-\frac{1}{9}(h - 2\,j)(h + j - 3\,\text{nu})(h + j + 3\,\text{nu})\,f[0, t]$$

In both cases we can choose h=2j

In[ ]:=
```
{dqp, dqm} = {deqp, deqm} /. h → 2 j /. ell ^ 2 → Abs[ell] ^ 2 // Simplify
```

Out[ ]=
$$\left\{-\left(\left(-4 + \text{nu}^2 + 4\,\text{ell}\,\pi\,t^2 + 4\,\text{ell}\,j\,\pi\,t^2 + 8\,m\,\pi\,t^2\,\text{Abs[ell]} + 4\,\pi^2\,t^4\,\text{Abs[ell]}^2\right) f[0, t]\right) + t\left(-3\,f^{(0,1)}[0, t] + t\,f^{(0,2)}[0, t]\right),\ -\left(\left(-4 + \text{nu}^2 + 4\,\text{ell}\,\pi\,t^2 + 4\,\text{ell}\,j\,\pi\,t^2 + 8\times(1 + m)\,\pi\,t^2\,\text{Abs[ell]} + 4\,\pi^2\,t^4\,\text{Abs[ell]}^2\right) f[0, t]\right) + t\left(-3\,f^{(0,1)}[0, t] + t\,f^{(0,2)}[0, t]\right)\right\}$$

Parameters

In[ ]:=
```
Clear[kap]
dqp1 = dqp /. m → -kap - ( j + 1) / 2 /. {8 ell → 8 Abs[ell], 16 ell → 16 Abs[ell]} // Simplify
dqm1 = dqm /. m → -kap - (- j + 1) / 2 /. {8 ell → -8 Abs[ell], 16 ell → -16 Abs[ell]} // Simplify
```

Out[ ]=
$$-\left(\left(-4 + \text{nu}^2 + 4\,\text{ell}\,\pi\,t^2 + 4\,\text{ell}\,j\,\pi\,t^2 - 4\times(1 + j + 2\,\text{kap})\,\pi\,t^2\,\text{Abs[ell]} + 4\,\pi^2\,t^4\,\text{Abs[ell]}^2\right) f[0, t]\right) + t\left(-3\,f^{(0,1)}[0, t] + t\,f^{(0,2)}[0, t]\right)$$

Out[ ]=
$$-\left(\left(-4 + \text{nu}^2 + 4\,\text{ell}\,\pi\,t^2 + 4\,\text{ell}\,j\,\pi\,t^2 + 4\times(1 + j - 2\,\text{kap})\,\pi\,t^2\,\text{Abs[ell]} + 4\,\pi^2\,t^4\,\text{Abs[ell]}^2\right) f[0, t]\right) + t\left(-3\,f^{(0,1)}[0, t] + t\,f^{(0,2)}[0, t]\right)$$

Expected form of solutions

In[ ]:=
```
Clear[wh, tau]
ff = t wh[2 Pi Abs[ell] t ^ 2];
{dqp2 , dqm2} = {dqp1 , dqm1} /. {f[0, t] → ff , f^(0,ee_)[0, t] :→ D[ff , {t, ee}]} /.
    t → Sqrt[tau] / Sqrt[2 Pi Abs[ell]] // Simplify
```

Out[ ]= $\left\{\left(\sqrt{\text{tau}}\ \left(-\left(\left(2\ \text{ell}\ (1+\text{j})\ \text{tau}+\left(-1+\text{nu}^2-2\times(1+\text{j}+2\ \text{kap})\ \text{tau}+\text{tau}^2\right)\text{Abs}[\text{ell}]\right)\text{wh}[\text{tau}]\right)+4\ \text{tau}^2\ \text{Abs}[\text{ell}]\ \text{wh}''[\text{tau}]\right)\right)\Big/\left(\sqrt{2\pi}\ \text{Abs}[\text{ell}]^{3/2}\right),\ \left(\sqrt{\text{tau}}\ \left(-\left(\left(2\ \text{ell}\ (1+\text{j})\ \text{tau}+\left(-1+\text{nu}^2+2\times(1+\text{j}-2\ \text{kap})\ \text{tau}+\text{tau}^2\right)\text{Abs}[\text{ell}]\right)\text{wh}[\text{tau}]\right)+4\ \text{tau}^2\ \text{Abs}[\text{ell}]\ \text{wh}''[\text{tau}]\right)\right)\Big/\left(\sqrt{2\pi}\ \text{Abs}[\text{ell}]^{3/2}\right)\right\}$

In[ ]:=
```
Solve[dqp2 == 0, wh''[tau]][[1]]
ftp = wh''[tau] / wh[tau] /. % /. Abs[ell] → ell // Simplify
Solve[dqm2 == 0, wh''[tau]][[1]]
ftm = wh''[tau] / wh[tau] /. % /. Abs[ell] → -ell // Simplify
```

Out[ ]= $\{\text{wh}''[\text{tau}]\to((2\ \text{ell tau}+2\ \text{ell j tau}-\text{Abs}[\text{ell}]+\text{nu}^2\ \text{Abs}[\text{ell}]-2\ \text{tau Abs}[\text{ell}]-2\ \text{j tau Abs}[\text{ell}]-4\ \text{kap tau Abs}[\text{ell}]+\text{tau}^2\ \text{Abs}[\text{ell}])\ \text{wh}[\text{tau}])/(4\ \text{tau}^2\ \text{Abs}[\text{ell}])\}$

Out[ ]= $\frac{-1+\text{nu}^2-4\ \text{kap tau}+\text{tau}^2}{4\ \text{tau}^2}$

Out[ ]= $\{\text{wh}''[\text{tau}]\to((2\ \text{ell tau}+2\ \text{ell j tau}-\text{Abs}[\text{ell}]+\text{nu}^2\ \text{Abs}[\text{ell}]+2\ \text{tau Abs}[\text{ell}]+2\ \text{j tau Abs}[\text{ell}]-4\ \text{kap tau Abs}[\text{ell}]+\text{tau}^2\ \text{Abs}[\text{ell}])\ \text{wh}[\text{tau}])/(4\ \text{tau}^2\ \text{Abs}[\text{ell}])\}$

Out[ ]= $\frac{-1+\text{nu}^2-4\ \text{kap tau}+\text{tau}^2}{4\ \text{tau}^2}$

In[ ]:=
```
Clear[X]
ft = {ftp , ftm} /. tau → 1 / X // Simplify ;
c0 = ft /. X → 0
c1 = Coefficient [ft, X] // Simplify
c2 = Coefficient [ft, X ^ 2] // Simplify
```

Out[ ]= $\left\{\frac{1}{4},\ \frac{1}{4}\right\}$

Out[ ]= $\{-\text{kap},\ -\text{kap}\}$

Out[ ]= $\left\{\frac{1}{4}\times(-1+\text{nu}^2),\ \frac{1}{4}\times(-1+\text{nu}^2)\right\}$

This confirms that we have the Whittaker differential equation with the parameters indicated in the text.