14b. Non-abelian case

First upward shift operator

We first establish the kernel relations, separately for eps = 1 and eps = -1

In[ ]:=
```
Clear[relp, relm]
F = tht[m[h, r]] × f[r, t] × Phi[h, p, r, p]
s31 = (sh[3, 1, F, subnab] // compr) /. r → r - 1 /. m[h, r - 2] → m[h, r] - eps // Simplify
relp[r_] = Solve[(s31 /. eps → 1) == 0, f[r, t]][[1]] // Simplify
relm[r_] = Solve[(s31 /. eps → -1) == 0, f[r, t]][[1]] // Simplify
```

Out[ ]= f[r, t] × Phi[h, p, r, p] × tht[m[h, r]]

Out[ ]= $\frac{1}{8\times(1+p)}$ Phi[3 + h, 1 + p, -1 + r, 1 + p] $\Big(2\,i\,\sqrt{2\pi}\,(2+p-r)\,t\,\sqrt{\text{Abs[ell]}}$ f[r, t] $\Big((1+\text{eps})\sqrt{\text{m[h, r]}}$ tht[-1 + m[h, r]] + (-1 + eps) $\sqrt{1+\text{m[h, r]}}$ tht[1 + m[h, r]]$\Big)$ + (p + r) $(2+h+2p-r+4\,\text{ell}\,\pi\,t^2)$ f[-2 + r, t] × tht[-eps + m[h, r]] + 2 (p + r) t tht[-eps + m[h, r]] $f^{(0,1)}$[-2 + r, t]$\Big)$

Out[ ]= {f[r, t] → $\big(i\,(p+r)\,((2+h+2p-r+4\,\text{ell}\,\pi\,t^2)$ f[-2 + r, t] + 2 t $f^{(0,1)}$[-2 + r, t]$)\big)\Big/\Big(4\sqrt{2\pi}\,(2+p-r)\,t\,\sqrt{\text{Abs[ell]}}\,\sqrt{\text{m[h, r]}}\Big)$}

Out[ ]= {f[r, t] → $-\Big(\big(i\,(p+r)\,((2+h+2p-r+4\,\text{ell}\,\pi\,t^2)$ f[-2 + r, t] + 2 t $f^{(0,1)}$[-2 + r, t]$)\big)\Big/\Big(4\sqrt{2\pi}\,(2+p-r)\,t\,\sqrt{\text{Abs[ell]}}\,\sqrt{1+\text{m[h, r]}}\Big)\Big)$}

Lowest component

In[ ]:=
```
tht[m[h, -p]] × f[-p, t] × Phi[h, p, -p, p]
sh[3, 1, %, subnab]
Coefficient[%, Phi[h + 3, p + 1, -p - 1, p + 1]] // Simplify
% /. eps → {1, -1} // Simplify
```

Out[ ]= f[-p, t] × Phi[h, p, -p, p] × tht[m[h, -p]]

Out[ ]= $\frac{1}{4}\times\Big(2\,i\,\sqrt{2\pi}\;t\;\sqrt{\text{Abs[ell]}}\;\text{f[-p, t]}\times\text{Phi[3 + h, 1 + p, -1 - p, 1 + p]}\Big((1+\text{eps})\sqrt{\text{m[h, -p]}}\;\text{tht[-1 + m[h, -p]]} + (-1+\text{eps})\sqrt{1+\text{m[h, -p]}}\;\text{tht[1 + m[h, -p]]}\Big) + \frac{1}{1+p}\,\text{Phi[3 + h, 1 + p, 1 - p, 1 + p]}\times\text{tht[m[h, -p]]}\Big((h + 3p + 4\,\text{ell}\,\pi\,t^2)\,\text{f[-p, t]} + 2\,t\,f^{(0,1)}\text{[-p, t]}\Big)\Big)$

Out[ ]= $i\sqrt{\frac{\pi}{2}}\;t\;\sqrt{\text{Abs[ell]}}\;\text{f[-p, t]}\Big((1+\text{eps})\sqrt{\text{m[h, -p]}}\;\text{tht[-1 + m[h, -p]]} + (-1+\text{eps})\sqrt{1+\text{m[h, -p]}}\;\text{tht[1 + m[h, -p]]}\Big)$

Out[ ]= $\Big\{i\sqrt{2\pi}\;t\;\sqrt{\text{Abs[ell]}}\;\text{f[-p, t]}\sqrt{\text{m[h, -p]}}\;\text{tht[-1 + m[h, -p]]},\; -i\sqrt{2\pi}\;t\;\sqrt{\text{Abs[ell]}}\;\text{f[-p, t]}\sqrt{1+\text{m[h, -p]}}\;\text{tht[1 + m[h, -p]]}\Big\}$

This may be zero for non-zero f_{-p} only if eps = 1 and $m[h, -p] = 0$.

So to consider further the case ε=1 and -p ≤ $r_0 < p$.

```
In[ ]:= f2p = f[r0 + 2, t] /. relp[r0 + 2] /. m[h, r0 + 2] → 1 // Simplify
eia =
 efeqn[h, p, r0, f, ell, 0, 1] /. f^(0,1)[2 + r0, t] → D[f2p, t] /. f[2 + r0, t] → f2p /. m[h, r0] → 0 //
  Simplify
Coefficient[%, f^(0,2)[r0, t]] // Simplify
```

Out[ ]= $(i(2+p+r0)((h+2p-r0+4\,ell\,\pi t^2) f[r0, t]+2t f^{(0,1)}[r0, t]))\Big/\left(4\sqrt{2\pi}(p-r0)t\sqrt{Abs[ell]}\right)$

Out[ ]= $\Big\{\frac{1}{12}(h^2-4j^2+6h(2+p-4\,ell\,\pi t^2)+3\times(16-4nu^2+4p^2-4r0+r0^2-16\,ell^2\pi^2t^4+2p(4+r0+4\,ell\,\pi t^2)))\, f[r0, t]+t((-1+p+r0)f^{(0,1)}[r0, t]+t f^{(0,2)}[r0, t]),$

$-\frac{1}{9}(h^3-2j^3+18j\,nu^2-9h^2r0+9j^2r0-3h(j^2+3nu^2-9r0^2)-27(-nu^2r0+r0^3+4\,ell\,p(2+p)\pi t^2-8\,ell\,\pi r0 t^2-4\,ell\,\pi r0^2t^2))f[r0, t]+\frac{3}{4}\times(2+p+r0)(-((h^2-4r0+3r0^2-16\,ell\,\pi r0t^2+16\,ell^2\pi^2t^4+2h(2+p-2r0+4\,ell\,\pi t^2)+p(8-6r0+8\,ell\,\pi t^2))f[r0, t])+4t((-1+p+r0)f^{(0,1)}[r0, t]+t f^{(0,2)}[r0, t]))\Big\}$

Out[ ]= $\{t^2, 3\times(2+p+r0)t^2\}$

```
In[ ]:= 3 × (2 + p + r0) eia[[1]] - eia[[2]] // Factor
```

Out[ ]= $\frac{1}{9}\times(6+h-2j+3p)\times(6+h+j-3nu+3p)\times(6+h+j+3nu+3p) f[r0, t]$

Case eps = 1 and r_0 = p.

```
In[ ]:= tht[0] × f[p, t] × Phi[h, p, p, p]
sh[3, 1, %, subnab] /. eps → 1 // Simplify
fp = f[p, t] /. DSolve[% == 0, f[p, t], t][[1]]
```

Out[ ]= f[p, t] × Phi[h, p, p, p] × tht[0]

Out[ ]= $\frac{1}{4}$ Phi[3 + h, 1 + p, 1 + p, 1 + p] × tht[0] $((h+p+4\,ell\,\pi t^2) f[p, t]+2t f^{(0,1)}[p, t])$

Out[ ]= $e^{\frac{1}{2}\times(-2\,ell\,\pi t^2-(h+p)\,Log[t])}\,c_1$

```
In[ ]:= efeqn[h, p, p, f, ell, 0, 1] /. f^(0,ee_)[p, t] :→ D[fp, {t, ee}] /. f[p, t] → fp // Simplify
% /. {j → (h - 3 p) / 2, nu → 2 + (h + p) / 2} // Simplify
```

Out[ ]= $\Big\{\frac{1}{3}e^{-ell\,\pi t^2}(6h+h^2-j^2+3\times(4-nu^2+2p+p^2))\,t^{\frac{1}{2}(-h-p)}c_1,$

$-\frac{1}{9}e^{-ell\,\pi t^2}(h-2j-3p)(h^2+j^2-9nu^2+2h(j-3p)-6jp+9p^2)\,t^{\frac{1}{2}(-h-p)}c_1\Big\}$

Out[ ]= {0, 0}

The K-type is on the boundary of the sector Sect[j].

In[ ]:=
```
{j1, nu1} = S1[{(h - 3 p) / 2, 2 + (h + p) / 2}]
{j2, nu2} = S2[{(h - 3 p) / 2, 2 + (h + p) / 2}]
```

Out[ ]= $\left\{\frac{1}{2}\times(6+h+3p), \frac{1}{2}\times(2+h-p)\right\}$

Out[ ]= $\{-3-h, 1+p\}$

In[ ]:=
```
(h + 6 + 3 p - 2 j1)
(h + 6 + 3 p - 2 j2) // Simplify
```

Out[ ]= 0

Out[ ]= $3\times(4+h+p)$

So j'=j1

Second upward shift operator

Kernel relations

In[ ]:=
```
Clear[relp, relm]
F = tht[m[h, r]] × f[r, t] × Phi[h, p, r, p]
sm31 = (sh[-3, 1, F, subnab] // compr) /. r → r + 1 /. m[h, r + 2] → m[h, r] + eps // Simplify
relp[r_] = Solve[(sm31 /. eps → 1) == 0, f[r, t]][[1]] // Simplify
relm[r_] = Solve[(sm31 /. eps → -1) == 0, f[r, t]][[1]] // Simplify
```

Out[ ]= f[r, t] × Phi[h, p, r, p] × tht[m[h, r]]

Out[ ]= $\frac{1}{8\times(1+p)}\text{Phi}[-3+h, 1+p, 1+r, 1+p]\Big(-2\,i\sqrt{2\pi}\,(2+p+r)\,t\sqrt{\text{Abs[ell]}}\;f[r,t]\Big((-1+\text{eps})\sqrt{m[h,r]}\;\text{tht}[-1+m[h,r]]+(1+\text{eps})\sqrt{1+m[h,r]}\;\text{tht}[1+m[h,r]]\Big)+(p-r)\,\text{tht}[\text{eps}+m[h,r]]\,\big((2-h+2p+r-4\,\text{ell}\,\pi\,t^2)\,f[2+r,t]+2\,t\,f^{(0,1)}[2+r,t]\big)\Big)$

Out[ ]= $\left\{f[r,t]\to-\Big(\big(i\,(p-r)\,((2-h+2p+r-4\,\text{ell}\,\pi\,t^2)\,f[2+r,t]+2\,t\,f^{(0,1)}[2+r,t])\big)\Big/\Big(4\sqrt{2\pi}\,(2+p+r)\,t\sqrt{\text{Abs[ell]}}\sqrt{1+m[h,r]}\Big)\Big)\right\}$

Out[ ]= $\left\{f[r,t]\to\big(i\,(p-r)\,((2-h+2p+r-4\,\text{ell}\,\pi\,t^2)\,f[2+r,t]+2\,t\,f^{(0,1)}[2+r,t])\big)\Big/\Big(4\sqrt{2\pi}\,(2+p+r)\,t\sqrt{\text{Abs[ell]}}\sqrt{m[h,r]}\Big)\right\}$

To consider eps=-1 and -p $\le r_0 \le p$

One component f_{-p} and $r_0 = -p$

In[ ]:=
```
tht[0] × f[-p, t] × Phi[h, p, -p, p]
sh[-3, 1, %, subnab] /. eps → -1
fmp = f[-p, t] /. DSolve[% == 0, f[-p, t], t][[1]]
```

Out[ ]= $f[-p, t] \times Phi[h, p, -p, p] \times tht[0]$

Out[ ]= $-\frac{1}{4}(h - p + 4\, ell\, \pi\, t^2)\, f[-p, t] \times Phi[-3 + h, 1 + p, -1 - p, 1 + p] \times tht[0] + \frac{1}{2}\, t\, Phi[-3 + h, 1 + p, -1 - p, 1 + p] \times tht[0]\, f^{(0,1)}[-p, t]$

Out[ ]= $e^{\frac{1}{2}\times(2\, ell\, \pi\, t^2 + (h-p)\, Log[t])}\, c_1$

In[ ]:=
```
eia = efeqn[h, p, -p, f, ell, 0, -1] /. f^(0,ee_)[-p, t] :→ D[fmp, {t, ee}] /. f[-p, t] → fmp /.
    ell → -Abs[ell] // Simplify
```

Out[ ]= $\left\{\frac{1}{3} e^{-\pi t^2 Abs[ell]} (-6 h + h^2 - j^2 + 3 \times (4 - nu^2 + 2 p + p^2))\, t^{\frac{h-p}{2}}\, c_1,\; -\frac{1}{9} e^{-\pi t^2 Abs[ell]} (h - 2 j + 3 p)(h^2 + j^2 - 9\, nu^2 + 6 j p + 9 p^2 + 2 h (j + 3 p))\, t^{\frac{h-p}{2}}\, c_1\right\}$

In[ ]:=
```
jj = (h + 3 p) / 2
nuu = 2 + (p - h) / 2
eia /. {j → jj, nu → nuu} // Simplify
```

Out[ ]= $\frac{1}{2}(h + 3 p)$

Out[ ]= $2 + \frac{1}{2}(-h + p)$

Out[ ]= {0, 0}

In[ ]:=
```
{j1, nu1} = S1[{jj, nuu}]
{j2, nu2} = S2[{jj, nuu}]
```

Out[ ]= {3 - h, 1 + p}

Out[ ]= $\left\{\frac{1}{2}(h - 3 \times (2 + p)),\; \frac{1}{2} \times (2 - h - p)\right\}$

In[ ]:=
```
h - 6 - 3 p - 2 j1 // Simplify
h - 6 - 3 p - 2 j2 // Simplify
```

Out[ ]= $3 \times (-4 + h - p)$

Out[ ]= 0

So we can take j'=j2

Cases - p < r0≤p

Consider the eigenfunction equations for r=r0

In[ ]:= **fr0m2 = f[r0 - 2, t] /. relm[r0 - 2] /. m[h, -2 + r0] → 1**

Out[ ]= $\left(i\,(2+p-r0)\left(\left(-h+2\,p+r0-4\,ell\,\pi\,t^2\right) f[r0,\,t]+2\,t\,f^{(0,1)}[r0,\,t]\right)\right)\Big/\left(4\,\sqrt{2\,\pi}\,(p+r0)\,t\,\sqrt{Abs[ell]}\right)$

In[ ]:= **eia = efeqn[h, p, r0, f, ell, 0, -1] /. $f^{(0,1)}$[r0 - 2, t] → D[fr0m2, t] /. f[-2 + r0, t] → fr0m2 /. ell → -Abs[ell] // Simplify**

Out[ ]= $\left\{\frac{1}{12}\times\left(48+h^2-4\,j^2-12\,nu^2+24\,p+12\,p^2-6\,h\,(2+p)+12\,r0-6\,p\,r0+3\,r0^2+24\,(h+p)\,\pi\,t^2\,Abs[ell]-48\,\pi^2\,t^4\,Abs[ell]^2\right) f[r0,\,t]+t\left((-1+p-r0)\,f^{(0,1)}[r0,\,t]+t\,f^{(0,2)}[r0,\,t]\right),\right.$
$-\frac{1}{9}\left((h-2\,j-3\,r0)\left(h^2+j^2-9\,nu^2+2\,h\,(j-3\,r0)-6\,j\,r0+9\,r0^2\right)+108\,\pi\left(2\,p+p^2-(-2+r0)\,r0\right) t^2\,Abs[ell]\right) f[r0,\,t]+\frac{3}{4}\times(2+p-r0)$
$\left.\left(\left((-4+h-3\,r0)\,(h-2\,p-r0)+8\,\pi\,(-h+p+2\,r0)\,t^2\,Abs[ell]+16\,\pi^2\,t^4\,Abs[ell]^2\right) f[r0,\,t]+4\,t\left((1-p+r0)\,f^{(0,1)}[r0,\,t]-t\,f^{(0,2)}[r0,\,t]\right)\right)\right\}$

In[ ]:= **Coefficient[eia, $f^{(0,2)}$[r0, t]] // Simplify**

Out[ ]= $\left\{t^2,\,-3\times(2+p-r0)\,t^2\right\}$

In[ ]:= **eia⟦2⟧ + (6 + 3 p - 3 r0) eia⟦1⟧ // Factor**

Out[ ]= $-\frac{1}{9}\times(-6+h-2\,j-3\,p)\times(-6+h+j-3\,nu-3\,p)\times(-6+h+j+3\,nu-3\,p)\,f[r0,\,t]$