18c. Eigenfunction?

Does this solution satisfy the eigenfunction equations?

Try first the K-Bessel function

```
In[ ]:= Clear[ff]
ff[r_, t] = t^(2 + p) ( -I beta / Abs[beta])^((r + p) / 2) BesselK[(h - r) / 2, 2 Pi Abs[beta] t]
```

Out[ ]= $(-i\,\text{beta})^{\frac{p+r}{2}}\, t^{2+p} \left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}} \text{BesselK}\left[\frac{h-r}{2}, 2\,\pi\, t\,\text{Abs[beta]}\right]$

```
In[ ]:= efeqa[h, p, r, f, beta] /. f[rr_, t] :> ff[rr, t] /.
   f^(0,ee_)[rr_, t] :> D[ff[rr, t], {t, ee}] // Simplify
eqs = (% /. Besseldn[1/2 × (4 + h - r)] /. Besselup[1/2 × (-4 + h - r)] // Simplify) /.
     Besseldn[1/2 × (2 + h - r)] /. Conjugate[beta] → Abs[beta]^2 / beta // Simplify
```

Out[▫]=

$$\Big\{\frac{1}{6}\,(-i\,\text{beta})^{\frac{p+r}{2}}\;t^{2+p}\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{1}{2}\times(2+p+r)}$$

$$\Big(\left(2\,h^2-2\,j^2-6\,\text{nu}^2+6\,p^2+h\,(3-6\,r)-3\,r+6\,r^2\right)\text{Abs[beta] BesselK}\Big[\frac{h-r}{2},\,2\,\pi\,t\,\text{Abs[beta]}\Big]-$$

$$6\,\pi\,t\,\text{Abs[beta]}^2\,\Big((-1+2\,p)\,\text{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\,2\,\pi\,t\,\text{Abs[beta]}\Big]+$$

$$(1-2\,r)\,\text{BesselK}\Big[\frac{1}{2}\times(2+h-r),\,2\,\pi\,t\,\text{Abs[beta]}\Big]\Big)+$$

$$12\,\text{beta}\,\pi\,(p-r)\,t\,\text{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\,2\,\pi\,t\,\text{Abs[beta]}\Big]\,\text{Conjugate[beta]}\Big),\;(-i\,\text{beta})^{\frac{p+r}{2}}$$

$$t^{2+p}\left(\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}}\left((h-2\,j-3\,r)\,(h+j+3\,\text{nu}-3\,r)\,(h+j-3\,(\text{nu}+r))+216\,\pi^2\,r\,t^2\,\text{Abs[beta]}^2\right)\right.$$

$$\text{BesselK}\Big[\frac{h-r}{2},\,2\,\pi\,t\,\text{Abs[beta]}\Big]-$$

$$27\,\pi\,(p+r)\,(2-h+3\,r)\,t\,\text{Abs[beta]}^{\frac{1}{2}\times(2-p-r)}\,\text{BesselK}\Big[\frac{1}{2}\times(2+h-r),\,2\,\pi\,t\,\text{Abs[beta]}\Big]-\frac{1}{\text{Abs[beta]}}$$

$$27\,\pi\,t\,\Big(-2\,(p+r)\,\text{Abs[beta]}^{2-\frac{p}{2}-\frac{r}{2}}\,\Big((2+p)\,\text{BesselK}\Big[\frac{1}{2}\times(2+h-r),\,2\,\pi\,t\,\text{Abs[beta]}\Big]-\pi\,t\,\text{Abs[beta]}$$

$$\Big(\text{BesselK}\Big[\frac{h-r}{2},\,2\,\pi\,t\,\text{Abs[beta]}\Big]+\text{BesselK}\Big[\frac{1}{2}\times(4+h-r),\,2\,\pi\,t\,\text{Abs[beta]}\Big]\Big)\Big)+$$

$$\text{beta}\,(2+h-3\,r)\,(-p+r)\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}}\text{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\,2\,\pi\,t\,\text{Abs[beta]}\Big]$$

$$\text{Conjugate[beta]}+2\,\text{beta}\,(p-r)\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}}$$

$$\Big((2+p)\,\text{BesselK}\Big[\frac{1}{2}\times(-2+h-r),\,2\,\pi\,t\,\text{Abs[beta]}\Big]-\pi\,t\,\text{Abs[beta]}\,\Big(\text{BesselK}\Big[\frac{1}{2}\times(-4+h-r),$$

$$2\,\pi\,t\,\text{Abs[beta]}\Big]+\text{BesselK}\Big[\frac{h-r}{2},\,2\,\pi\,t\,\text{Abs[beta]}\Big]\Big)\Big)\,\text{Conjugate[beta]}\Big)\Big)\Big\}$$

Out[▫]=

$$\Big\{\frac{1}{3}\,(-i\,\text{beta})^{\frac{p+r}{2}}\,(h^2-j^2-3\,\text{nu}^2+3\,p^2)\,t^{2+p}\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}}\text{BesselK}\Big[\frac{h-r}{2},\,2\,\pi\,t\,\text{Abs[beta]}\Big],$$

$$(-i\,\text{beta})^{\frac{p+r}{2}}\,\left(h^3-2\,j^3+18\,j\,\text{nu}^2-3\,h\,(j^2+3\,\text{nu}^2-9\,p^2)-9\,h^2\,r+9\,j^2\,r+27\,(\text{nu}^2-p^2)\,r\right)$$

$$t^{2+p}\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}}\text{BesselK}\Big[\frac{h-r}{2},\,2\,\pi\,t\,\text{Abs[beta]}\Big]\Big\}$$

In[▫]:=

```
eqs /. j → -h /. nu → p // Simplify
```

Out[▫]=

```
{0, 0}
```

Indeed, we have an eigenfunction with spectral parameters (-h,p).

Next the I-Bessel function.

In[ ]:=
```
Clear[ff]
ff[r_, t] = t^(2 + p) ( I beta / Abs[beta])^((r + p) / 2) BesselI[(h - r) / 2, 2 Pi Abs[beta] t]
```

Out[ ]= $(i\,\mathrm{beta})^{\frac{p+r}{2}}\, t^{2+p} \left(\frac{1}{\mathrm{Abs[beta]}}\right)^{\frac{p+r}{2}} \mathrm{BesselI}\left[\frac{h-r}{2}, 2\,\pi\, t\, \mathrm{Abs[beta]}\right]$

```
In[ ]:= efeqa[h, p, r, f, beta] /. f[rr_, t] :> ff[rr, t] /.
          f^(0,ee_)[rr_, t] :> D[ff[rr, t], {t, ee}] // Simplify
        eqs = (% /. Besseldn[1/2 (4 + h - r)] /. Besselup[1/2 (-4 + h - r)] // Simplify) /.
            Besseldn[1/2 (2 + h - r)] /. Conjugate[beta] -> Abs[beta]^2 / beta // Simplify
```

Out[ ]=
$$\left\{\frac{1}{3}(i\,\text{beta})^{\frac{p+r}{2}}\,t^{2+p}\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{1}{2}(2+p+r)}\Big((h^2-j^2-3\,h\,r+3(-nu^2+p^2+r^2))\,\text{Abs[beta] BesselI}\left[\frac{h-r}{2},\,2\pi\,t\,\text{Abs[beta]}\right]+6\pi\,t\,\text{Abs[beta]}^2\left(p\,\text{BesselI}\left[\frac{1}{2}(-2+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]-r\,\text{BesselI}\left[\frac{1}{2}(2+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]\right)+6\,\text{beta}\,\pi(-p+r)\,t\,\text{BesselI}\left[\frac{1}{2}(-2+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]\text{Conjugate[beta]}\Big),\ (i\,\text{beta})^{\frac{p+r}{2}}\,t^{2+p}\Big(\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}}((h-2\,j-3\,r)(h+j+3\,nu-3\,r)(h+j-3(nu+r))+216\pi^2\,r\,t^2\,\text{Abs[beta]}^2)\,\text{BesselI}\left[\frac{h-r}{2},\,2\pi\,t\,\text{Abs[beta]}\right]+27\pi(p+r)(2-h+3\,r)\,t\,\text{Abs[beta]}^{\frac{1}{2}(2-p-r)}\,\text{BesselI}\left[\frac{1}{2}(2+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]+27\pi\,t\Big(-2(p+r)\,\text{Abs[beta]}^{\frac{1}{2}(2-p-r)}\Big((2+p)\,\text{BesselI}\left[\frac{1}{2}(2+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]+\pi\,t\,\text{Abs[beta]}\left(\text{BesselI}\left[\frac{h-r}{2},\,2\pi\,t\,\text{Abs[beta]}\right]+\text{BesselI}\left[\frac{1}{2}(4+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]\right)\Big)+\text{beta}(2+h-3\,r)(-p+r)\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{1}{2}(2+p+r)}\text{BesselI}\left[\frac{1}{2}(-2+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]\text{Conjugate[beta]}+2\,\text{beta}(p-r)\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{1}{2}(2+p+r)}\Big((2+p)\,\text{BesselI}\left[\frac{1}{2}(-2+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]+\pi\,t\,\text{Abs[beta]}\left(\text{BesselI}\left[\frac{1}{2}(-4+h-r),\,2\pi\,t\,\text{Abs[beta]}\right]+\text{BesselI}\left[\frac{h-r}{2},\,2\pi\,t\,\text{Abs[beta]}\right]\right)\Big)\text{Conjugate[beta]}\Big)\Big)\right\}$$

Out[ ]=
$$\left\{\frac{1}{3}(i\,\text{beta})^{\frac{p+r}{2}}(h^2-j^2-3\,nu^2+3\,p^2)\,t^{2+p}\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}}\text{BesselI}\left[\frac{h-r}{2},\,2\pi\,t\,\text{Abs[beta]}\right],\ (i\,\text{beta})^{\frac{p+r}{2}}(h^3-2\,j^3+18\,j\,nu^2-3\,h(j^2+3\,nu^2-9\,p^2)-9\,h^2\,r+9\,j^2\,r+27(nu^2-p^2)\,r)\,t^{2+p}\left(\frac{1}{\text{Abs[beta]}}\right)^{\frac{p+r}{2}}\text{BesselI}\left[\frac{h-r}{2},\,2\pi\,t\,\text{Abs[beta]}\right]\right\}$$

In[ ]:=
```
eqs /. j → -h /. nu → p // Simplify
```
Out[ ]=
```
{0, 0}
```

Both elements behave under ZU(**g**) according to the character $\psi=\psi[-h,p]$.