19a. First downward shift operator

In[ ]:=
```
F = tht[m[r]] × f[r, t] × Phi[h, p, r, p]
```

Out[ ]=
```
f[r, t] × Phi[h, p, r, p] × tht[m[r]]
```

In[ ]:=
```
sh[3, -1, F, subnab] // compr // Simplify;
kr3m1 = % /. r → r + 1 /. m[r + 2] → m[r] + eps // Simplify
```

Out[ ]=

$$-\frac{1}{4\times(1+p)}\,p\,\text{Phi}[3+h,\,-1+p,\,1+r,\,-1+p]\Big(\big(4-h+2p+r-4\,\text{ell}\,\pi\,t^2\big)\,f[r,t]\times\text{tht}[m[r]]+2\,i\,\sqrt{2\pi}\,t\,\sqrt{\text{Abs}[\text{ell}]}\,f[2+r,t]\Big((1+\text{eps})\sqrt{\text{eps}+m[r]}\,\text{tht}[-1+\text{eps}+m[r]]+(-1+\text{eps})\sqrt{1+\text{eps}+m[r]}\,\text{tht}[1+\text{eps}+m[r]]\Big)-2\,t\,\text{tht}[m[r]]\,f^{(0,1)}[r,t]\Big)$$

In[ ]:=
```
kr3m1 /. eps → 1 /. Abs[ell] → ell // Simplify;
kr3m1psub = Solve[% == 0, f[r + 2, t]]〚1〛
(f[r + 2, t] /. %) == (2 t f^(0,1)[r, t] + (h - 2 p - r - 4 + 4 Pi ell t^2) f[r, t]) /
    (4 I t Sqrt[2 Pi ell (1 + m[r])]) /. (pp_ qq_)^ee_ :→ pp^ee qq^ee // Simplify
```

Out[ ]=

$$\Big\{f[2+r,t]\to-\Big(\big(i\,(-4\,f[r,t]+h\,f[r,t]-2\,p\,f[r,t]-r\,f[r,t]+4\,\text{ell}\,\pi\,t^2\,f[r,t]+2\,t\,f^{(0,1)}[r,t])\big)\Big/\big(4\sqrt{\text{ell}}\,\sqrt{2\pi}\,t\,\sqrt{1+m[r]}\big)\Big)\Big\}$$

Out[ ]=
```
True
```

This is valid for m[r] ≥ 0 and r+2≤p

In[ ]:=
```
kr3m1 /. eps → -1 // Simplify;

kr3m1msub = Solve[% == 0, f[r + 2, t]]〚1〛 // Simplify
(f[r + 2, t] /. %) == -(2 t f^(0,1)[r, t] + (h - 2 p - r - 4 + 4 Pi ell t^2) f[r, t]) /
   (4 I t Sqrt[2 Pi Abs[ell] m[r]]) // Simplify
```

Out[ ]=

$$\Big\{f[2+r,t]\to\frac{i\,\big((-4+h-2p-r+4\,\text{ell}\,\pi\,t^2)\,f[r,t]+2\,t\,f^{(0,1)}[r,t]\big)}{4\sqrt{2\pi}\,t\,\sqrt{\text{Abs}[\text{ell}]}\,\sqrt{m[r]}}\Big\}$$

Out[ ]=
```
True
```

Valid for m(r)≥1

In[ ]:=
```
kr3m1 /. eps → -1 /. r → r0 /. m[r0] → 0 // Simplify
```

Out[ ]=

$$-\frac{1}{4\times(1+p)}\,p\,\text{Phi}[3+h,\,-1+p,\,1+r0,\,-1+p]\times\text{tht}[0]\,\Big(\big(4-h+2p+r0-4\,\text{ell}\,\pi\,t^2\big)\,f[r0,t]-2\,t\,f^{(0,1)}[r0,t]\Big)$$

Valid for r=r0 between -p and p