19b. Second downward shift operator

In[∘]:=
```
F = tht[m[r]] × f[r, t] × Phi[h, p, r, p]
```

Out[∘]=
$$f[r, t] \times \text{Phi}[h, p, r, p] \times \text{tht}[m[r]]$$

In[∘]:=
```
sh[-3, -1, F, subnab] // compr // Simplify;
krm3m1 = % /. r → r - 1 /. m[r - 2] → m[r] - eps // Simplify
```

Out[∘]=
$$\frac{1}{4 \times (1+p)} p\, \text{Phi}[-3+h, -1+p, -1+r, -1+p] \Big(2 i \sqrt{2\pi}\; t \sqrt{\text{Abs}[\text{ell}]}\; f[-2+r, t] \Big((-1+\text{eps}) \sqrt{-\text{eps}+m[r]}\; \text{tht}[-1-\text{eps}+m[r]] + (1+\text{eps}) \sqrt{1-\text{eps}+m[r]}\; \text{tht}[1-\text{eps}+m[r]]\Big) - \text{tht}[m[r]] \left(\left(4+h+2p-r+4\,\text{ell}\,\pi\, t^2\right) f[r, t] - 2t f^{(0,1)}[r, t]\right)\Big)$$

In[∘]:=
```
krm3m1 /. eps → 1 /. Abs[ell] → ell // Simplify;
krm3m1psub = f[r - 2, t] /. Solve[% == 0, f[r - 2, t]]〚1〛 // Simplify
% == ((h + 2 p - r + 4 + 4 Pi ell t^2) f[r, t] - 2 t f^(0,1)[r, t]) / (4 I t Sqrt[2 Pi ell m[r]]) /.
  (pp_ qq_)^ee_ :→ pp^ee qq^ee // Simplify
```

Out[∘]=
$$-\frac{i \left(\left(4+h+2p-r+4\,\text{ell}\,\pi\, t^2\right) f[r, t] - 2t f^{(0,1)}[r, t]\right)}{4 \sqrt{\text{ell}}\; \sqrt{2\pi}\; t \sqrt{m[r]}}$$

Out[∘]= True

Valid for m[r]≥0

In[∘]:=
```
krm3m1 /. eps → -1 // Simplify
krm3m1msub = Solve[% == 0, f[r - 2, t]]〚1〛 // Simplify
```

Out[∘]=
$$-\frac{1}{4 \times (1+p)} p\, \text{Phi}[-3+h, -1+p, -1+r, -1+p] \times \text{tht}[m[r]] \left(\left(4+h+2p-r+4\,\text{ell}\,\pi\, t^2\right) f[r, t] + 4 i \sqrt{2\pi}\; t \sqrt{\text{Abs}[\text{ell}]}\; f[-2+r, t] \sqrt{1+m[r]} - 2t f^{(0,1)}[r, t]\right)$$

Out[∘]=
$$\left\{f[-2+r, t] \to \frac{i \left(\left(4+h+2p-r+4\,\text{ell}\,\pi\, t^2\right) f[r, t] - 2t f^{(0,1)}[r, t]\right)}{4 \sqrt{2\pi}\; t \sqrt{\text{Abs}[\text{ell}]}\; \sqrt{1+m[r]}}\right\}$$

In[∘]:=
```
(f[r - 2, t] /. krm3m1msub) == -((h + 2 p - r + 4 + 4 Pi ell t^2) f[r, t] - 2 t f^(0,1)[r, t]) /
    (4 I t Sqrt[2 Pi Abs[ell] (1 + m[r])]) // Simplify
```

Out[∘]= True

Valid for m[r]≥0

In[∘]:=
```
krm3m1 /. eps → 1 /. r → r0 /. m[r0] → 0 // Simplify
```

Out[∘]=
$$-\frac{1}{4 \times (1+p)} p\, \text{Phi}[-3+h, -1+p, -1+r0, -1+p] \times \text{tht}[0] \left(\left(4+h+2p-r0+4\,\text{ell}\,\pi\, t^2\right) f[r0, t] - 2t f^{(0,1)}[r0, t]\right)$$

Valid for $|r_0| \le p$