20c. Downward shift operator Sh(3,-1)

Note that this shift operator is considered on $x^{0,p}$, on which we know that the other shift operator vanishes.

```
In[ ]:= Clear[m, m0, h, r0, wh]
m[h_, r_] := m0 + (eps / 6) (3 r + 2 j - h) // Simplify
whsub = {wh^(0,0,2)[kp_, s_, tau_] :> (1 / 4 - kp / tau + (s ^ 2 - 1 / 4) tau ^ (-2)) wh[kp, s, tau]};
```

Case ε=1

```
In[ ]:= F = tht[m[h, p]] × f[p, t] × Phi[h, p, p, p] + tht[m[h, p - 2]] × f[p - 2, t] × Phi[h, p, p - 2, p] /.
      eps → 1 /. h → 2 j - 3 p // Simplify

Out[ ]= f[-2 + p, t] × Phi[2 j - 3 p, p, -2 + p, p] × tht[-1 + m0 + p] +
  f[p, t] × Phi[2 j - 3 p, p, p, p] × tht[m0 + p]
```

Determine relation between components

```
In[ ]:= sh[-3, -1, F, subnab] /. eps → 1 // Simplify
fp2m =
  f[p - 2, t] /. Solve[Coefficient[%, Phi[-3 - 3 p + 2 j, -1 + p, -1 + p, -1 + p]] == 0, f[p - 2, t]][[
      1]] // Simplify
```

Out[ ]= $\frac{1}{2\times(1+p)}$ p (−((2 + j − p + 2 ell π t²) f[p, t] × Phi[−3 + 2 j − 3 p, −1 + p, −1 + p, −1 + p] × tht[m0 + p]) +
f[−2 + p, t] (−((3 + j − p + 2 ell π t²) Phi[−3 + 2 j − 3 p, −1 + p, −3 + p, −1 + p] × tht[−1 + m0 + p]) +
2 ⅈ √(m0 + p) √(2 π) t √(Abs[ell]) Phi[−3 + 2 j − 3 p, −1 + p, −1 + p, −1 + p] × tht[m0 + p]) +
t (Phi[−3 + 2 j − 3 p, −1 + p, −3 + p, −1 + p] × tht[−1 + m0 + p] f^(0,1)[−2 + p, t] +
Phi[−3 + 2 j − 3 p, −1 + p, −1 + p, −1 + p] × tht[m0 + p] f^(0,1)[p, t]))

Out[ ]= $\frac{ⅈ\,(-((2 + j - p + 2\,ell\,\pi\,t^2)\,f[p,\,t]) + t\,f^{(0,1)}[p,\,t])}{2\,\sqrt{m0 + p}\,\sqrt{2\,\pi}\,t\,\sqrt{Abs[ell]}}$

Apply downward shift operator and substitute for f_{p-2}

In[ ]:=
```
sh[3, -1, F, subnab] /. eps → 1 /. f[-2 + p, t] → fp2m /. f^(0,1)[-2 + p, t] → D[fp2m, t] // Simplify
co = Coefficient[%, tht[m0 + p - 1] × Phi[3 - 3 p + 2 j, -1 + p, -1 + p, -1 + p]] // Simplify
```

Out[ ]=
```
-(((p (8 ⅈ (m0 + p) Sqrt[2 π] t Sqrt[Abs[ell]] f[p, t] × Phi[3 + 2 j - 3 p, -1 + p, -1 + p, -1 + p] × tht[-1 + m0 + p] +
      4 Sqrt[-1 + m0 + p] Phi[3 + 2 j - 3 p, -1 + p, -3 + p, -1 + p] ×
       tht[-2 + m0 + p] ((2 + j - p + 2 ell π t^2) f[p, t] - t f^(0,1)[p, t]) +
      (ⅈ Sqrt[2/π] Phi[3 + 2 j - 3 p, -1 + p, -1 + p, -1 + p] × tht[-1 + m0 + p]
        ((-4 + j^2 - 4 p - 4 j p + 3 p^2 + 4 ell π t^2 + 4 ell j π t^2 - 8 ell p π t^2 + 4 ell^2 π^2 t^4) f[p,
          t] + t ((3 + 2 p) f^(0,1)[p, t] - t f^(0,2)[p, t]))) / (t Sqrt[Abs[ell]]))) / (8 × (1 + p) Sqrt[m0 + p]))
```

Out[ ]=
```
-((ⅈ p ((-4 + j^2 - 4 p - 4 j p + 3 p^2 + 4 ell π t^2 +
       4 ell j π t^2 - 8 ell p π t^2 + 4 ell^2 π^2 t^4 + 8 (m0 + p) π t^2 Abs[ell]) f[p, t] +
    t ((3 + 2 p) f^(0,1)[p, t] - t f^(0,2)[p, t]))) / (4 × (1 + p) Sqrt[m0 + p] Sqrt[2 π] t Sqrt[Abs[ell]]))
```

In[ ]:=
```
Clear[kap0]
wh0 = t^(p + 1) wh[kap0, nu/2, 2 Pi Abs[ell] t^2]
quot =
 co/(t^p wh[kap0, nu/2, 2 Pi Abs[ell] t^2]) /. f[p, t] → wh0 /. f^(0,ee_)[p, t] :→ D[wh0, {t, ee}] //.
     whsub /. Abs[ell] → ell /. kap0 → -m0 - (j + 1)/2 // Factor
```

Out[ ]=
```
t^(1+p) wh[kap0, nu/2, 2 π t^2 Abs[ell]]
```

Out[ ]=
$$-\frac{i\, p\,(-j - nu + 2\,p)\,(-j + nu + 2\,p)}{4\,\sqrt{ell}\,(1 + p)\,\sqrt{m0 + p}\,\sqrt{2\,\pi}}$$

This goes in the sixth box in Table 4.9.

Case ε=-1, 1≤p ≤ m_0

With the same approach as in the previous case

In[ ]:=
```
F = tht[m[h, p]] × f[p, t] × Phi[h, p, p, p] +
    tht[m[h, p - 2]] × f[p - 2, t] × Phi[h, p, p - 2, p] /. eps → -1 /. h → 2 j - 3 p
```

Out[ ]=
```
f[-2 + p, t] × Phi[2 j - 3 p, p, -2 + p, p] × tht[m0 + 1/6 × (6 - 6 p)] +
 f[p, t] × Phi[2 j - 3 p, p, p, p] × tht[m0 - p]
```

In[ ]:=
```
sh[-3, -1, F, subnab] /. eps → -1 // Simplify
fp2m =
 f[p - 2, t] /. Solve[Coefficient[%, Phi[-3 - 3 p + 2 j, -1 + p, -1 + p, -1 + p]] == 0, f[p - 2, t]][[
    1]] // Simplify
```

Out[ ]=
$$\frac{1}{2\times(1+p)}\, p\Big(-2\,i\,\sqrt{1+m0-p}\,\sqrt{2\pi}\;t\,\sqrt{\text{Abs[ell]}}\;f[-2+p,t]\times\text{Phi}[-3+2j-3p,-1+p,-1+p,-1+p]\times \text{tht}[m0-p]-\left(2+j-p+2\,\text{ell}\,\pi t^2\right)f[p,t]\times\text{Phi}[-3+2j-3p,-1+p,-1+p,-1+p]\times \text{tht}[m0-p]-3\,f[-2+p,t]\times\text{Phi}[-3+2j-3p,-1+p,-3+p,-1+p]\times\text{tht}[1+m0-p]- j\,f[-2+p,t]\times\text{Phi}[-3+2j-3p,-1+p,-3+p,-1+p]\times\text{tht}[1+m0-p]+ p\,f[-2+p,t]\times\text{Phi}[-3+2j-3p,-1+p,-3+p,-1+p]\times\text{tht}[1+m0-p]- 2\,\text{ell}\,\pi t^2\,f[-2+p,t]\times\text{Phi}[-3+2j-3p,-1+p,-3+p,-1+p]\times\text{tht}[1+m0-p]+ t\,\text{Phi}[-3+2j-3p,-1+p,-3+p,-1+p]\times\text{tht}[1+m0-p]\,f^{(0,1)}[-2+p,t]+ t\,\text{Phi}[-3+2j-3p,-1+p,-1+p,-1+p]\times\text{tht}[m0-p]\,f^{(0,1)}[p,t]\Big)$$

Out[ ]=
$$\frac{i\left(\left(2+j-p+2\,\text{ell}\,\pi t^2\right)f[p,t]-t\,f^{(0,1)}[p,t]\right)}{2\sqrt{1+m0-p}\,\sqrt{2\pi}\;t\,\sqrt{\text{Abs[ell]}}}$$

In[ ]:=
```
sh[3, -1, F, subnab] /. eps → -1 /. {f[p - 2, t] → fp2m, f^(0,1)[-2 + p, t] → D[fp2m, t]} // Simplify
co = Coefficient[%, tht[1 + m0 - p] × Phi[3 - 3 p + 2 j, p - 1, p - 1, p - 1]] // Simplify
```

Out[ ]=
$$\Big(i\,p\Big(8\sqrt{2}\,(1+m0-p)\,\pi t^2\,\text{Abs[ell]}\,f[p,t]\times\text{Phi}[3+2j-3p,-1+p,-1+p,-1+p]\times\text{tht}[1+m0-p]+ 4\,i\sqrt{2+m0-p}\,\sqrt{\pi}\;t\,\sqrt{\text{Abs[ell]}}\;\text{Phi}[3+2j-3p,-1+p,-3+p,-1+p]\times \text{tht}[2+m0-p]\left(\left(2+j-p+2\,\text{ell}\,\pi t^2\right)f[p,t]-t\,f^{(0,1)}[p,t]\right)+ \sqrt{2}\;\text{Phi}[3+2j-3p,-1+p,-1+p,-1+p]\times\text{tht}[1+m0-p] \left(\left(-4+j^2-4p-4jp+3p^2+4\,\text{ell}\,\pi t^2+4\,\text{ell}\,j\,\pi t^2-8\,\text{ell}\,p\,\pi t^2+4\,\text{ell}^2\pi^2t^4\right)f[p,t]+ t\left((3+2p)\,f^{(0,1)}[p,t]-t\,f^{(0,2)}[p,t]\right)\right)\Big)\Big)\Big/\left(8\sqrt{1+m0-p}\,(1+p)\,\sqrt{\pi}\;t\,\sqrt{\text{Abs[ell]}}\right)$$

Out[ ]=
$$\Big(i\,p\Big(\big(-4+j^2-4p-4jp+3p^2+4\,\text{ell}\,\pi t^2+4\,\text{ell}\,j\,\pi t^2- 8\,\text{ell}\,p\,\pi t^2+4\,\text{ell}^2\pi^2t^4+8\times(1+m0-p)\,\pi t^2\,\text{Abs[ell]}\big)f[p,t]+ t\left((3+2p)\,f^{(0,1)}[p,t]-t\,f^{(0,2)}[p,t]\right)\Big)\Big)\Big/\left(4\sqrt{1+m0-p}\,(1+p)\,\sqrt{2\pi}\;t\,\sqrt{\text{Abs[ell]}}\right)$$

In[ ]:=
```
Clear[wh, kap0]
wh0 = t ^ (p + 1) wh[kap0, nu / 2, 2 Pi Abs[ell] t ^ 2]
quot =
 co / (t ^ p wh[kap0, nu / 2, 2 Pi Abs[ell] t ^ 2]) /. f[p, t] → wh0 /. f^(0,ee_)[p, t] :→ D[wh0, {t, ee}] /.
    whsub /. ell → -Abs[ell] /. kap0 → -m0 - (-j + 1) / 2 // Factor
```

Out[ ]=
$$t^{1+p}\,\text{wh}\left[\text{kap0},\frac{\text{nu}}{2},2\pi t^2\,\text{Abs[ell]}\right]$$

Out[ ]=
$$\frac{i\,p\,(-j-\text{nu}+2p)\,(-j+\text{nu}+2p)}{4\sqrt{1+m0-p}\,(1+p)\,\sqrt{2\pi}\;\sqrt{\text{Abs[ell]}}}$$

Box 7 on the right in Table 4.9

Case ε=-1, p≥ m_0, $r_0 > -p$

```
In[ ]:= F = tht[0] × f[r0, t] × Phi[h, p, r0, p] + tht[1] × f[r0 - 2, t] × Phi[h, p, r0 - 2, p]

Out[ ]= f[r0, t] × Phi[h, p, r0, p] × tht[0] + f[-2 + r0, t] × Phi[h, p, -2 + r0, p] × tht[1]
```

```
In[ ]:= sh[-3, -1, F, subnab] /. eps → -1 // Simplify
fr0m = f[r0 - 2, t] /.
    Solve[Coefficient[%, Phi[-3 + h, -1 + p, -1 + r0, -1 + p]] == 0, f[r0 - 2, t]][[1]] // Simplify
```

$$\text{Out[ ]=}\ -\frac{1}{4\times(1+p)}\,p\Big(4\,i\,\sqrt{2\pi}\ t\ \sqrt{\text{Abs[ell]}}\ f[-2+r0,t]\times \text{Phi}[-3+h,-1+p,-1+r0,-1+p]\times \text{tht}[0]+$$
$$(4+h+2p-r0+4\,\text{ell}\,\pi t^2)\,f[r0,t]\times\text{Phi}[-3+h,-1+p,-1+r0,-1+p]\times\text{tht}[0]+$$
$$6\,f[-2+r0,t]\times\text{Phi}[-3+h,-1+p,-3+r0,-1+p]\times\text{tht}[1]+$$
$$h\,f[-2+r0,t]\times\text{Phi}[-3+h,-1+p,-3+r0,-1+p]\times\text{tht}[1]+$$
$$2p\,f[-2+r0,t]\times\text{Phi}[-3+h,-1+p,-3+r0,-1+p]\times\text{tht}[1]-$$
$$r0\,f[-2+r0,t]\times\text{Phi}[-3+h,-1+p,-3+r0,-1+p]\times\text{tht}[1]+$$
$$4\,\text{ell}\,\pi t^2\,f[-2+r0,t]\times\text{Phi}[-3+h,-1+p,-3+r0,-1+p]\times\text{tht}[1]-$$
$$2t\,\text{Phi}[-3+h,-1+p,-3+r0,-1+p]\times\text{tht}[1]\,f^{(0,1)}[-2+r0,t]-$$
$$2t\,\text{Phi}[-3+h,-1+p,-1+r0,-1+p]\times\text{tht}[0]\,f^{(0,1)}[r0,t]\Big)$$

$$\text{Out[ ]=}\ \frac{i\left((4+h+2p-r0+4\,\text{ell}\,\pi t^2)\,f[r0,t]-2t\,f^{(0,1)}[r0,t]\right)}{4\sqrt{2\pi}\ t\ \sqrt{\text{Abs[ell]}}}$$

```
In[ ]:= sh[3, -1, F, subnab] /. eps → -1 /.
    {f[-2 + r0, t] → fr0m, f^(0,1)[-2 + r0, t] → D[fr0m, t]} // Simplify
co = Coefficient[%, tht[0] × Phi[3 + h, -1 + p, 1 + r0, -1 + p]] /. h → 2 j - 3 p /. ell → -Abs[ell] //
    Simplify
```

$$\text{Out[ ]=}\ \Big(p\Big(32\,i\sqrt{2}\,\pi t^2\,\text{Abs[ell]}\,f[r0,t]\times\text{Phi}[3+h,-1+p,-1+r0,-1+p]\times\text{tht}[1]+8\sqrt{\pi}\ t\ \sqrt{\text{Abs[ell]}}$$
$$\Big(f[r0,t]\Big((-4+h-2p-r0+4\,\text{ell}\,\pi t^2)\,\text{Phi}[3+h,-1+p,1+r0,-1+p]\times\text{tht}[0]-$$
$$\sqrt{2}\,(4+h+2p-r0+4\,\text{ell}\,\pi t^2)\,\text{Phi}[3+h,-1+p,-3+r0,-1+p]\times\text{tht}[2]\Big)+$$
$$2t\Big(\text{Phi}[3+h,-1+p,1+r0,-1+p]\times\text{tht}[0]+\sqrt{2}\,\text{Phi}[3+h,-1+p,-3+r0,-1+p]\times$$
$$\text{tht}[2]\Big)\,f^{(0,1)}[r0,t]\Big)+i\sqrt{2}\,\text{Phi}[3+h,-1+p,-1+r0,-1+p]\times\text{tht}[1]$$
$$\big((-16+h^2-16p-4p^2-2h\,r0+r0^2+16\,\text{ell}\,\pi t^2+8\,\text{ell}\,h\,\pi t^2-8\,\text{ell}\,\pi\,r0\,t^2+16\,\text{ell}^2\pi^2t^4)$$
$$f[r0,t]+4t\big((3+2p)\,f^{(0,1)}[r0,t]-t\,f^{(0,2)}[r0,t]\big)\big)\Big)\Big)\Big/\Big(32\times(1+p)\sqrt{\pi}\ t\ \sqrt{\text{Abs[ell]}}\Big)$$

$$\text{Out[ ]=}\ \frac{1}{4\times(1+p)}\left(-p\,(4-2j+5p+r0+4\pi t^2\,\text{Abs[ell]})\,f[r0,t]+2p\,t\,f^{(0,1)}[r0,t]\right)$$

Work with the three separate Whittaker functions

```
In[ ]:= Clear[wh, kap0]
        wh0 = t^(m0 + 1) wh[kap, nu/2, 2 Pi Abs[ell] t^2]
        quot = co/(t^(m0 + 1) wh[kap + 1, nu/2, 2 Pi Abs[ell] t^2]) /.
            {f[r0, t] → wh0, f^(0,1)[r0, t] → D[wh0, t]} /. m0 → (r0 + p)/2 // Simplify
```

Out[ ]= $t^{1+m0}\,\mathrm{wh}\left[\mathrm{kap},\ \frac{\mathrm{nu}}{2},\ 2\,\pi\,t^2\,\mathrm{Abs}[\mathrm{ell}]\right]$

Out[ ]= $\left(-p\,(1-j+2\,p+2\,\pi\,t^2\,\mathrm{Abs}[\mathrm{ell}])\,\mathrm{wh}\left[\mathrm{kap},\ \frac{\mathrm{nu}}{2},\ 2\,\pi\,t^2\,\mathrm{Abs}[\mathrm{ell}]\right]+4\,p\,\pi\,t^2\,\mathrm{Abs}[\mathrm{ell}]\,\mathrm{wh}^{(0,0,1)}\left[\mathrm{kap},\ \frac{\mathrm{nu}}{2},\ 2\,\pi\,t^2\,\mathrm{Abs}[\mathrm{ell}]\right]\right)\Big/\left(2\times(1+p)\,\mathrm{wh}\left[1+\mathrm{kap},\ \frac{\mathrm{nu}}{2},\ 2\,\pi\,t^2\,\mathrm{Abs}[\mathrm{ell}]\right]\right)$

```
In[ ]:= quot /. wh → WhittakerW /. kap → -p + (j - 1)/2 // Simplify
```

Out[ ]= $-\frac{p}{1+p}$

```
In[ ]:= quot /. wh → WhittakerV //. Whrel /. kap → -p + (j - 1)/2 // Factor
```

Out[ ]= $\frac{p\,(-j-\mathrm{nu}+2\,p)\,(-j+\mathrm{nu}+2\,p)}{4\times(1+p)}$

```
In[ ]:= quot /. wh → WhittakerM /. kap → -p + (j - 1)/2 // Simplify
```

Out[ ]= $\frac{(j+\mathrm{nu}-2\,p)\,p}{2\times(1+p)}$

Last box of Table 4.9.

Case $\varepsilon = -1,\ p > m_0, r_0 = -p$

In this case there is only one component

```
In[ ]:= F = tht[0] × f[-p, t] × Phi[h, p, -p, p]
```

Out[ ]= f[-p, t] × Phi[h, p, -p, p] × tht[0]

```
In[ ]:= sh[3, -1, F, subnab]
        co = Coefficient[%, tht[0] × Phi[3 + h, -1 + p, 1 - p, -1 + p]] // Simplify
```

Out[ ]= $\frac{1}{4\times(1+p)}\,p\,\mathrm{Phi}[3+h,\ -1+p,\ 1-p,\ -1+p]\times\mathrm{tht}[0]\,\left((-4+h-p+4\,\mathrm{ell}\,\pi\,t^2)\,f[-p,\ t]+2\,t\,f^{(0,1)}[-p,\ t]\right)$

Out[ ]= $\frac{p\,\left((-4+h-p+4\,\mathrm{ell}\,\pi\,t^2)\,f[-p,\ t]+2\,t\,f^{(0,1)}[-p,\ t]\right)}{4\times(1+p)}$

```
In[ ]:= Clear[wh, kap0]
        wh0 = t ^ (m0 + 1) wh[kap, nu / 2, 2 Pi Abs[ell] t ^ 2]
        quot = co / (t ^ (m0 + 1) wh[kap + 1, nu / 2, 2 Pi Abs[ell] t ^ 2]) /.
              {f[-p, t] → wh0, f^(0,1)[-p, t] → D[wh0, t]} //.
            {ell → -Abs[ell], h → 2 j - 3 p, j → 2 kap + 2 p + 1, m0 → 0} // Simplify
```

Out[ ]= $t^{1+m0}\,\mathrm{wh}\left[kap, \frac{nu}{2}, 2\,\pi\,t^2\,Abs[ell]\right]$

Out[ ]= $\left(p\left(\left(kap-\pi\,t^2\,Abs[ell]\right)\mathrm{wh}\left[kap, \frac{nu}{2}, 2\,\pi\,t^2\,Abs[ell]\right]+ 2\,\pi\,t^2\,Abs[ell]\,\mathrm{wh}^{(0,0,1)}\left[kap, \frac{nu}{2}, 2\,\pi\,t^2\,Abs[ell]\right]\right)\right)\Big/\left((1+p)\,\mathrm{wh}\left[1+kap, \frac{nu}{2}, 2\,\pi\,t^2\,Abs[ell]\right]\right)$

```
In[ ]:= quot /. wh → WhittakerW // Simplify
```

Out[ ]= $-\frac{p}{1+p}$

```
In[ ]:= quot /. wh → WhittakerV //. Whrel /. kap → -p + (j - 1) / 2 // Factor
```

Out[ ]= $\frac{p\,(-j-nu+2\,p)\,(-j+nu+2\,p)}{4\times(1+p)}$

```
In[ ]:= quot /. wh → WhittakerM /. kap → -p + (j - 1) / 2 // Simplify
```

Out[ ]= $\frac{(j+nu-2\,p)\,p}{2\times(1+p)}$

Last box in Table 4.9