20d. Downward shift operator Sh(-3,-1)

This concerns the upper part of Table 4.9.

In[ ]:=
```
whsub = {wh^(0,0,2)[kp_, s_, tau_] :> (1/4 - kp/tau + (s^2 - 1/4) tau^(-2)) wh[kp, s, tau]};
```

Case $\varepsilon = 1$, $1 \le p \le m_0$

The same methods as in 20c.

We consider families $x^{p,0}$.

In[ ]:=
```
Clear[r, f]
```

In[ ]:=
```
F =
 tht[m[h, -p]] × f[-p, t] × Phi[h, p, -p, p] + tht[m[h, -p + 2]] × f[-p + 2, t] × Phi[h, p, -p + 2, p] /.
   h → 2 j + 3 p /. eps → 1 // Simplify
```

Out[ ]=
```
f[-p, t] × Phi[2 j + 3 p, p, -p, p] × tht[m0 - p] + f[2 - p, t] × Phi[2 j + 3 p, p, 2 - p, p] × tht[1 + m0 - p]
```

In[ ]:=
```
sh[3, -1, F, subnab] /. eps → 1 // Simplify
Coefficient[%, Phi[3 + 3 p + 2 j, p - 1, 1 - p, p - 1]] // Simplify
fm2p = f[2 - p, t] /. Solve[% == 0, f[2 - p, t]][[1]] // Simplify
```

Out[ ]=

$$\frac{1}{2\times(1+p)}\,p\,\Big(-2\,i\,\sqrt{1+m0-p}\,\sqrt{2\,\pi}\,t\,\sqrt{\mathrm{Abs[ell]}}\;f[2-p,t]\times \mathrm{Phi}[3+2j+3p,-1+p,1-p,-1+p]\times \mathrm{tht}[m0-p]+$$
$$(-2+j+p+2\,\mathrm{ell}\,\pi\,t^2)\,f[-p,t]\times \mathrm{Phi}[3+2j+3p,-1+p,1-p,-1+p]\times \mathrm{tht}[m0-p]-$$
$$3\,f[2-p,t]\times \mathrm{Phi}[3+2j+3p,-1+p,3-p,-1+p]\times \mathrm{tht}[1+m0-p]+$$
$$j\,f[2-p,t]\times \mathrm{Phi}[3+2j+3p,-1+p,3-p,-1+p]\times \mathrm{tht}[1+m0-p]+$$
$$p\,f[2-p,t]\times \mathrm{Phi}[3+2j+3p,-1+p,3-p,-1+p]\times \mathrm{tht}[1+m0-p]+$$
$$2\,\mathrm{ell}\,\pi\,t^2\,f[2-p,t]\times \mathrm{Phi}[3+2j+3p,-1+p,3-p,-1+p]\times \mathrm{tht}[1+m0-p]+$$
$$t\,\mathrm{Phi}[3+2j+3p,-1+p,3-p,-1+p]\times \mathrm{tht}[1+m0-p]\,f^{(0,1)}[2-p,t]+$$
$$t\,\mathrm{Phi}[3+2j+3p,-1+p,1-p,-1+p]\times \mathrm{tht}[m0-p]\,f^{(0,1)}[-p,t]\Big)$$

Out[ ]=

$$\frac{1}{2\times(1+p)}\,p\,\mathrm{tht}[m0-p]\,\Big(-2\,i\,\sqrt{1+m0-p}\,\sqrt{2\,\pi}\,t\,\sqrt{\mathrm{Abs[ell]}}\;f[2-p,t]+(-2+j+p+2\,\mathrm{ell}\,\pi\,t^2)\,f[-p,t]+t\,f^{(0,1)}[-p,t]\Big)$$

Out[ ]=

$$-\frac{i\,\big((-2+j+p+2\,\mathrm{ell}\,\pi\,t^2)\,f[-p,t]+t\,f^{(0,1)}[-p,t]\big)}{2\,\sqrt{1+m0-p}\,\sqrt{2\,\pi}\,t\,\sqrt{\mathrm{Abs[ell]}}}$$

In[ ]:=
```
sh[-3, -1, F, subnab] /. eps → 1 /. f[2 - p, t] → fm2p /. f^(0,1)[2 - p, t] → D[fm2p, t] // Simplify
co = Coefficient[%, tht[1 + m0 - p] × Phi[3 p + 2 j - 3, p - 1, 1 - p, p - 1]] // Simplify
```

Out[ ]=
$$\left(i\, p \left(8 \sqrt{2}\, (1 + m0 - p)\, \pi\, t^2\, \text{Abs[ell]}\, \text{f[-p, t]} \times \text{Phi[-3 + 2 j + 3 p, -1 + p, 1 - p, -1 + p]} \times \text{tht[1 + m0 - p]} - 4\, i \sqrt{2 + m0 - p}\, \sqrt{\pi}\, t\, \sqrt{\text{Abs[ell]}}\, \text{Phi[-3 + 2 j + 3 p, -1 + p, 3 - p, -1 + p]} \times \text{tht[2 + m0 - p]} \left(\left(-2 + j + p + 2\, \text{ell}\, \pi\, t^2\right) \text{f[-p, t]} + t\, f^{(0,1)}\text{[-p, t]}\right) + \sqrt{2}\, \text{Phi[-3 + 2 j + 3 p, -1 + p, 1 - p, -1 + p]} \times \text{tht[1 + m0 - p]} \left(\left(-4 + j^2 - 4 p + 3 p^2 - 4\, \text{ell}\, \pi\, t^2 + 8\, \text{ell}\, p\, \pi\, t^2 + 4\, \text{ell}^2 \pi^2 t^4 + 4 j \left(p + \text{ell}\, \pi\, t^2\right)\right) \text{f[-p, t]} + t \left((3 + 2 p)\, f^{(0,1)}\text{[-p, t]} - t\, f^{(0,2)}\text{[-p, t]}\right)\right)\right)\right) \Big/ \left(8 \sqrt{1 + m0 - p}\, (1 + p) \sqrt{\pi}\, t \sqrt{\text{Abs[ell]}}\right)$$

Out[ ]=
$$\left(i\, p \left(\left(-4 + j^2 - 4 p + 3 p^2 - 4\, \text{ell}\, \pi\, t^2 + 8\, \text{ell}\, p\, \pi\, t^2 + 4\, \text{ell}^2 \pi^2 t^4 + 4 j \left(p + \text{ell}\, \pi\, t^2\right) + 8 \times (1 + m0 - p)\, \pi\, t^2\, \text{Abs[ell]}\right) \text{f[-p, t]} + t \left((3 + 2 p)\, f^{(0,1)}\text{[-p, t]} - t\, f^{(0,2)}\text{[-p, t]}\right)\right)\right) \Big/ \left(4 \sqrt{1 + m0 - p}\, (1 + p) \sqrt{2 \pi}\, t \sqrt{\text{Abs[ell]}}\right)$$

In[ ]:=
```
Clear[wh, kap0]
wh0 = t ^ (p + 1) wh[kap0, nu / 2, 2 Pi Abs[ell] t ^ 2]
co /. f[-p, t] → wh0 /. f^(0,ee_)[-p, t] :→ D[wh0, {t, ee}] /. whsub // Simplify
quot =
  % / (t ^ p wh[kap0, nu / 2, 2 Pi Abs[ell] t ^ 2]) /. Abs[ell] → ell /. kap0 → -m0 - (j + 1) / 2 // Factor
```

Out[ ]=
$$t^{1+p}\, \text{wh}\left[\text{kap0}, \frac{\text{nu}}{2}, 2 \pi t^2\, \text{Abs[ell]}\right]$$

Out[ ]=
$$\left(i\, p\, t^p \left(j^2 - \text{nu}^2 + 4 p^2 - 4\, \text{ell}\, \pi\, t^2 + 8\, \text{ell}\, p\, \pi\, t^2 + 4\, \text{ell}^2 \pi^2 t^4 + 4 j \left(p + \text{ell}\, \pi\, t^2\right) + 8 \times (1 + \text{kap0} + m0 - p)\, \pi\, t^2\, \text{Abs[ell]} - 4 \pi^2 t^4\, \text{Abs[ell]}^2\right) \text{wh}\left[\text{kap0}, \frac{\text{nu}}{2}, 2 \pi t^2\, \text{Abs[ell]}\right]\right) \Big/ \left(4 \sqrt{1 + m0 - p}\, (1 + p) \sqrt{2 \pi} \sqrt{\text{Abs[ell]}}\right)$$

Out[ ]=
$$\frac{i\, p\, (j - \text{nu} + 2 p)(j + \text{nu} + 2 p)}{4 \sqrt{\text{ell}} \sqrt{1 + m0 - p}\, (1 + p) \sqrt{2 \pi}}$$

Box 2 on the right in Table 4.9

Case $\varepsilon = 1,\ p > m_0, r_0 < p$

In[ ]:=
```
F = tht[m[h, r0]] × f[r0, t] × Phi[h, p, r0, p] + tht[m[h, r0 + 2]] × f[r0 + 2, t] × Phi[h, p, r0 + 2, p] /.
    h → 2 j + 3 p /. eps → 1 /. m0 → (p - r0) / 2 // Simplify
```

Out[ ]=
```
f[r0, t] × Phi[2 j + 3 p, p, r0, p] × tht[0] + f[2 + r0, t] × Phi[2 j + 3 p, p, 2 + r0, p] × tht[1]
```

```
In[ ]:= sh[3, -1, F, subnab] /. eps → 1 // Simplify;
        Coefficient[%, Phi[+3 + 3 p + 2 j, p - 1, r0 + 1, p - 1]]
        fr0p = f[r0 + 2, t] /. Solve[% == 0, f[r0 + 2, t]][[1]] // Simplify
```

Out[ ]= $\frac{1}{4\times(1+p)} p \left((-4 + 2 j + p - r0 + 4\, ell\, \pi\, t^2)\, f[r0, t] \times tht[0] - 4 i \sqrt{2\pi}\, t \sqrt{Abs[ell]}\, f[2 + r0, t] \times tht[0] + 2\, t\, tht[0]\, f^{(0,1)}[r0, t]\right)$

Out[ ]= $-\left(\left(i \left((-4 + 2 j + p - r0 + 4\, ell\, \pi\, t^2)\, f[r0, t] + 2\, t\, f^{(0,1)}[r0, t]\right)\right) \Big/ \left(4 \sqrt{2\pi}\, t \sqrt{Abs[ell]}\right)\right)$

```
In[ ]:=
        sh[-3, -1, F, subnab] /. eps → 1 /. f[r0 + 2, t] → fr0p /.
           f^(0,ee_)[2 + r0, t] :→ D[fr0p, {t, ee}] // Simplify
        co = Coefficient[%, tht[0] × Phi[-3 + 3 p + 2 j, -1 + p, -1 + r0, -1 + p]] // Simplify
```

Out[ ]= $\Big(p \Big(32 i \sqrt{2}\, \pi\, t^2\, Abs[ell]\, f[r0, t] \times Phi[-3 + 2 j + 3 p, -1 + p, 1 + r0, -1 + p] \times tht[1] - 8 \sqrt{\pi}\, t \sqrt{Abs[ell]} \Big(f[r0, t] \Big((4 + 2 j + 5 p - r0 + 4\, ell\, \pi\, t^2)\, Phi[-3 + 2 j + 3 p, -1 + p, -1 + r0, -1 + p] \times tht[0] + \sqrt{2}\, (4 - 2 j - p + r0 - 4\, ell\, \pi\, t^2)\, Phi[-3 + 2 j + 3 p, -1 + p, 3 + r0, -1 + p] \times tht[2]\Big) - 2\, t \Big(Phi[-3 + 2 j + 3 p, -1 + p, -1 + r0, -1 + p] \times tht[0] + \sqrt{2}\, Phi[-3 + 2 j + 3 p, -1 + p, 3 + r0, -1 + p] \times tht[2]\Big) f^{(0,1)}[r0, t]\Big) + i \sqrt{2}\, Phi[-3 + 2 j + 3 p, -1 + p, 1 + r0, -1 + p] \times tht[1] \Big(\big(-16 + 4 j^2 + 5 p^2 + r0^2 - 16\, ell\, \pi\, t^2 - 8\, ell\, \pi\, r0\, t^2 + 16\, ell^2\, \pi^2\, t^4 - 2 p\, (8 + 3 r0 - 12\, ell\, \pi\, t^2) + 4 j\, (3 p - r0 + 4\, ell\, \pi\, t^2)\big)\, f[r0, t] + 4\, t \big((3 + 2 p)\, f^{(0,1)}[r0, t] - t\, f^{(0,2)}[r0, t]\big)\Big)\Big)\Big) \Big/ \left(32 \times (1 + p) \sqrt{\pi}\, t \sqrt{Abs[ell]}\right)$

Out[ ]= $-\frac{1}{4\times(1+p)} p \left((4 + 2 j + 5 p - r0 + 4\, ell\, \pi\, t^2)\, f[r0, t] - 2\, t\, f^{(0,1)}[r0, t]\right)$

```
In[ ]:= Clear[wh, kap0]
        wh0 = t^(m0 + 1) wh[kap, nu/2, 2 Pi Abs[ell] t^2]
        quot =
         co/(t^(m0 + 1) wh[kap + 1, nu/2, 2 Pi Abs[ell] t^2]) /. f[r0, t] → wh0 /. f^(0,1)[r0, t] → D[wh0, t] /.
             j → -2 kap - 2 p - 1 /. Abs[ell] → ell /. r0 → p - 2 m0 // Simplify
```

Out[ ]= $t^{1+m0}\, wh\left[kap, \frac{nu}{2}, 2\pi\, t^2\, Abs[ell]\right]$

Out[ ]= $\left(p \left((kap - ell\, \pi\, t^2)\, wh\left[kap, \frac{nu}{2}, 2\, ell\, \pi\, t^2\right] + 2\, ell\, \pi\, t^2\, wh^{(0,0,1)}\left[kap, \frac{nu}{2}, 2\, ell\, \pi\, t^2\right]\right)\right) \Big/ \left((1 + p)\, wh\left[1 + kap, \frac{nu}{2}, 2\, ell\, \pi\, t^2\right]\right)$

```
In[ ]:= quot /. wh → WhittakerW // Simplify
```

Out[ ]= $-\frac{p}{1+p}$

In[ ]:= `quot /. wh → WhittakerV /. Whrel /. kap → -p - (j + 1) / 2 // Factor`

Out[ ]= $\frac{p\,(j - nu + 2\,p)\,(j + nu + 2\,p)}{4 \times (1 + p)}$

In[ ]:= `quot /. wh → WhittakerM /. kap → -p - (j + 1) / 2 // Simplify`

Out[ ]= $-\frac{p\,(j - nu + 2\,p)}{2 \times (1 + p)}$

Box 3 on the right in Table 4.9

Case $\varepsilon = 1$, $p > m_0$, $r_0 = p$

Only one component

In[ ]:= `F = tht[0] × f[p, t] × Phi[h, p, p, p] /. h → 2 j + 3 p /. eps → 1 // Simplify`

Out[ ]= f[p, t] × Phi[2 j + 3 p, p, p, p] × tht[0]

In[ ]:= `sh[-3, -1, F, subnab]`
`co = Coefficient[%, tht[0] × Phi[-3 + 3 p + 2 j, -1 + p, -1 + p, -1 + p]] // Simplify`

Out[ ]= $-\frac{1}{2 \times (1 + p)}$ p Phi[-3 + 2 j + 3 p, -1 + p, -1 + p, -1 + p] × tht[0] $((j + 2 \times (1 + p + ell\,\pi\,t^2))$ f[p, t] $- t\, f^{(0,1)}$[p, t])

Out[ ]= $-\frac{p\,((j + 2 \times (1 + p + ell\,\pi\,t^2))\, f[p, t] - t\, f^{(0,1)}[p, t])}{2 \times (1 + p)}$

In[ ]:= `Clear[wh, kap0]`
`wh0 = t ^ (m0 + 1) wh[kap, nu / 2, 2 Pi Abs[ell] t ^ 2]`
`quot =`
`co / (t ^ (m0 + 1) wh[kap + 1, nu / 2, 2 Pi Abs[ell] t ^ 2]) /. f[p, t] → wh0 /. f^(0,1)[p, t] → D[wh0, t] /.`
`m0 → 0 /. Abs[ell] → ell /. j → -2 kap - 2 p - 1 // Simplify`

Out[ ]= $t^{1+m0}\, wh\left[kap, \frac{nu}{2}, 2\,\pi\,t^2\, Abs[ell]\right]$

Out[ ]= $\left(p\left((kap - ell\,\pi\,t^2)\, wh\left[kap, \frac{nu}{2}, 2\,ell\,\pi\,t^2\right] + 2\,ell\,\pi\,t^2\, wh^{(0,0,1)}\left[kap, \frac{nu}{2}, 2\,ell\,\pi\,t^2\right]\right)\right) \Big/ \left((1 + p)\, wh\left[1 + kap, \frac{nu}{2}, 2\,ell\,\pi\,t^2\right]\right)$

In[ ]:= `quot /. wh → WhittakerW // Simplify`

Out[ ]= $-\frac{p}{1 + p}$

In[ ]:= `quot /. wh → WhittakerV //. Whrel /. kap → -p - (j + 1) / 2 // Factor`

Out[ ]= $\frac{p\,(j - nu + 2\,p)\,(j + nu + 2\,p)}{4 \times (1 + p)}$

In[ ]:=
```
quot /. wh → WhittakerM /. kap → -p - (j + 1) / 2 // Simplify
```

Out[ ]=
$$-\frac{p\,(j - nu + 2\,p)}{2\times(1+p)}$$

Box 3 on the right in Table 4.9

Case ε = -1

In[ ]:=
```
F =
 tht[m[h, -p]] × f[-p, t] × Phi[h, p, -p, p] + tht[m[h, -p + 2]] × f[-p + 2, t] × Phi[h, p, -p + 2, p] /.
   h → 2 j + 3 p /. eps → -1 // Simplify
```

Out[ ]=
$$f[2-p,\ t]\times Phi[2\,j+3\,p,\ p,\ 2-p,\ p]\times tht[-1+m0+p] + f[-p,\ t]\times Phi[2\,j+3\,p,\ p,\ -p,\ p]\times tht[m0+p]$$

In[ ]:=
```
sh[3, -1, F, subnab]
Coefficient[%, Phi[3 + 3 p + 2 j, -1 + p, 1 - p, -1 + p]] /. eps → -1 // Simplify
fm2p = f[2 - p, t] /. Solve[% == 0, f[2 - p, t]] // Simplify
```

Out[ ]=
$$\frac{1}{2\times(1+p)}\,p\,\Big((-3+j+p+2\,ell\,\pi\,t^{2})\,f[2-p,\ t]\times Phi[3+2\,j+3\,p,\ -1+p,\ 3-p,\ -1+p]\times tht[-1+m0+p] -$$
$$2\,f[-p,\ t]\times Phi[3+2\,j+3\,p,\ -1+p,\ 1-p,\ -1+p]\times tht[m0+p] +$$
$$j\,f[-p,\ t]\times Phi[3+2\,j+3\,p,\ -1+p,\ 1-p,\ -1+p]\times tht[m0+p] +$$
$$p\,f[-p,\ t]\times Phi[3+2\,j+3\,p,\ -1+p,\ 1-p,\ -1+p]\times tht[m0+p] +$$
$$2\,ell\,\pi\,t^{2}\,f[-p,\ t]\times Phi[3+2\,j+3\,p,\ -1+p,\ 1-p,\ -1+p]\times tht[m0+p] -$$
$$i\,\sqrt{2\,\pi}\,t\,\sqrt{Abs[ell]}\,f[2-p,\ t]\times Phi[3+2\,j+3\,p,\ -1+p,\ 1-p,\ -1+p]$$
$$\left((1+eps)\,\sqrt{-1+m0+p}\,tht[-2+m0+p] + (-1+eps)\,\sqrt{m0+p}\,tht[m0+p]\right) +$$
$$t\,Phi[3+2\,j+3\,p,\ -1+p,\ 3-p,\ -1+p]\times tht[-1+m0+p]\,f^{(0,1)}[2-p,\ t] +$$
$$t\,Phi[3+2\,j+3\,p,\ -1+p,\ 1-p,\ -1+p]\times tht[m0+p]\,f^{(0,1)}[-p,\ t]\Big)$$

Out[ ]=
$$\frac{1}{2\times(1+p)}\,p\,tht[m0+p]$$
$$\left(2\,i\,\sqrt{m0+p}\,\sqrt{2\,\pi}\,t\,\sqrt{Abs[ell]}\,f[2-p,\ t] + (-2+j+p+2\,ell\,\pi\,t^{2})\,f[-p,\ t] + t\,f^{(0,1)}[-p,\ t]\right)$$

Out[ ]=
$$\left\{\frac{i\,\left((-2+j+p+2\,ell\,\pi\,t^{2})\,f[-p,\ t] + t\,f^{(0,1)}[-p,\ t]\right)}{2\,\sqrt{m0+p}\,\sqrt{2\,\pi}\,t\,\sqrt{Abs[ell]}}\right\}$$

```
In[ ]:= sh[-3, -1, F, subnab] /. f[2 - p, t] → fm2p /. f^(0,1)[2 - p, t] → D[fm2p, t] /. eps → -1 // Simplify
co = Coefficient[%, tht[-1 + m0 + p] × Phi[-3 + 3 p + 2 j, p - 1, 1 - p, p - 1]] // Simplify
```

Out[ ]= $\Big\{\Big\{\Big(p\Big(-8\, i\,(m0+p)\,\sqrt{2\pi}\ t\ \sqrt{\text{Abs[ell]}}\ f[-p,t]\times \text{Phi}[-3+2j+3p,-1+p,1-p,-1+p]\times \text{tht}[-1+m0+p]+4\sqrt{-1+m0+p}\ \text{Phi}[-3+2j+3p,-1+p,3-p,-1+p]\times \text{tht}[-2+m0+p]\left((-2+j+p+2\,\text{ell}\,\pi t^2)\,f[-p,t]+t\,f^{(0,1)}[-p,t]\right)-\Big(i\sqrt{\frac{2}{\pi}}\ \text{Phi}[-3+2j+3p,-1+p,1-p,-1+p]\times \text{tht}[-1+m0+p]\left((-4+j^2-4p+3p^2-4\,\text{ell}\,\pi t^2+8\,\text{ell}\,p\,\pi t^2+4\,\text{ell}^2\pi^2 t^4+4j(p+\text{ell}\,\pi t^2))\,f[-p,t]+t\left((3+2p)\,f^{(0,1)}[-p,t]-t\,f^{(0,2)}[-p,t]\right)\right)\Big)\Big/\Big(t\sqrt{\text{Abs[ell]}}\Big)\Big)\Big)\Big/\Big(8\times(1+p)\sqrt{m0+p}\Big)\Big\}\Big\}$

Out[ ]= $\Big\{\Big\{-\Big(\Big(i\,p\,\Big((-4+j^2-4p+3p^2-4\,\text{ell}\,\pi t^2+8\,\text{ell}\,p\,\pi t^2+4\,\text{ell}^2\pi^2 t^4+4j(p+\text{ell}\,\pi t^2)+8(m0+p)\,\pi t^2\,\text{Abs[ell]})\,f[-p,t]+t\left((3+2p)\,f^{(0,1)}[-p,t]-t\,f^{(0,2)}[-p,t]\right)\Big)\Big)\Big/\Big(4\times(1+p)\sqrt{m0+p}\ \sqrt{2\pi}\ t\ \sqrt{\text{Abs[ell]}}\Big)\Big)\Big\}\Big\}$

```
In[ ]:= Clear[wh, kap0]
wh0 = t^(p + 1) wh[kap0, nu / 2, 2 Pi Abs[ell] t^2]
quot =
  co / (t^p wh[kap0, nu / 2, 2 Pi Abs[ell] t^2]) /. f[-p, t] → wh0 /. f^(0,ee_)[-p, t] :→ D[wh0, {t, ee}] /.
    whsub /. {ell → -Abs[ell], kap0 → -m0 - (-j + 1) / 2} // Factor
```

Out[ ]= $t^{1+p}\,\text{wh}\left[\text{kap0},\frac{\text{nu}}{2},2\pi t^2\,\text{Abs[ell]}\right]$

Out[ ]= $\left\{\left\{-\frac{i\,p\,(j-\text{nu}+2p)\,(j+\text{nu}+2p)}{4\times(1+p)\sqrt{m0+p}\ \sqrt{2\pi}\ \sqrt{\text{Abs[ell]}}}\right\}\right\}$

Box 4 on the right in Table 4.9