21a Differential equations for components

One component

If m[h, eps p] = 0 there is only one component, f[eps p,t]

Case eps=1

In[ ]:=
```
F = tht[0] × f[p, t] × Phi[h, p, p, p]
sh[3, -1, F, subnab] /. eps → 1 (* is of course 0 *)
sh[-3, -1, F, subnab] /. eps → 1

fp = f[p, t] /. DSolve[% == 0, f[p, t], t][[1]] /. C[1] → 1 // Simplify
```

Out[ ]= f[p, t] × Phi[h, p, p, p] × tht[0]

Out[ ]= 0

Out[ ]= $-\frac{1}{4\times(1+p)}$ p Phi[-3 + h, -1 + p, -1 + p, -1 + p] × tht[0] $((4+h+p+4\,\text{ell}\,\pi\,t^2)$ f[p, t] $- 2\,t\,f^{(0,1)}$[p, t])

Out[ ]= $e^{\text{ell}\,\pi\,t^2}\,t^{\frac{1}{2}\times(4+h+p)}$

In[ ]:=
```
fp ^ (-1) t ^ (p + 1) WhittakerV [kap[h, p], -s[h, p], 2 Pi ell t ^ 2] //. parmsub /. eps → 1 /.
    m[h, p] → 0 /. Whrel // Simplify
(* use that V_κ,s is even in s *)
% /. (ell t ^ 2) ^ ee_ :→ ell ^ ee t ^ (2 ee) // Simplify
```

Out[ ]= $-e^{\frac{1}{4}\,i\,(2+h-p)\,\pi}\,(2\,\pi)^{\frac{1}{4}\times(2+h-p)}\,t^{\frac{1}{2}\times(-2-h+p)}\,(\text{ell}\,t^2)^{\frac{1}{4}\times(2+h-p)}$

Out[ ]= $-e^{\frac{1}{4}\,i\,(2+h-p)\,\pi}\,\text{ell}^{\frac{1}{4}\times(2+h-p)}\,(2\,\pi)^{\frac{1}{4}\times(2+h-p)}$

This is a non-zero quantity not depending on t.

Case eps = -1

```
In[ ]:= F = tht[0] × f[-p, t] × Phi[h, p, -p, p]
sh[-3, -1, F, subnab] /. eps → -1 (* is of course 0 *)
sh[3, -1, F, subnab] /. eps → -1 /. ell → -Abs[ell]

fmp = f[-p, t] /. DSolve[% == 0, f[-p, t], t][[1]] /. C[1] → 1 // Simplify
```

Out[ ]= f[-p, t] × Phi[h, p, -p, p] × tht[0]

Out[ ]= 0

Out[ ]= $\frac{1}{4\times(1+p)}$ p Phi[3 + h, -1 + p, 1 - p, -1 + p] × tht[0] $\left((-4+h-p-4\pi t^2 \text{Abs[ell]})\, f[-p,t] + 2\, t\, f^{(0,1)}[-p,t]\right)$

Out[ ]= $e^{\pi t^2 \text{Abs[ell]}}\, t^{\frac{1}{2}\times(4-h+p)}$

```
In[ ]:= fmp ^ (-1) t ^ (p + 1) WhittakerV [kap[h, -p], s[h, -p], 2 Pi Abs[ell] t ^ 2] //. parmsub /.
        m[h, -p] → 0 /. eps → -1 //. Whrel /. (t ^ 2) ^ ee_ :→ t ^ (2 ee) // Simplify
```

Out[ ]= $-e^{-\frac{1}{4} i (-2+h+p)\pi}\, (2\pi)^{\frac{1}{4}\times(2-h-p)}\, \text{Abs[ell]}^{\frac{1}{4}\times(2-h-p)}$

Copies of kernel relations from 19

In[ ]:=

```
Clear[krnab3m1, krnabm3m1, f]
krnab3m1[1] = f[2 + r, t] == -((ⅈ ((-4 + h - 2 p - r + 4 ell π t^2) f[r, t] + 2 t f^(0,1)[r, t])) /
      (4 Sqrt[2 π] t Sqrt[Abs[ell]] Sqrt[1 + m[h, r]]));

krnab3m1[-1] = f[2 + r, t] ==
   (ⅈ ((-4 + h - 2 p - r + 4 ell π t^2) f[r, t] + 2 t f^(0,1)[r, t])) / (4 Sqrt[2 π] t Sqrt[Abs[ell]] Sqrt[m[h, r]]);

krnabm3m1[1] = f[-2 + r, t] ==
   -((ⅈ ((4 + h + 2 p - r + 4 ell π t^2) f[r, t] - 2 t f^(0,1)[r, t])) / (4 Sqrt[2 π] t Sqrt[ell] Sqrt[m[h, r]]));

krnabm3m1[-1] = f[-2 + r, t] ==
   (ⅈ ((4 + h + 2 p - r + 4 ell π t^2) f[r, t] - 2 t f^(0,1)[r, t])) / (4 Sqrt[2 π] t Sqrt[Abs[ell]] Sqrt[1 + m[h, r]]);
```

More than one component, case eps=1

We consider two successive components f_r and f_{r+2} and combine the results of the kernel relations. First we solve f_{r+2} from one of the kernel relations, obtaining an expression in terms of f_r and its derivative.

```
In[ ]:= frp = f[r + 2, t] /. Solve[krnab3m1[1], f[r + 2, t]][[1]] // Simplify
```

Out[ ]=
$$-\frac{i\left(\left(-4+h-2\,p-r+4\,\text{ell}\,\pi\,t^{2}\right)f[r,t]+2\,t\,f^{(0,1)}[r,t]\right)}{4\sqrt{2\pi}\;t\;\sqrt{\text{Abs[ell]}}\;\sqrt{1+m[h,r]}}$$

We insert this expression in the other kernel relation. This results in a second order differential equation.

```
In[ ]:= deq =
  krnabm3m1[1] /. r → r + 2 /. f^(0,1)[2 + r, t] → D[frp, t] /. f[r + 2, t] → frp //. parmsub /. eps → 1 //
   Simplify
```

Out[ ]=
$$f[r,t]==\left(-\left(\left(-16+h^{2}-16\,p-4\,p^{2}-2\,h\,r+r^{2}-16\,\text{ell}\,\pi\,t^{2}+8\,\text{ell}\,h\,\pi\,t^{2}-8\,\text{ell}\,\pi\,r\,t^{2}+16\,\text{ell}^{2}\,\pi^{2}\,t^{4}\right)f[r,t]\right)+4\,t\left(-\left((3+2\,p)\,f^{(0,1)}[r,t]\right)+t\,f^{(0,2)}[r,t]\right)\right)\Big/\left(32\sqrt{\text{ell}}\;\pi\,t^{2}\sqrt{\text{Abs[ell]}}\;(1+m[h,r])\right)$$

Go over to expected form

```
In[ ]:= Clear[wh, tau]
deq /. f^(0,ee_)[r, t] :→ D[t^(p + 1) wh[2 Pi ell t^2], {t, ee}] /. f[r, t] → t^(p + 1) wh[2 Pi ell t^2] /.
   t → Sqrt[tau] / Sqrt[2 Pi ell] // Simplify
Solve[%, wh''[tau]][[1]] /. Abs[ell] → ell // Simplify
(wh''[tau] /. %) / wh[tau] // Simplify
% == 1/4 - kap[h, r] / tau + (s[h, r]^2 - 1/4) / tau^2 //. parmsub /. eps → 1 // Simplify
```

Out[ ]=
$$\frac{1}{16\sqrt{2\pi}\;\sqrt{\text{tau}}}\left(\frac{\sqrt{\text{tau}}}{\sqrt{\text{ell}}}\right)^{p}\left(\frac{2^{4-\frac{p}{2}}\,\pi^{-p/2}\,\text{tau}\;\text{wh[tau]}}{\sqrt{\text{ell}}}+\left((2\,\pi)^{-p/2}\left(\left(-4+h^{2}-2\,h\,r+r^{2}-8\,\text{tau}+4\,h\,\text{tau}-4\,r\,\text{tau}+4\,\text{tau}^{2}\right)\text{wh[tau]}-16\,\text{tau}^{2}\,\text{wh}''[\text{tau}]\right)\right)\Big/\left(\sqrt{\text{Abs[ell]}}\;(1+m[h,r])\right)\right)==0$$

Out[ ]=
$$\left\{\text{wh}''[\text{tau}]\to\frac{1}{16\,\text{tau}^{2}}\left(-4+h^{2}-2\,h\,r+r^{2}+8\,\text{tau}+4\,h\,\text{tau}-4\,r\,\text{tau}+4\,\text{tau}^{2}+16\,\text{tau}\,m[h,r]\right)\text{wh[tau]}\right\}$$

Out[ ]=
$$\frac{1}{16\,\text{tau}^{2}}\left(-4+h^{2}-2\,h\,r+r^{2}+8\,\text{tau}+4\,h\,\text{tau}-4\,r\,\text{tau}+4\,\text{tau}^{2}+16\,\text{tau}\,m[h,r]\right)$$

Out[ ]= True

Indeed the Whittaker differential equation .

More than one component, case eps=-1

In[ ]:=
```
frp = f[r + 2, t] /. Solve[krnab3m1[-1], f[r + 2, t]][[1]] // Simplify
```

Out[ ]=
$$\frac{i\left(\left(-4+h-2p-r+4\,\mathrm{ell}\,\pi t^2\right) f[r,t]+2t\,f^{(0,1)}[r,t]\right)}{4\sqrt{2\pi}\,t\,\sqrt{\mathrm{Abs}[\mathrm{ell}]}\,\sqrt{m[h,r]}}$$

In[ ]:=
```
deq = krnabm3m1[-1] /. r → r + 2 /. f^(0,1)[2 + r, t] → D[frp, t] /. f[r + 2, t] → frp //. parmsub /.
    eps → -1 /. ell → -Abs[ell] // Simplify
```

Out[ ]=
$$\left(f[r,t]\left(-16+h^2-16p-4p^2-2hr+r^2+16\pi^2 t^4\,\mathrm{Abs}[\mathrm{ell}]^2+8\pi t^2\,\mathrm{Abs}[\mathrm{ell}]\,(2-h+r+4\,m[h,r])\right)+4t\left((3+2p)\,f^{(0,1)}[r,t]-t\,f^{(0,2)}[r,t]\right)\right)/\left(t\,\mathrm{Abs}[\mathrm{ell}]\,m[h,r]\right)=0$$

In[ ]:=
```
Clear[wh, tau]
deq /. f^(0,ee_)[r, t] :→ D[t^(p + 1) wh[2 Pi Abs[ell] t^2], {t, ee}] /.
   f[r, t] → t^(p + 1) wh[2 Pi Abs[ell] t^2] /. t → Sqrt[tau] / Sqrt[2 Pi Abs[ell]] // Simplify
Solve[%, wh''[tau]][[1]] // Simplify
(wh''[tau] /. %) / wh[tau] // Simplify
% == 1/4 - kap[h, r]/tau + (s[h, r]^2 - 1/4)/tau^2 //. parmsub /. eps → -1 // Simplify
```

Out[ ]=
$$\frac{1}{m[h,r]}\,\mathrm{tau}^{p/2}\,\mathrm{Abs}[\mathrm{ell}]^{-1-\frac{p}{2}}\left(\left(-4+h^2+r^2+8\,\mathrm{tau}+4r\,\mathrm{tau}+4\,\mathrm{tau}^2-2h(r+2\,\mathrm{tau})+16\,\mathrm{tau}\,m[h,r]\right)\mathrm{wh}[\mathrm{tau}]-16\,\mathrm{tau}^2\,\mathrm{wh}''[\mathrm{tau}]\right)=0$$

Out[ ]=
$$\left\{\mathrm{wh}''[\mathrm{tau}]\to\frac{1}{16\,\mathrm{tau}^2}\left(-4+h^2+r^2+8\,\mathrm{tau}+4r\,\mathrm{tau}+4\,\mathrm{tau}^2-2h(r+2\,\mathrm{tau})+16\,\mathrm{tau}\,m[h,r]\right)\mathrm{wh}[\mathrm{tau}]\right\}$$

Out[ ]=
$$\frac{1}{16\,\mathrm{tau}^2}\left(-4+h^2+r^2+8\,\mathrm{tau}+4r\,\mathrm{tau}+4\,\mathrm{tau}^2-2h(r+2\,\mathrm{tau})+16\,\mathrm{tau}\,m[h,r]\right)$$

Out[ ]= True