21b Check kernel relations and determination of coefficients

We suppose that one component has the form $f_r = t^{p+1}\, \mathrm{wh}_{\kappa(r),s(r)}(2\pi \mid l \mid t^2)$

and try to determine the components f_{r-2} and f_{r+2} under the assumption m(h,r±2)≠0

In[ ]:=
```
Clear[kap, s, eps]
subkaps = {kap[r_ + 2] :> kap[r] - eps / 2,
    kap[r_ - 2] :> kap[r] + eps / 2, s[r_ + 2] :> s[r] - 1 / 2, s[r_ - 2] :> s[r] + 1 / 2};
```

We use the kernel relations given in 21a.

General starting formula

In[ ]:=
```
Clear[p, m, wh, fr, frp2, frm2, qoW, qoV]
fr[r_, t_] := t ^ (p + 1)
   {WhittakerW[kap[r], s[r], 2 Pi Abs[ell] t ^ 2], WhittakerV[kap[r], s[r], 2 Pi Abs[ell] t ^ 2]}
frp2 = fr[r + 2, t] {qoW, qoV} /. subkaps;
frm2 = fr[r - 2, t] {qoW, qoV} /. subkaps;
```

In[ ]:=
```
Clear[coV, voW]
coV[r_] = (-1) ^ (m[h, r]) Factorial[m[h, r]] ^ (-1 / 2)
coW[r_] = I ^ (m[h, r]) Factorial[m[h, r]] ^ (1 / 2)
```

Out[ ]= $\frac{(-1)^{m[h,r]}}{\sqrt{m[h,\ r]!}}$

Out[ ]= $i^{m[h,r]}\sqrt{m[h,\ r]!}$

Case ε=1, up

In[ ]:=
```
krnab3m1[1] /. f[2 + r, t] → frp2 /. f → fr /. subkaps /. eps → 1 /. Abs[ell] → ell /. Whrel /.
        Whups[kap[r] - 1 / 2, s[r] - 1 / 2] /. Whups[kap[r] + 1 / 2, s[r] - 1 / 2] /. Whup[kap[r] - 1] /.
     h - r → 4 s[r] //. {(ell pp_) ^ ee_ :> ell ^ ee pp ^ ee, (t ^ ff_) ^ ee_ :> t ^ (ff ee)} // Simplify;
rel = % /. {2 kap[r] →  -2 m[h, r] - 2 eps s[r] - 1} /. eps → 1 // Simplify
```

Out[ ]= $\Big\{\Big(i\, t^p \left(1 + m[h,\ r] + i\, qoW \sqrt{1 + m[h,\ r]}\right) \left(\left(m[h,\ r] + 2\left(ell\, \pi\, t^2 + s[r]\right)\right) WhittakerW\left[kap[r],\ s[r],\ 2\, ell\, \pi\, t^2\right] - WhittakerW\left[1 + kap[r],\ s[r],\ 2\, ell\, \pi\, t^2\right]\right)\Big) \Big/ \left(\sqrt{ell}\ \sqrt{2\pi}\ (1 + m[h,\ r])^{3/2}\right),$
$\left(i\, t^p \left(1 + qoV \sqrt{1 + m[h,\ r]}\right)\right) \left(\left(-2 + 8\, ell\, \pi\, t^2 - 4\, kap[r] + 4\, s[r]\right) WhittakerV\left[kap[r],\ s[r],\ 2\, ell\, \pi\, t^2\right] + \left(1 + 4\, kap[r] + 4\, kap[r]^2 - 4\, s[r]^2\right) WhittakerV\left[1 + kap[r],\ s[r],\ 2\, ell\, \pi\, t^2\right]\right) \Big/ \left(4 \sqrt{ell}\ \sqrt{2\pi}\ \sqrt{1 + m[h,\ r]}\right)\Big\} == \{0,\ 0\}$

In[ ]:=
```
rel /. {qoW → I Sqrt[1 + m[h, r]], qoV → -1 / Sqrt[1 + m[h, r]]} // Simplify
```

Out[ ]= True

```
In[ ]:= {coW[r + 2] / coW[r] == I Sqrt[1 + m[h, r]], coV[r + 2] / coV[r] == -1 / Sqrt[1 + m[h, r]]} //. parmsub /.
          eps → 1 /. Factorial[1 + nn_] :→ (nn + 1) Factorial[nn] //.
        {(pp_ qq_)^ee_ :→ pp^ee qq^ee} // Simplify
Out[ ]= {True, True}
```

Case ε=1, down

```
In[ ]:=
krnabm3m1[1] /. f[-2 + r, t] → frm2 /. f → fr /. subkaps /. eps → 1 /. Abs[ell] → ell //. Whrel /.
       Whdns[kap[r] + 1 / 2, s[r] + 1 / 2] /. Whdn[kap[r] + 1] //.
    {(ell pp_)^ee_ :→ ell^ee pp^ee, (t^ff_)^ee_ :→ t^(ff ee)} // Simplify;
% /. {qoW → (I Sqrt[1 + m[h, r - 2]])^(-1), qoV → (-(1 + m[h, r - 2])^(-1 / 2))^(-1)} /.
      m[h, r - 2] → m[h, r] - 1 /. s[r - 2] → s[r] + 1 / 2 // Simplify;
% //. {s[r] → (h - r) / 4, kap[r] → -m[h, r] - s[r] - 1 / 2} // Simplify
Out[ ]= True
```

Case ε=1, relation for $r = r_0$

In the case that $-p \le r_0 < p$ there is still a relation to check.

```
In[ ]:= Clear[r0]
krnabm3m1[1]
```

$$\text{Out}[\,]= f[-2+r,t] == -\frac{i\left(\left(4+h+2p-r+4\,\text{ell}\,\pi t^2\right) f[r,t]-2t\,f^{(0,1)}[r,t]\right)}{4\sqrt{\text{ell}}\,\sqrt{2\pi}\,t\,\sqrt{m[h,r]}}$$

```
In[ ]:= i ((4 + h + 2 p - r + 4 ell π t^2) f[r, t] - 2 t f^(0,1)[r, t]) /. r → r0 /. f → fr // Simplify;
% {coW, coV} /. Whrel //. {kap[r0] → -m[h, r0] - s[r0] - 1 / 2, m[h, r0] → 0, s[r0] → (h - r0) / 4} /.
   Abs[ell] → ell // Simplify
% /. h - r0 → r0 - h //. Whrel // Simplify
```

$$\text{Out}[\,]= \left\{4\,i\,\text{coW}\,t^{1+p}\,\text{WhittakerW}\left[\frac{1}{4}\times(2-h+r0),\frac{h-r0}{4},2\,\text{ell}\,\pi t^2\right],0\right\}$$

$$\text{Out}[\,]= \left\{i\,2^{\frac{1}{4}\times(10-h+r0)}\,\text{coW}\,e^{-\text{ell}\,\pi t^2}\,\pi^{\frac{1}{4}\times(2-h+r0)}\,t^{1+p}\left(\text{ell}\,t^2\right)^{\frac{1}{4}\times(2-h+r0)},0\right\}$$

So the W-Whittaker function does not yield a solution.

Case ε=-1, down

```
In[ ]:= krnabm3m1[-1] /. f[-2 + r, t] → frm2 /. f → fr /. subkaps /. eps → -1 /. ell → -Abs[ell] //.
          Whrel /. Whdns[kap[r] - 1 / 2, s[r] + 1 / 2] /. Whdn[kap[r] + 1] /.
     {(ell pp_)^ee_ :→ ell ^ee pp^ee, (t^ff_)^ee_ :→ t^(ff ee)} // Simplify;
rel = (% //. {h - r → 4 s[r], -I h + I r → -4 I s[r], 2 I kap[r] → -2 I m[h, r] + 2 I s[r] - I,
          4 I kap[r] → -4 I m[h, r] + 4 I s[r] - 2 I} // Simplify) /.
     {2 I kap[r] → -2 I m[h, r] + 2 I s[r] - I} // Simplify
```

Out[ ]= $\left\{\left(t^{p}\left(-i - i\, m[h, r] + qoW \sqrt{1 + m[h, r]}\right)\right.\right.$
$\left((-1 + 2\, kap[r] + 2\, s[r])\, WhittakerW[-1 + kap[r], s[r], 2\pi t^{2} Abs[ell]] + 2\, WhittakerW[kap[r], s[r], 2\pi t^{2} Abs[ell]]\right)\Big) \Big/ \left(2\sqrt{2\pi}\sqrt{Abs[ell]}\sqrt{1 + m[h, r]}\right),$
$-\Big(\Big(i\, t^{p}\left(1 + qoV \sqrt{1 + m[h, r]}\right) \times \big(2\, WhittakerV[-1 + kap[r], s[r], 2\pi t^{2} Abs[ell]] +$
$(1 - 2\, kap[r] + 2\, s[r])\, WhittakerV[kap[r], s[r], 2\pi t^{2} Abs[ell]]\big)\Big) \Big/$
$\left(2\sqrt{2\pi}\sqrt{Abs[ell]}\sqrt{1 + m[h, r]}\right)\Big)\Big\}$ == {0, 0}

```
In[ ]:= rel /. {qoW → I Sqrt[1 + m[h, r]], qoV → -1 / Sqrt[1 + m[h, r]]} // Simplify
```

Out[ ]= True

```
In[ ]:= {coW[r - 2] / coW[r] == I Sqrt[1 + m[h, r]], coV[r - 2] / coV[r] == -1 / Sqrt[1 + m[h, r]]} //. parmsub /.
       eps → -1 /. Factorial[1 + nn_] :→ Factorial[nn] (1 + nn) //.
   {(pp_ qq_)^ee_ :→ pp^ee qq^ee, (1 / pp_)^ee_ :→ pp^(-ee)} // Simplify
```

Out[ ]= {True, True}

Case ε=-1, up

```
In[ ]:= krnab3m1[-1] /. f[2 + r, t] → frp2 /. f → fr /. subkaps /. {qoW → (I Sqrt[1 + m[h, r + 2]])^(-1),
```

$\quad qoV \to (-1 / Sqrt[1 + m[h, r + 2]])^{\wedge}(-1),\ qoM \to \left(-\dfrac{i\sqrt{1 + m[h, r + 2]}}{1 + 2\, s[r + 2]}\right)^{\wedge}(-1)\Big\}$ /.

```
         m[h, r + 2] → m[h, r] - 1 /. eps → -1 /. ell → -Abs[ell] //. Whrel /.
      Whups[kap[r] + 1 / 2, s[r] - 1 / 2] //. {(ell pp_)^ee_ :→ ell ^ee pp^ee,
      (t^ff_)^ee_ :→ t^(ff ee)} // Simplify;
% //. {h - r → 4 s[r], kap[r] → -m[h, r] + s[r] - 1 / 2} // Simplify;
% /. s[rr_] → (h - rr) / 4 // Simplify
```

Out[ ]= True

Case ε=-1, relation for $r = r_0$

In[ ]:=
```
krnab3m1[-1]
```

Out[ ]=
$$f[2+r,t] == \frac{i\left(\left(-4+h-2p-r+4\,ell\,\pi\,t^2\right)f[r,t]+2\,t\,f^{(0,1)}[r,t]\right)}{4\sqrt{2\pi}\;t\;\sqrt{Abs[ell]}\;\sqrt{m[h,r]}}$$

In[ ]:=
```
Clear[r0]
((-4 + h - 2 p - r + 4 ell π t^2) f[r, t] + 2 t f^(0,1)[r, t]) /. r → r0 /. f → fr // Simplify;
% {coW, coV} //. Whrel /. eps → -1 /. ell → -Abs[ell] /. kap[r0] → s[r0] - 1/2 // Simplify;
% /. s[r0] → (h - r0)/4 /. Whrel // Simplify
```

Out[ ]=
$$\left\{-2^{\frac{1}{4}\times(10+h-r0)}\,coW\,e^{-\pi\,t^2\,Abs[ell]}\,\pi^{\frac{1}{4}\times(2+h-r0)}\,t^{1+p}\,\left(t^2\right)^{\frac{1}{4}\times(2+h-r0)}\,Abs[ell]^{\frac{1}{4}\times(2+h-r0)},\;0\right\}$$

So the W-Whittaker functions do not satisfy the kernel relations for these values of r_0.