22c. Case $0 < m_0(j_2) < p$, and $\varepsilon = -1$

In[ ]:=
```
Clear[r0]
F = tht[0] × f[r0, t] × Phi[h, p, r0, p] + tht[1] × f[r0 - 2, t] × Phi[h, p, r0 - 2, p]
```

Out[ ]=
```
f[r0, t] × Phi[h, p, r0, p] × tht[0] + f[-2 + r0, t] × Phi[h, p, -2 + r0, p] × tht[1]
```

We use that one shift operator gives zero:

In[ ]:=
```
sh[-3, -1, F, subnab] /. eps → -1 // Simplify
Coefficient[%, Phi[-3 + h, -1 + p, -1 + r0, -1 + p]]
fr0m2 = f[r0 - 2, t] /. Solve[% == 0, f[r0 - 2, t]][[1]] // Simplify
```

Out[ ]=
$$-\frac{1}{4\times(1+p)} p \Big(4 i \sqrt{2\pi}\, t \sqrt{\text{Abs[ell]}}\, f[-2+r0, t] \times \text{Phi}[-3+h, -1+p, -1+r0, -1+p] \times \text{tht}[0] + (4+h+2p-r0+4\,\text{ell}\,\pi t^2)\, f[r0, t] \times \text{Phi}[-3+h, -1+p, -1+r0, -1+p] \times \text{tht}[0] + 6 f[-2+r0, t] \times \text{Phi}[-3+h, -1+p, -3+r0, -1+p] \times \text{tht}[1] + h f[-2+r0, t] \times \text{Phi}[-3+h, -1+p, -3+r0, -1+p] \times \text{tht}[1] + 2p f[-2+r0, t] \times \text{Phi}[-3+h, -1+p, -3+r0, -1+p] \times \text{tht}[1] - r0 f[-2+r0, t] \times \text{Phi}[-3+h, -1+p, -3+r0, -1+p] \times \text{tht}[1] + 4\,\text{ell}\,\pi t^2 f[-2+r0, t] \times \text{Phi}[-3+h, -1+p, -3+r0, -1+p] \times \text{tht}[1] - 2t\,\text{Phi}[-3+h, -1+p, -3+r0, -1+p] \times \text{tht}[1]\, f^{(0,1)}[-2+r0, t] - 2t\,\text{Phi}[-3+h, -1+p, -1+r0, -1+p] \times \text{tht}[0]\, f^{(0,1)}[r0, t]\Big)$$

Out[ ]=
$$-\frac{1}{4\times(1+p)} p \Big(4 i \sqrt{2\pi}\, t \sqrt{\text{Abs[ell]}}\, f[-2+r0, t] \times \text{tht}[0] + (4+h+2p-r0+4\,\text{ell}\,\pi t^2)\, f[r0, t] \times \text{tht}[0] - 2t\,\text{tht}[0]\, f^{(0,1)}[r0, t]\Big)$$

Out[ ]=
$$\frac{i\left((4+h+2p-r0+4\,\text{ell}\,\pi t^2)\, f[r0, t] - 2t\, f^{(0,1)}[r0, t]\right)}{4\sqrt{2\pi}\, t\sqrt{\text{Abs[ell]}}}$$

Next we use the other shift operator

In[ ]:=
```
Fp = sh[3, -1, F, subnab] /. eps → -1 // Simplify
fp[r0 - 1, t] = Coefficient[%, Phi[3 + h, -1 + p, 1 + r0, -1 + p]] / tht[0] // Simplify
```

Out[ ]=
$$-\frac{1}{4\times(1+p)} p\Big(f[r0,t]\big((4-h+2p+r0-4\,ell\,\pi t^2)\,Phi[3+h,-1+p,1+r0,-1+p]\times tht[0]-4\,i\sqrt{2\pi}\,t\sqrt{Abs[ell]}\,Phi[3+h,-1+p,-1+r0,-1+p]\times tht[1]\big)+ f[-2+r0,t]\big((2-h+2p+r0-4\,ell\,\pi t^2)\,Phi[3+h,-1+p,-1+r0,-1+p]\times tht[1]- 8\,i\sqrt{\pi}\,t\sqrt{Abs[ell]}\,Phi[3+h,-1+p,-3+r0,-1+p]\times tht[2]\big)- 2t\big(Phi[3+h,-1+p,-1+r0,-1+p]\times tht[1]\,f^{(0,1)}[-2+r0,t]+ Phi[3+h,-1+p,1+r0,-1+p]\times tht[0]\,f^{(0,1)}[r0,t]\big)\Big)$$

Out[ ]=
$$\frac{p\,\big((-4+h-2p-r0+4\,ell\,\pi t^2)\,f[r0,t]+2t\,f^{(0,1)}[r0,t]\big)}{4\times(1+p)}$$

We know the lowest order term of b_{υ}^{-}.

Hence we have with a non-zero factor **cnz**:

In[ ]:=
```
Clear[cnz]
eq = ((-4 + h - 2 p - r0 + 4 ell π t^2) f[r0, t] + 2 t f^(0,1)[r0, t]) ==
    cnz t^(0 + 1) WhittakerV[-p + 1 + (j2 - 1) / 2, nu2, 2 Pi Abs[ell] t^2] /.
  ell → -Abs[ell] /. hpsub // Simplify
```

Out[ ]=
$$(-4+h-2p-r0-4\pi t^2\,Abs[ell])\,f[r0,t]+2t\,f^{(0,1)}[r0,t]== cnz\,t\,WhittakerV\Big[\frac{1}{4}\times(2+h-p),\ \frac{1}{2}Abs[h-p],\ 2\pi t^2\,Abs[ell]\Big]$$

We solve for the derivative of f_{r_0}

In[ ]:=
```
fr0d = f^(0,1)[r0, t] /. Solve[eq, f^(0,1)[r0, t]][[1]] // Simplify
```

Out[ ]=
$$\frac{1}{2t}\Big((4-h+2p+r0+4\pi t^2\,Abs[ell])\,f[r0,t]+ cnz\,t\,WhittakerV\Big[\frac{1}{4}\times(2+h-p),\ \frac{1}{2}Abs[h-p],\ 2\pi t^2\,Abs[ell]\Big]\Big)$$

This can be inserted in the eigenfunction equations, together with the expressions for f_{r_0+2}

In[ ]:=
```
ei =
  efeqn[h, p, r0, f, ell, 0, -1] /. j → j2 /. nu → nu2 /. f[r0 - 2, t] → fr0m2 /. f^(0,1)[-2 + r0, t] →
        D[fr0m2, t] /. f^(0,2)[r0, t] → D[fr0d, t] /. f^(0,1)[r0, t] → fr0d //. Whrel /.
    hpsub /. ell → -Abs[ell] /. Abs[h - p]^2 → (h - p)^2 // Simplify
```

Out[ ]=
$$\Big\{\frac{1}{16}\,\text{cnz}\,t\Big(-4\times(4+2\,h-p+r0-8\,\pi\,t^2\,\text{Abs[ell]})\,\text{WhittakerV}\Big[\frac{1}{4}\times(2+h-p),\ \frac{1}{2}\,\text{Abs}[h-p],\ 2\,\pi\,t^2\,\text{Abs[ell]}\Big]+(16-3\,h^2-8\,p-3\,p^2+h\,(8+6\,p))\,\text{WhittakerV}\Big[\frac{1}{4}\times(6+h-p),\ \frac{1}{2}\,\text{Abs}[h-p],\ 2\,\pi\,t^2\,\text{Abs[ell]}\Big]\Big),$$
$$-\frac{3}{16}\,\text{cnz}\,(p+r0)\,t\Big(4\times(4+2\,h-p+r0-8\,\pi\,t^2\,\text{Abs[ell]})\,\text{WhittakerV}\Big[\frac{1}{4}\times(2+h-p),\ \frac{1}{2}\,\text{Abs}[h-p],\ 2\,\pi\,t^2\,\text{Abs[ell]}\Big]+(-16+3\,h^2+8\,p+3\,p^2-2\,h\,(4+3\,p))\,\text{WhittakerV}\Big[\frac{1}{4}\times(6+h-p),\ \frac{1}{2}\,\text{Abs}[h-p],\ 2\,\pi\,t^2\,\text{Abs[ell]}\Big]\Big)\Big\}$$

These expressions should be zero. We insert the asymptotic behavior of the V-Whittaker function.

In[ ]:=
```
Clear[tau]
eia = ei /. t → Sqrt[tau] / Sqrt[2 Pi Abs[ell]] /. Abs[ell] → ell /.
    WhittakerV[kap_, s_, tau_] :→ -E^(-Pi I kap) tau^(-kap) E^(tau / 2) // Simplify
```

Out[ ]=
$$\Big\{\frac{1}{16\,\sqrt{\text{ell}}\,\sqrt{2\,\pi}}\,i\,\text{cnz}\,e^{-\frac{1}{4}\,i\,(h\,\pi-p\,\pi+2\,i\,\text{tau})}\,\text{tau}^{\frac{1}{4}\times(-4-h+p)}\,(3\,h^2+3\,p^2+4\,p\,(2+\text{tau})-2\,h\,(4+3\,p+4\,\text{tau})-4\times(4+(4+r0)\,\text{tau}-4\,\text{tau}^2)),$$
$$\frac{1}{16\,\sqrt{\text{ell}}\,\sqrt{2\,\pi}}\,3\,i\,\text{cnz}\,e^{-\frac{1}{4}\,i\,(h\,\pi-p\,\pi+2\,i\,\text{tau})}\,(p+r0)\,\text{tau}^{\frac{1}{4}\times(-4-h+p)}\,(3\,h^2+3\,p^2+4\,p\,(2+\text{tau})-2\,h\,(4+3\,p+4\,\text{tau})-4\times(4+(4+r0)\,\text{tau}-4\,\text{tau}^2))\Big\}$$

In[ ]:=
```
eia E^(-tau / 2) tau^(-1 + (h - p) / 4) // Simplify
Limit[%, tau → Infinity]
```

Out[ ]=
$$\Big\{\Big(i\,\text{cnz}\,e^{-\frac{1}{4}\,i\,(h-p)\,\pi}\,(3\,h^2+3\,p^2+4\,p\,(2+\text{tau})-2\,h\,(4+3\,p+4\,\text{tau})-4\times(4+(4+r0)\,\text{tau}-4\,\text{tau}^2))\Big)\Big/\Big(16\,\sqrt{\text{ell}}\,\sqrt{2\,\pi}\,\text{tau}^2\Big),\ \Big(3\,i\,\text{cnz}\,e^{-\frac{1}{4}\,i\,(h-p)\,\pi}\,(p+r0)\,(3\,h^2+3\,p^2+4\,p\,(2+\text{tau})-2\,h\,(4+3\,p+4\,\text{tau})-4\times(4+(4+r0)\,\text{tau}-4\,\text{tau}^2))\Big)\Big/\Big(16\,\sqrt{\text{ell}}\,\sqrt{2\,\pi}\,\text{tau}^2\Big)\Big\}$$

Out[ ]=
$$\Big\{\frac{i\,\text{cnz}\,e^{-\frac{1}{4}\,i\,(h-p)\,\pi}}{\sqrt{\text{ell}}\,\sqrt{2\,\pi}},\ \frac{e^{-\frac{1}{4}\,i\,(h-p)\,\pi}\,(48\,i\,\text{cnz}\,p+48\,i\,\text{cnz}\,r0)}{16\,\sqrt{\text{ell}}\,\sqrt{2\,\pi}}\Big\}$$

So **cnz** has to vanish.