22f. Case $0 \le m_0(j_1) < m_0(j_2)$ and $\varepsilon = -1$

In[ ]:=
```
m0[j1] /. mrsub /. eps → -1
m0[j2] /. mrsub /. eps → -1
m0[j1] - m0[j2] /. mrsub /. eps → -1 // Simplify
```

Out[ ]= $\frac{1}{2}(-p + r0)$

Out[ ]= $p + r0$

Out[ ]= $\frac{1}{2} \times (-3\,p - r0)$

This shows that $r_0 \ge p$.

The lowest component of F is f_{-p} and the highest component is f_p. These orders are different, since p≥1.

We denote F^+ by **Fp**, and F^- by **Fm**.

The components of $F^\pm$ depend on two components of F.

In[ ]:=
```
Clear[F, Fp, Fm, f, fph, fml]
F = tht[m[h, p]] × f[p, t] × Phi[h, p, p, p] + tht[m[h, p] + 1] × f[p - 2, t] × Phi[h, p, p - 2, p]
sh[3, -1, F, subnab] /. eps → -1 /. j → j2 // Simplify
fph = (* highest component of Fp *)
 Coefficient[%, Phi[3 + h, -1 + p, -1 + p, -1 + p]] / tht[m[h, p] + 1] // Simplify
```

Out[ ]= f[p, t] × Phi[h, p, p, p] × tht[m[h, p]] + f[-2 + p, t] × Phi[h, p, -2 + p, p] × tht[1 + m[h, p]]

Out[ ]= $\frac{1}{4 \times (1 + p)}$ $\Big(4\,i\,p\,\sqrt{2\,\pi}\;t\,\sqrt{\text{Abs[ell]}}\,\Big(\text{f[p, t]}\,\sqrt{1 + \text{m[h, p]}}\;\text{Phi[3 + h, -1 + p, -1 + p, -1 + p]} \times \text{tht[1 + m[h, p]]} + \text{f[-2 + p, t]}\,\sqrt{2 + \text{m[h, p]}}\;\text{Phi[3 + h, -1 + p, -3 + p, -1 + p]} \times \text{tht[2 + m[h, p]]}\Big) + p\,\text{Phi[3 + h, -1 + p, -1 + p, -1 + p]} \times \text{tht[1 + m[h, p]]}\,\big(\big(-2 + h - 3\,p + 4\,\text{ell}\,\pi\,t^2\big)\,\text{f[-2 + p, t]} + 2\,t\,f^{(0,1)}\text{[-2 + p, t]}\big)\Big)$

Out[ ]= $\frac{1}{4 \times (1 + p)}\,p\,\Big(\big(-2 + h - 3\,p + 4\,\text{ell}\,\pi\,t^2\big)\,\text{f[-2 + p, t]} + 2\,t\,\Big(2\,i\,\sqrt{2\,\pi}\,\sqrt{\text{Abs[ell]}}\;\text{f[p, t]}\,\sqrt{1 + \text{m[h, p]}} + f^{(0,1)}\text{[-2 + p, t]}\Big)\Big)$

In[∘]:=

```
F = tht[m[h, -p]] × f[-p, t] × Phi[h, p, -p, p] + tht[m[h, -p] - 1] × f[2 - p, t] × Phi[h, p, 2 - p, p]
sh[-3, -1, F, subnab] /. eps → -1 /. j → j1 // Simplify
fml = (* lowest component of Fm *)
 Coefficient[%, Phi[-3 + h, -1 + p, 1 - p, -1 + p]] / tht[-1 + m[h, -p]] // Simplify
```

Out[∘]= f[2 - p, t] × Phi[h, p, 2 - p, p] × tht[-1 + m[h, -p]] + f[-p, t] × Phi[h, p, -p, p] × tht[m[h, -p]]

Out[∘]= $-\frac{1}{4\times(1+p)} p \Big(4 i \sqrt{2\pi}\ t \sqrt{\text{Abs[ell]}} \big(\text{f[2 - p, t]} \sqrt{-1+\text{m[h, -p]}}\ \text{Phi[-3 + h, -1 + p, 3 - p, -1 + p]} \times \text{tht[-2 + m[h, -p]]} + \text{f[-p, t]} \sqrt{\text{m[h, -p]}}\ \text{Phi[-3 + h, -1 + p, 1 - p, -1 + p]} \times \text{tht[-1 + m[h, -p]]}\big) + \text{Phi[-3 + h, -1 + p, 1 - p, -1 + p]} \times \text{tht[-1 + m[h, -p]]} \big((2 + h + 3 p + 4\ \text{ell}\ \pi t^2)\ \text{f[2 - p, t]} - 2 t\ f^{(0,1)}\text{[2 - p, t]}\big)\Big)$

Out[∘]= $-\frac{1}{4\times(1+p)} p \Big((2 + h + 3 p + 4\ \text{ell}\ \pi t^2)\ \text{f[2 - p, t]} + 4 i \sqrt{2\pi}\ t \sqrt{\text{Abs[ell]}}\ \text{f[-p, t]} \sqrt{\text{m[h, -p]}} - 2 t\ f^{(0,1)}\text{[2 - p, t]}\Big)$

We assume that the derivatives $F^+$and F^- are a linear combination of basis functions, with determining components as indicated in Table 4.17.
The K-type of F^+corresponds to a point on the left boundary of the sector Sect(j_2).
For F^+ we have to use the lower part of Table 4.17, applying it with $x^{0,p-1}$.

In[∘]:=

```
Clear[cop, cup]
eqp = (fph == t^(p - 1 + 1) (cop WhittakerW[-m0[j2] - (j2 + 1) / 2, nu2 / 2, 2 Pi Abs[ell] t^2] +
      cup WhittakerV[-m0[j2] - (j2 + 1) / 2, nu2 / 2, 2 Pi Abs[ell] t^2]))
```

Out[∘]= $\frac{1}{4\times(1+p)} p \Big((-2 + h - 3 p + 4\ \text{ell}\ \pi t^2)\ \text{f[-2 + p, t]} + 2 t \big(2 i \sqrt{2\pi} \sqrt{\text{Abs[ell]}}\ \text{f[p, t]} \sqrt{1+\text{m[h, p]}} + f^{(0,1)}\text{[-2 + p, t]}\big)\Big) == t^p \Big(\text{cup WhittakerV}\Big[\frac{1}{2}\times(-1 - \text{j2}) - \text{m0[j2]}, \frac{\text{nu2}}{2}, 2\pi t^2\ \text{Abs[ell]}\Big] + \text{cop WhittakerW}\Big[\frac{1}{2}\times(-1 - \text{j2}) - \text{m0[j2]}, \frac{\text{nu2}}{2}, 2\pi t^2\ \text{Abs[ell]}\Big]\Big)$

For F^-we deal with a K-type on the right boundary of the sector Sect(j_1).

In[ ]:=
```
Clear[com, cum]
eqm = (fml == t^(p - 1 + 1) (com WhittakerW[-m0[j1] - (j1 + 1) / 2, nu1 / 2, 2 Pi Abs[ell] t^2] +
     cum WhittakerV[-m0[j1] - (j1 + 1) / 2, nu1 / 2, 2 Pi Abs[ell] t^2]))
```

Out[ ]=

$$-\frac{1}{4\times(1+p)} p \left(\left(2+h+3p+4\,\text{ell}\,\pi t^2\right) f[2-p,t] + 4\,i\sqrt{2\pi}\, t\sqrt{\text{Abs[ell]}}\, f[-p,t]\sqrt{m[h,-p]} - 2\,t\, f^{(0,1)}[2-p,t]\right) == t^p \left(\text{cum WhittakerV}\left[\frac{1}{2}\times(-1-j1)-m0[j1],\ \frac{nu1}{2},\ 2\pi t^2\,\text{Abs[ell]}\right] + \text{com WhittakerW}\left[\frac{1}{2}\times(-1-j1)-m0[j1],\ \frac{nu1}{2},\ 2\pi t^2\,\text{Abs[ell]}\right]\right)$$

The quantities **cop**, **cup**, **com** and **cum** are unknown coefficients.
Table 4.5 expresses the parameter κ_0 in terms of $m_0(w')$ and $m_0(w'')$.

The resulting equations **eqm** and **eqp** involve the components $f_{\pm p}$, $f_{\pm(p-2)}$ and their derivatives. We solve the equations for $f_{\pm(p-2)}$'.

In[ ]:=
```
solp = Solve[eqp, f^(0,1)[-2 + p, t]][[1]] // Simplify
solm = Solve[eqm, f^(0,1)[2 - p, t]][[1]] // Simplify
```

Out[ ]=

$$\left\{f^{(0,1)}[-2+p,t] \to -\frac{\left(-2+h-3p+4\,\text{ell}\,\pi t^2\right) f[-2+p,t]}{2t} - 2\,i\sqrt{2\pi}\sqrt{\text{Abs[ell]}}\, f[p,t]\sqrt{1+m[h,p]} + \frac{1}{p} 2\times(1+p)\, t^{-1+p}\left(\text{cup WhittakerV}\left[-\frac{1}{2}-\frac{j2}{2}-m0[j2],\ \frac{nu2}{2},\ 2\pi t^2\,\text{Abs[ell]}\right] + \text{cop WhittakerW}\left[-\frac{1}{2}-\frac{j2}{2}-m0[j2],\ \frac{nu2}{2},\ 2\pi t^2\,\text{Abs[ell]}\right]\right)\right\}$$

Out[ ]=

$$\left\{f^{(0,1)}[2-p,t] \to \frac{\left(2+h+3p+4\,\text{ell}\,\pi t^2\right) f[2-p,t]}{2t} + 2\,i\sqrt{2\pi}\sqrt{\text{Abs[ell]}}\, f[-p,t]\sqrt{m[h,-p]} + \frac{1}{p} 2\times(1+p)\, t^{-1+p}\left(\text{cum WhittakerV}\left[-\frac{1}{2}-\frac{j1}{2}-m0[j1],\ \frac{nu1}{2},\ 2\pi t^2\,\text{Abs[ell]}\right] + \text{com WhittakerW}\left[-\frac{1}{2}-\frac{j1}{2}-m0[j1],\ \frac{nu1}{2},\ 2\pi t^2\,\text{Abs[ell]}\right]\right)\right\}$$

We take the eigenfunction equations for the components of F, of order p for the + case, and of order -p for the - case.

```
In[ ]:= eip =
  efeqn[h, p, p, f, ell, m[p], -1] /. solp /. nu → nu2 /. j → j2 /. {eps → -1, ell → -Abs[ell]} /.
     m[p] → m[h, p] /. m[h, x_] :→ m0[j2] - (1 / 6) × (3 x + 2 j2 - h) // Simplify
  eim = efeqn[h, p, -p, f, ell, m[-p], -1] /. solm /. nu → nu1 /. j → j1 /.
      {eps → -1, ell → -Abs[ell]} /. m[-p] → m[h, -p] /.
    m[h, x_] :→ m0[j1] - (1 / 6) × (3 x + 2 j1 - h) // Simplify
```

Out[ ]=

$$\Big\{\frac{1}{12}\Big(f[p,t]\,(48+h^2-4\,j2^2-12\,nu2^2-6\,h\,p+9\,p^2-48\,\pi^2\,t^4\,Abs[ell]^2+8\,\pi\,t^2\,Abs[ell]\,(-6+h+4\,j2+9\,p-12\,m0[j2]))+4\,t\Big(-4\,i\,p\,\sqrt{3\,\pi}\,\sqrt{Abs[ell]}\,f[-2+p,t]\,\sqrt{6+h-2\,j2-3\,p+6\,m0[j2]}-9\,f^{(0,1)}[p,t]+3\,t\,f^{(0,2)}[p,t]\Big)\Big),$$
$$-\frac{1}{9}\,(h-2\,j2-3\,p)\,(h^2+j2^2-9\,nu2^2+2\,h\,(j2-3\,p)-6\,j2\,p+9\,p^2)\,f[p,t]+8\,i\,(1+p)\,\sqrt{3\,\pi}\,t^{1+p}\,\sqrt{Abs[ell]}\,\sqrt{6+h-2\,j2-3\,p+6\,m0[j2]}\Big(cup\,WhittakerV\Big[-\frac{1}{2}-\frac{j2}{2}-m0[j2],\frac{nu2}{2},2\,\pi\,t^2\,Abs[ell]\Big]+cop\,WhittakerW\Big[-\frac{1}{2}-\frac{j2}{2}-m0[j2],\frac{nu2}{2},2\,\pi\,t^2\,Abs[ell]\Big]\Big)\Big\}$$

Out[ ]=

$$\Big\{\frac{1}{12}\Big(f[-p,t]\,(48+h^2-4\,j1^2-12\,nu1^2+6\,h\,p+9\,p^2-48\,\pi^2\,t^4\,Abs[ell]^2+8\,\pi\,t^2\,Abs[ell]\,(-6+h+4\,j1-9\,p-12\,m0[j1]))+4\,t\Big(4\,i\,p\,\sqrt{3\,\pi}\,\sqrt{Abs[ell]}\,f[2-p,t]\,\sqrt{h-2\,j1+3\,p+6\,m0[j1]}-9\,f^{(0,1)}[-p,t]+3\,t\,f^{(0,2)}[-p,t]\Big)\Big),$$
$$-\frac{1}{9}\,(h-2\,j1+3\,p)\,(h^2+j1^2-9\,nu1^2+6\,j1\,p+9\,p^2+2\,h\,(j1+3\,p))\,f[-p,t]+8\,i\,(1+p)\,\sqrt{3\,\pi}\,t^{1+p}\,\sqrt{Abs[ell]}\,\sqrt{h-2\,j1+3\,p+6\,m0[j1]}\Big(cum\,WhittakerV\Big[-\frac{1}{2}-\frac{j1}{2}-m0[j1],\frac{nu1}{2},2\,\pi\,t^2\,Abs[ell]\Big]+com\,WhittakerW\Big[-\frac{1}{2}-\frac{j1}{2}-m0[j1],\frac{nu1}{2},2\,\pi\,t^2\,Abs[ell]\Big]\Big)\Big\}$$

The following quantities should vanish.

In[ ]:=
```
ei = {eip[[2]], eim[[2]]} //. hpsub /. Abs[xx_]^2 → xx^2 // Simplify
```

Out[ ]=
$$\Big\{24\, i\,(1+p)\,\sqrt{2\pi}\,t^{1+p}\,\sqrt{\mathrm{Abs[ell]}}\,\sqrt{1-p+\mathrm{m0}\Big[\frac{1}{2}(h+3p)\Big]}\,\Big(\mathrm{cup\ WhittakerV}\Big[\frac{1}{4}\times\Big(-2-h-3p-4\,\mathrm{m0}\Big[\frac{1}{2}(h+3p)\Big]\Big),\ \frac{1}{4}\mathrm{Abs}[h-p],\ 2\pi t^2\,\mathrm{Abs[ell]}\Big]+\mathrm{cop\ WhittakerW}\Big[\frac{1}{4}\times\Big(-2-h-3p-4\,\mathrm{m0}\Big[\frac{1}{2}(h+3p)\Big]\Big),\ \frac{1}{4}\mathrm{Abs}[h-p],\ 2\pi t^2\,\mathrm{Abs[ell]}\Big]\Big),$$
$$24\, i\,(1+p)\,\sqrt{2\pi}\,t^{1+p}\,\sqrt{\mathrm{Abs[ell]}}\,\sqrt{p+\mathrm{m0}\Big[\frac{1}{2}(h-3p)\Big]}\,\Big(\mathrm{cum\ WhittakerV}\Big[\frac{1}{4}\times\Big(-2-h+3p-4\,\mathrm{m0}\Big[\frac{1}{2}(h-3p)\Big]\Big),\ \frac{1}{4}\mathrm{Abs}[h+p],\ 2\pi t^2\,\mathrm{Abs[ell]}\Big]+\mathrm{com\ WhittakerW}\Big[\frac{1}{4}\times\Big(-2-h+3p-4\,\mathrm{m0}\Big[\frac{1}{2}(h-3p)\Big]\Big),\ \frac{1}{4}\mathrm{Abs}[h+p],\ 2\pi t^2\,\mathrm{Abs[ell]}\Big]\Big)\Big\}$$

Use the asymptotic behavior of the Whittaker functions

In[ ]:=
```
ei /. WhittakerW[kp_, s_, tau_] :→ 0 /. WhittakerV[kp_, s_, tau_] :→ tau^(-kp) Exp[tau / 2] /.
   (Abs[ell] t^2)^ee_ :→ (Abs[ell])^ee t^(2 ee) // Simplify
Simplify[E^(-Pi Abs[ell] t^2) %] /. (t^2)^ee_ :→ t^(2 ee) // Simplify
```

Out[ ]=
$$\Big\{3\, i\, 2^{\frac{1}{4}\times(16+h+3p)+\mathrm{m0}[\frac{1}{2}(h+3p)]}\,\mathrm{cup}\,e^{\pi t^2\,\mathrm{Abs[ell]}}\,(1+p)\,\pi^{\frac{1}{4}\times(4+h+3p+4\,\mathrm{m0}[\frac{1}{2}(h+3p)])}\,t^{1+p}\,(t^2)^{\frac{1}{4}\times(2+h+3p+4\,\mathrm{m0}[\frac{1}{2}(h+3p)])}\,\mathrm{Abs[ell]}^{\frac{1}{4}\times(4+h+3p+4\,\mathrm{m0}[\frac{1}{2}(h+3p)])}\,\sqrt{1-p+\mathrm{m0}\Big[\frac{1}{2}(h+3p)\Big]},$$
$$3\, i\, 2^{\frac{1}{4}\times(16+h-3p)+\mathrm{m0}[\frac{1}{2}(h-3p)]}\,\mathrm{cum}\,e^{\pi t^2\,\mathrm{Abs[ell]}}\,(1+p)\,\pi^{\frac{1}{4}\times(4+h-3p+4\,\mathrm{m0}[\frac{1}{2}(h-3p)])}\,t^{1+p}\,(t^2)^{\frac{1}{4}\times(2+h-3p+4\,\mathrm{m0}[\frac{1}{2}(h-3p)])}\,\mathrm{Abs[ell]}^{\frac{1}{4}\times(4+h-3p+4\,\mathrm{m0}[\frac{1}{2}(h-3p)])}\,\sqrt{p+\mathrm{m0}\Big[\frac{1}{2}(h-3p)\Big]}\Big\}$$

Out[ ]=
$$\Big\{3\, i\, 2^{\frac{1}{4}\times(16+h+3p)+\mathrm{m0}[\frac{1}{2}(h+3p)]}\,\mathrm{cup}\,(1+p)\,\pi^{\frac{1}{4}\times(4+h+3p+4\,\mathrm{m0}[\frac{1}{2}(h+3p)])}\,t^{\frac{1}{2}\times(4+h+5p+4\,\mathrm{m0}[\frac{1}{2}(h+3p)])}\,\mathrm{Abs[ell]}^{\frac{1}{4}\times(4+h+3p+4\,\mathrm{m0}[\frac{1}{2}(h+3p)])}\,\sqrt{1-p+\mathrm{m0}\Big[\frac{1}{2}(h+3p)\Big]},$$
$$3\, i\, 2^{\frac{1}{4}\times(16+h-3p)+\mathrm{m0}[\frac{1}{2}(h-3p)]}\,\mathrm{cum}\,(1+p)\,\pi^{\frac{1}{4}\times(4+h-3p+4\,\mathrm{m0}[\frac{1}{2}(h-3p)])}\,t^{\frac{1}{2}\times(4+h-p+4\,\mathrm{m0}[\frac{1}{2}(h-3p)])}\,\mathrm{Abs[ell]}^{\frac{1}{4}\times(4+h-3p+4\,\mathrm{m0}[\frac{1}{2}(h-3p)])}\,\sqrt{p+\mathrm{m0}\Big[\frac{1}{2}(h-3p)\Big]}\Big\}$$

This shows that the coefficients **cum** and **cup** vanish.

```
In[ ]:= ei /. {cum → 0, cup → 0} /. WhittakerW[kp_, s_, tau_] :→ tau ^ kp Exp[-tau / 2] /.
          (t^2) ^ ee_ :→ t ^ (2 ee) // Simplify
```

Out[ ]=

$$\left\{3\,i\,2^{3-\frac{h}{4}-\frac{3p}{4}-m0\left[\frac{1}{2}(h+3p)\right]}\,\mathrm{cop}\,e^{-\pi t^2\,\mathrm{Abs[ell]}}\,(1+p)\,\pi^{\frac{1}{4}\left(-h-3p-4\,m0\left[\frac{1}{2}(h+3p)\right]\right)}\,t^{\frac{1}{2}\left(-h-p-4\,m0\left[\frac{1}{2}(h+3p)\right]\right)}\,\mathrm{Abs[ell]}^{\frac{1}{4}\left(-h-3p-4\,m0\left[\frac{1}{2}(h+3p)\right]\right)}\sqrt{1-p+m0\left[\frac{1}{2}(h+3p)\right]},\ 3\,i\,2^{3-\frac{h}{4}+\frac{3p}{4}-m0\left[\frac{1}{2}(h-3p)\right]}\,\mathrm{com}\,e^{-\pi t^2\,\mathrm{Abs[ell]}}\,(1+p)\,\pi^{\frac{1}{4}\left(-h+3p-4\,m0\left[\frac{1}{2}(h-3p)\right]\right)}\,t^{\frac{1}{2}\left(-h+5p-4\,m0\left[\frac{1}{2}(h-3p)\right]\right)}\,\mathrm{Abs[ell]}^{\frac{1}{4}\left(-h+3p-4\,m0\left[\frac{1}{2}(h-3p)\right]\right)}\sqrt{p+m0\left[\frac{1}{2}(h-3p)\right]}\right\}$$

This shows that the coefficient **cop** and **cup** vanish as well.

The conclusion is that the determining coefficients of F^{+} and F^{-} are zero, which completes the proof in this case.