25c. Types IF and FI

See §4.5.3.2

In[ ]:=
```
Clear[gamsub , A, B]
gamsub[zz_] := {Gamma[zz] → Pi Gamma[1 - zz]^(-1) Sin[Pi zz]^(-1)};
```

Case a

In[ ]:=
```
ca[p_, r_, nua_] =
 Sin[Pi (nua + j) / 2] Sin[Pi (nua - j) / 2]^(-1) c[p, r, nua] /. gamsub[1 + 1/2 (-nua + p - r - j)] /.
    gamsub[1 + 1/2 (nua + p - r - j)] //.
   {Csc[π (1 + 1/2 (-j - nua + p - r))] → Csc[π (1/2 (-j - nua))] (-1)^(1 + (p - r) / 2),
    Sin[π (1 + 1/2 (-j + nua + p - r))] → Sin[π (1 + 1/2 (-j + nua))] (-1)^(1 + (p - r) / 2),
    Sin[(1 + 1/2 (-j + nua)) π] → -Sin[(1/2 (-j + nua)) π],
    Csc[1/2 (-j - nua) π] → - Csc[1/2 (j + nua) π]} // Simplify
```

Out[ ]=

$$\left((-1)^{p-r}\,\mathrm{Gamma}\left[\frac{1}{2}\,(j-nua-p+r)\right]\mathrm{Gamma}\left[\frac{1}{2}\times(2+j-nua+p+r)\right]\right)\Big/\left(\mathrm{Gamma}\left[\frac{1}{2}\,(j+nua-p+r)\right]\mathrm{Gamma}\left[\frac{1}{2}\times(2+j+nua+p+r)\right]\right)$$

In[ ]:=
```
ca[p, r, nu] //. lsub[2 j, 0] /. (-1)^(2 b) → 1 // Simplify
% /. {j → 2 B + nu + 2} // Simplify
```

Out[ ]=

$$\frac{\left(\frac{1}{2}\times(-2-2\,b+j-nu)\right)!\,\left(a+\frac{j}{2}-\frac{nu}{2}\right)!}{\left(\frac{1}{2}\times(2\,a+j+nu)\right)!\times\left(\frac{1}{2}\times(-2-2\,b+j+nu)\right)!}$$

Out[ ]=

$$\frac{(1+a+B)!\,(-b+B)!}{(1+a+B+nu)!\,(-b+B+nu)!}$$

Case c

In[ ]:=
```
Clear[cc]
cc[p_, r_, nua_] = Sin[Pi (j - nua) / 2]^(-1) c[p, r, nua] /. gamsub[1 + 1/2 (nua + p - r - j)] /.
  Sin[π (1 + 1/2 (nua + p - r - j))] → (-1)^((p - r) / 2) Sin[Pi (j - nua) / 2]
```

Out[ ]=
$$\left(i^{p-r}\,\text{Gamma}\left[1+\frac{1}{2}(-j-\text{nua}+p-r)\right]\text{Gamma}\left[\frac{1}{2}(j-\text{nua}-p+r)\right]\text{Gamma}\left[1+\frac{1}{2}(j-\text{nua}+p+r)\right]\right)\Big/\left(\pi\,\text{Gamma}\left[1+\frac{1}{2}(j+\text{nua}+p+r)\right]\right)$$

In[ ]:=
```
cc[p, r, -nu] //. lsub[(j + 3 nu) / 2, (j - nu) / 2] // Simplify
% /. I^(2 b + ee_) :> (-1)^b I^ee /. nu → B + 1 // Simplify
```

Out[ ]=
$$\frac{i^{2\,b+j-\text{nu}}\,b!\,(-1-b+\text{nu})!\times\left(\frac{1}{2}\times(2\,a+j+\text{nu})\right)!}{\pi\left(a+\frac{j}{2}-\frac{\text{nu}}{2}\right)!}$$

Out[ ]=
$$\frac{(-1)^{1+b}\,i^{1-B+j}\,b!\,(-b+B)!\left(\frac{1}{2}\times(1+2\,a+B+j)\right)!}{\pi\left(\frac{1}{2}\times(-1+2\,a-B+j)\right)!}$$

This gives a posidefinite form if B=0, hence ν=1.

Can there be a positive definite sesquilinear form if $\nu \geq 2$?

In[ ]:=
```
ph0 = phi[h0, p0, r0, p0, -nu]
ph1 = phi[h0 - 3, p0 + 1, r0 - 1, p0 + 1, -nu]
```

Out[ ]=
$$t^{2-\text{nu}}\,\text{Phi}[h0,\,p0,\,r0,\,p0]$$

Out[ ]=
$$t^{2-\text{nu}}\,\text{Phi}[-3+h0,\,1+p0,\,-1+r0,\,1+p0]$$

In[ ]:=
```
qup = sh[-3, 1, ph0, subtriv] / ph1 // Simplify
qdn = sh[3, -1, ph1, subtriv] / ph0 // Simplify
```

Out[ ]=
$$\frac{(2+p0-r0)\times(4-h0-2\,\text{nu}+2\,p0+r0)}{8\times(1+p0)}$$

Out[ ]=
$$-\frac{(1+p0)\times(4-h0+2\,\text{nu}+2\,p0+r0)}{4\times(2+p0)}$$

In[ ]:=
```
qup qdn /. r0 → (h0 - 2 j) / 3 /. {h0 → (j + 3 nu) / 2, p0 → (j - nu) / 2} // Factor
% /. j → nu + pos + 2 // Simplify
```

Out[ ]=
$$\frac{(2+j-\text{nu})\times(-1+\text{nu})}{4+j-\text{nu}}$$

Out[ ]=
$$\frac{(-1+\text{nu})\times(4+\text{pos})}{6+\text{pos}}$$

Positive if $v >= 2$.

Then the sum cannot be zero.

Case d

In[ ]:=
```
cd[p_, r_, nua_] = Sin[Pi (nua - j)]^(-1) c[p, r, nua] /. gamsub[1 + 1/2 (nua + p - r - j)] /.
    Sin[π (1 + 1/2 (nua + p - r - j))] → (-1)^((p - r)/2) Sin[Pi (-nua + j)] // Simplify
```

Out[ ]= $-\left(\left(i^{p-r}\,\mathrm{Gamma}\left[\frac{1}{2}\times(2-j-\mathrm{nua}+p-r)\right]\mathrm{Gamma}\left[\frac{1}{2}(j-\mathrm{nua}-p+r)\right]\mathrm{Gamma}\left[\frac{1}{2}\times(2+j-\mathrm{nua}+p+r)\right]\right)\Big/\left(\pi\,\mathrm{Gamma}\left[\frac{1}{2}\times(2+j+\mathrm{nua}+p+r)\right]\right)\right)$

In[ ]:=
```
cd[p, r, -j] //. lsub[2 j, 0] /. nu → j /. j → B + 1 /. I^(2 b) → (-1)^b // Simplify
```

Out[ ]= $\frac{(-1)^{1+b}\, b!\,(1+a+B)!\,(-b+B)!}{\pi\, a!}$

The factorials are defined. The factor $(-1)^b$prohibits positive definite, unless B=1, hence v=1

Can there be a positive definite sesquilinear form if $v \geq 2$?

In[ ]:=
```
ph0 = phi[h0, p0, r0, p0, -j]
ph1 = phi[h0 - 3, p0 + 1, r0 - 1, p0 + 1, -j]
```

Out[ ]= t^{2-j} Phi[h0, p0, r0, p0]

Out[ ]= t^{2-j} Phi[-3 + h0, 1 + p0, -1 + r0, 1 + p0]

In[ ]:=
```
qup = sh[-3, 1, ph0, subtriv] / ph1 // Simplify
qdn = sh[3, -1, ph1, subtriv] / ph0 // Simplify
```

Out[ ]= $\frac{(2+p0-r0)\times(4-h0-2\,j+2\,p0+r0)}{8\times(1+p0)}$

Out[ ]= $-\frac{(1+p0)\times(4-h0+2\,j+2\,p0+r0)}{4\times(2+p0)}$

In[ ]:=
```
qup qdn /. r0 → (h0 - 2 j) / 3 /. {h0 → 2 j, p0 → 0} // Factor
```

Out[ ]= $\frac{1}{2}\times(-1+j)$

This is positive if j=j ≥ 2. Hence no positive definite form.

Case e

In[ ]:=
```
Clear[ce]
ce[p_, r_, nua_] =
 Sin[Pi (j - nua) / 2] Sin[Pi (j + nua) / 2] ^ (-1) c[p, r, nua] /. gamsub[1 + 1/2 (nua + p + r + j)] /.
     gamsub[1 + 1/2 (-nua + p + r + j)] /.
    Csc[π (1 + 1/2 (-nua + p + r + j))] → Csc[π (1/2 (-nua + j))] (-1) ^ (1 + (p + r) / 2) /.
   Sin[π (1 + 1/2 (nua + p + r + j))] → Sin[π (1/2 (nua + j))] (-1) ^ (1 + (p + r) / 2) // Simplify
```

Out[ ]=
$$\left((-1)^{p+r}\,\text{Gamma}\left[\frac{1}{2}(-j-\text{nua}-p-r)\right]\text{Gamma}\left[\frac{1}{2}\times(2-j-\text{nua}+p-r)\right]\right)\Big/\left(\text{Gamma}\left[\frac{1}{2}(-j+\text{nua}-p-r)\right]\text{Gamma}\left[\frac{1}{2}\times(2-j+\text{nua}+p-r)\right]\right)$$

In[ ]:=
```
(ce[p, r, nu] //. lsub[2 j, 0] // Simplify) /. (-1) ^ (2 a) → 1 /. j → -2 A - nu - 2 // Simplify
```

Out[ ]=
$$\frac{(-a+A)!\,(1+A+b)!}{(-a+A+nu)!\,(1+A+b+nu)!}$$

This is positive as long as a≤-(nu+j)/2-1

Case g

In[ ]:=
```
Clear[cg]
cg[p_, r_, nua_] = Sin[Pi (nua + j) / 2] ^ (-1) c[p, r, nua] /. gamsub[1 + 1/2 (nua + p + r + j)] /.
   Sin[π (1 + 1/2 (nua + p + r + j))] → -(-1) ^ ((p + r) / 2) Sin[Pi (nua + j) / 2] // Simplify
```

Out[ ]=
$$-\left(\left(i^{p+r}\,\text{Gamma}\left[\frac{1}{2}(-j-\text{nua}-p-r)\right]\text{Gamma}\left[\frac{1}{2}\times(2-j-\text{nua}+p-r)\right]\text{Gamma}\left[\frac{1}{2}\times(2+j-\text{nua}+p+r)\right]\right)\Big/\left(\pi\,\text{Gamma}\left[\frac{1}{2}\times(2-j+\text{nua}+p-r)\right]\right)\right)$$

In[ ]:=
```
cg[p, r, -nu] //. lsub[(j - 3 nu) / 2, -(j + nu) / 2] // Simplify
% /. ⅈ^(2 a-nu-j) → (-1) ^ a I ^ (-nu - j) //. {nu → A + 1, j → -(nu + 2 + pos)} // Simplify
```

Out[ ]= $-\frac{i^{2a-j-nu}\, a!\left(b-\frac{j}{2}+\frac{nu}{2}\right)!\,(-1-a+nu)!}{\pi\left(b-\frac{j}{2}-\frac{nu}{2}\right)!}$

Out[ ]= $\frac{(-1)^a\, i^{pos}\, a!\,(-a+A)!\left(2+A+b+\frac{pos}{2}\right)!}{\pi\left(1+b+\frac{pos}{2}\right)!}$

If A=ν-1 ≥1, then the sign changes make the form indefinite. For ν=1, we get a multiple of a positive definite form.

Can there be a positive definite sesquilinear form if ν≥2?

In[ ]:=
```
ph0 = phi[h0, p0, r0, p0, -nu]
ph1 = phi[h0 + 3, p0 + 1, r0 + 1, p0 + 1, -nu]
```

Out[ ]= t^{2-nu} Phi[h0, p0, r0, p0]

Out[ ]= t^{2-nu} Phi[3 + h0, 1 + p0, 1 + r0, 1 + p0]

In[ ]:=
```
qup = sh[3, 1, ph0, subtriv] / ph1 // Simplify
qdn = sh[-3, -1, ph1, subtriv] / ph0 // Simplify
```

Out[ ]= $\frac{(4+h0-2\,nu+2\,p0-r0)\times(2+p0+r0)}{8\times(1+p0)}$

Out[ ]= $-\frac{(1+p0)\times(4+h0+2\,nu+2\,p0-r0)}{4\times(2+p0)}$

Check from the definitions that these shift operators are conjugate.

In[ ]:=
```
qup qdn /. r0 → (h0 - 2 j) / 3 /. {h0 → (j - 3 nu) / 2, p0 → -(j + nu) / 2} // Factor
% /. j → -(nu + 2 + pos) // Simplify
```

Out[ ]= $\frac{(-1+nu)\times(-2+j+nu)}{-4+j+nu}$

Out[ ]= $\frac{(-1+nu)\times(4+pos)}{6+pos}$

Positive if $\nu \geq 2$

Case h

```
In[ ]:= Clear[ch]
ch[p_, r_, nua_] = Sin[Pi (nua + j) / 2]^(-1) c[p, r, nua] /. gamsub[1 + 1/2 (nua + p + r + j)] /.
  Sin[π (1 + 1/2 (nua + p + r + j))] → -(-1)^((p + r) / 2) Sin[Pi (nua + j) / 2]
```

Out[ ]= $$-\left(\left(i^{p+r}\,\text{Gamma}\left[1+\frac{1}{2}(-j-\text{nua}+p-r)\right]\text{Gamma}\left[\frac{1}{2}(-j-\text{nua}-p-r)\right]\text{Gamma}\left[1+\frac{1}{2}(j-\text{nua}+p+r)\right]\right)\Big/\left(\pi\,\text{Gamma}\left[1+\frac{1}{2}(-j+\text{nua}+p-r)\right]\right)\right)$$

```
In[ ]:= ch[p, r, j] //. lsub[2 j, 0] /. I^(2 a) → (-1)^a /. j → -A - 1 // Simplify
```

Out[ ]= $$\frac{(-1)^{1+a}\,a!\,(-a+A)!\,(1+A+b)!}{\pi\,b!}$$

The sesquilinear form is positive definite only if ν=-1

Can there be a positive definite sesquilinear form if $\nu \geq 2$?

```
In[ ]:= ph0 = phi[h0, p0, r0, p0, j]
ph1 = phi[h0 + 3, p0 + 1, r0 + 1, p0 + 1, j]
```

Out[ ]= t^{2+j} Phi[h0, p0, r0, p0]

Out[ ]= t^{2+j} Phi[3 + h0, 1 + p0, 1 + r0, 1 + p0]

```
In[ ]:= qup = sh[3, 1, ph0, subtriv] / ph1 // Simplify
qdn = sh[-3, -1, ph1, subtriv] / ph0 // Simplify
```

Out[ ]= $$\frac{(4+h0+2j+2p0-r0)\times(2+p0+r0)}{8\times(1+p0)}$$

Out[ ]= $$-\frac{(1+p0)\times(4+h0-2j+2p0-r0)}{4\times(2+p0)}$$

```
In[ ]:= qup qdn /. r0 → (h0 - 2 j) / 3 /. {h0 → 2 j, p0 → 0} /. j → -Abs[j] // Factor
```

Out[ ]= $$\frac{1}{2}\times(-1+\text{Abs}[j])$$

Positive if j <-1