26a. Checks for Proposition 5.6

Factor of automorphy

In[ ]:=
```
UT = 2^(-1/2) {{1, 0, -I}, {0, 2, 0}, {1, 0, I}};
(* in (2.9) *)
```

In[ ]:=
```
Clear[h, jf]
jf[g_List, {z_, u_}] := Block[{m}, m = Inverse[UT].g.UT;
  m[[3, 1]] z + m[[3, 2]] u + m[[3, 3]] // Simplify]
```

In[ ]:=
```
Clear[ma, a, b]
ma[x_] := Table[x[i, j], {i, 1, 3}, {j, 1, 3}]
```

In[ ]:=
```
jf[ma[a], actX[ma[b], {z, u}]] × jf[ma[b], {z, u}] == jf[ma[a].ma[b], {z, u}]
```

Out[ ]= True

So j in (5.15) is a factor of automorphy.

Transformation behavior

We want to check that if $F \mapsto F_1$ corresponds to $f \mapsto f_1$=L(g)g then

Use that

In[ ]:=
```
actX[nm[x, y, r], {I, 0}]
```

Out[ ]= $\{2 r + i (1 + x^2 + y^2), i x + y\}$

First some substitutions

In[ ]:=
```
Clear[f, f1, F, F1, fsub, z, u, tau, zsub]
fsub[f_, F_] :=
 {f[ns[xx_, yy_, rr_] ** as[tt_]] :> tt^(-h/2) F[actX[nm[xx, yy, rr].am[tt], {I, 0}]]}
sub1 = {x -> Im[u], y -> Re[u], r -> Re[z]/2, t -> tau^(1/2)};
sub2 = {Re[zz_] :> (zz + Conjugate[zz])/2,
    Im[zz_] :> (zz - Conjugate[zz])/(2 I), u Conjugate[u] -> Im[z] - tau,
    (pp_ qq_)^ee_ :> pp^ee qq^ee, (pp_^ff_)^ee_ :> pp^(ff ee)};
zsub[xx_] := Simplify[xx /. sub1 //. sub2] /. u Conjugate[u] -> (z - Conjugate[z])/(2 I) - tau //
  Simplify
```

sub1 is based on the following formula

In[ ]:=
```
actX[nm[x, y, r].am[t], {I, 0}]
```

Out[ ]= $\{2 r + i (t^2 + x^2 + y^2), i x + y\}$

Check for g ∈ N

```
In[ ]:= rel = f[ns[x1, y1, r1] ** ns[x, y, r] ** as[t]] == f1[ns[x, y, r] ** as[t]] //. Gsub /. fsub[f, F] /.
          fsub[f1, F1] // zsub // Simplify

        actX[nm[x1, y1, r1], {z, u}] // Simplify
        jf[nm[x1, y1, r1], {z, u}]
```

Out[ ]= $tau^{-h/4}\left(F\left[\left\{2\,r1+2\,u\,x1+i\,x1^{2}+2\,i\,u\,y1+i\,y1^{2}+z,\ u+i\,x1+y1\right\}\right]-F1[\{z,u\}]\right)==0$

Out[ ]= $\left\{2\,r1+i\,x1^{2}+2\,u\,(x1+i\,y1)+i\,y1^{2}+z,\ u+i\,x1+y1\right\}$

Out[ ]= 1

Check for g ∈ A

```
In[ ]:= rel =
          f[as[t1] ** ns[x, y, r] ** as[t]] == f1[ns[x, y, r] ** as[t]] //. Gsub /. fsub[f, F] /. fsub[f1, F1] //
            zsub // Simplify;
        % /. Conjugate[t1] → t1

        actX[am[t1], {z, u}]
        jf[am[t1], {z, u}]
```

Out[ ]= $t1^{-h/2}\,tau^{-h/4}\,F\left[\left\{t1^{2}\,z,\ t1\,u\right\}\right]==tau^{-h/4}\,F1[\{z,u\}]$

Out[ ]= $\left\{t1^{2}\,z,\ t1\,u\right\}$

Out[ ]= $\frac{1}{t1}$

Check for m ∈ M

```
In[ ]:= rel = f[ms[zt] ** ns[b, r] ** as[t]] == f1[ns[b, r] ** as[t]] //. Gsub /. f[gg_ ** ms[zt]] :→ zt ^ h f[gg]
```

Out[ ]= $zt^{h}\,f\left[ns\left[b\,zt^{3},\ r\right]**as[t]\right]==f1[ns[b,r]**as[t]]$

In[ ]:=
```
rel /. ns[b_, r_] :> ns[Re[b], Im[b], r] /. fsub[f, F] /. fsub[f1, F1] // Simplify
% //. {ⅈ Im[b zt^3]^2 + ⅈ Re[b zt^3]^2 → I (x^2 + y^2), Im[bb_] + I Re[bb_] :> I Conjugate[bb],
    Conjugate[zt] → zt^(-1), ⅈ Im[b]^2 + ⅈ Re[b]^2 → I (x^2 + y^2), Conjugate[b] → x - I y} // zsub
actX[mm[zt], {u, z}]
jf[mm[zt], {z, u}]
```

Out[ ]=
$$t^{-h/2} \left(zt^h F\left[\left\{2 r + i t^2 + i \operatorname{Im}[b\, zt^3]^2 + i \operatorname{Re}[b\, zt^3]^2,\ \operatorname{Im}[b\, zt^3] + i \operatorname{Re}[b\, zt^3]\right\}\right] - F1\left[\left\{2 r + i t^2 + i \operatorname{Im}[b]^2 + i \operatorname{Re}[b]^2,\ \operatorname{Im}[b] + i \operatorname{Re}[b]\right\}\right]\right) == 0$$

Out[ ]=
$$tau^{-h/4} \left(zt^h F\left[\left\{z, \frac{u}{zt^3}\right\}\right] - F1[\{z, u\}]\right) == 0$$

Out[ ]=
$$\left\{u, \frac{z}{zt^3}\right\}$$

Out[ ]=
```
zt
```

Check for w

In[ ]:=
```
Clear[dd]
f[ws ** ns[b, r] ** as[t]] == f1[ns[b, r] ** as[t]] /.
  ws ** ns[b, r] ** as[t] → ns[bb, rr] ** as[tt] ** kk // Simplify
rel = % /. f[gg_ ** kk] :> f[gg] Conjugate[dd]^(h/2) Abs[dd]^(-h/2) /.
   {bb → b/dd, rr → -r/Abs[d]^2, tt → t/Abs[dd]} // Simplify
```

Out[ ]=
```
f[ns[bb, rr] ** as[tt] ** kk] == f1[ns[b, r] ** as[t]]
```

Out[ ]=
$$f1[ns[b, r] ** as[t]] == \operatorname{Abs}[dd]^{-h/2} \operatorname{Conjugate}[dd]^{h/2} f\left[ns\left[\frac{b}{dd}, -\frac{r}{\operatorname{Abs}[d]^2}\right] ** as\left[\frac{t}{\operatorname{Abs}[dd]}\right]\right]$$

In[ ]:=
```
rel1 = rel /. ns[b, r] → ns[Re[b], Im[b], r] /.
        ns[b/dd, -r/Abs[d]^2] → ns[Re[b/dd], Im[b/dd], -r/Abs[dd]^2] /. fsub[f, F] /.
     fsub[f1, F1] /. Im[xx_] + I Re[xx_] :> I Conjugate[xx] /.
   Im[b/dd]^2 + Re[b/dd]^2 → Abs[b]^2/Abs[dd]^2 // Simplify
```

Out[ ]=
$$t^{-h/2} \left(\operatorname{Conjugate}[dd]^{h/2} F\left[\left\{\frac{i\,(2\, i\, r + t^2 + \operatorname{Abs}[b]^2)}{\operatorname{Abs}[dd]^2},\ \frac{i \operatorname{Conjugate}[b]}{\operatorname{Conjugate}[dd]}\right\}\right] - F1\left[\left\{2 r + i t^2 + i \operatorname{Im}[b]^2 + i \operatorname{Re}[b]^2,\ i \operatorname{Conjugate}[b]\right\}\right]\right) == 0$$

In[ ]:=
```
2 I r + t^2 + x^2 + y^2 // zsub
```

Out[ ]=
```
ⅈ Conjugate[z]
```

In[ ]:=
```
rel1 //. {2 I r + t^2 + Abs[b]^2 → I Conjugate[z], dd → I Conjugate[z],
    2 r + i t^2 + i Im[b]^2 + i Re[b]^2 → I Conjugate[dd], b → x + I y} // Simplify
rel2 = % /. Abs[z]^(-2) → z^(-1) Conjugate[z]^(-1) /. Conjugate[xi_] :> xi /. y → u - I x //
  Simplify
```

Out[ ]= $t^{-h/2}\left((-i\,z)^{h/2}\,F\left[\left\{-\frac{\text{Conjugate}[z]}{\text{Abs}[z]^2},\ -\frac{\text{Conjugate}[x]-i\,\text{Conjugate}[y]}{z}\right\}\right]-F1[\{z,\ i\,\text{Conjugate}[x]+\text{Conjugate}[y]\}]\right)==0$

Out[ ]= $t^{-h/2}\left((-i\,z)^{h/2}\,F\left[\left\{-\frac{1}{z},\ \frac{i\,u}{z}\right\}\right]-F1[\{z,\ u\}]\right)==0$

In[ ]:=
```
rel2
jf[wm, {z, u}]^(h/2)
actX[wm, {z, u}]
```

Out[ ]= $t^{-h/2}\left((-i\,z)^{h/2}\,F\left[\left\{-\frac{1}{z},\ \frac{i\,u}{z}\right\}\right]-F1[\{z,\ u\}]\right)==0$

Out[ ]= $(-i\,z)^{h/2}$

Out[ ]= $\left\{-\frac{1}{z},\ -\frac{i\,u}{z}\right\}$

Differentiation

In[ ]:=
```
ff = f[ns[x, y, r] ** as[t]] × Phi[h, 0, 0, 0] /. fsub[f, F] /.
    F[{z_, u_}] → F[z, u, Conjugate[z], Conjugate[u]] /. Conjugate[xx_] :> xx
```

Out[ ]= $t^{-h/2}\,F\left[2\,r+i\left(t^2+x^2+y^2\right),\ i\,x+y,\ 2\,r-i\left(t^2+x^2+y^2\right),\ -i\,x+y\right]\times \text{Phi}[h,\ 0,\ 0,\ 0]$

Substitution rule

In[ ]:=
```
Clear[subNA]
subNA[xx_] := xx /. {Rna[HHr, ff_] :> t D[ff, t], Rna[XX1, ff_] :> t D[ff, x] - t y D[ff, r],
    Rna[XX2, ff_] :> t D[ff, y] + t x D[ff, r], Rna[XX0, ff_] :> (t^2/2) D[ff, r]} // Simplify
```

In[ ]:=
```
eR[Z31, ff, subNA] /. sub1 //. {Im[u]^2 → Abs[u]^2 - Re[u]^2, tau → Im[z] - Abs[u]^2,
     Re[zz_] + I Im[zz_] :> zz, Re[zz_] - I Im[zz_] :> Conjugate[zz]} // Simplify;
z31f = % /. Im[z] → tau + Abs[u]^2 // Simplify
```

Out[ ]= $\text{tau}^{\frac{2-h}{4}}\left(-2\,i\,\sqrt{\text{tau}}\ \text{Phi}[3+h,\ 1,\ 1,\ 1]\,F^{(0,0,1,0)}[z,\ u,\ \text{Conjugate}[z],\ \text{Conjugate}[u]]+\text{Phi}[3+h,\ 1,\ -1,\ 1]\left(i\,F^{(0,0,0,1)}[z,\ u,\ \text{Conjugate}[z],\ \text{Conjugate}[u]]+2\,u\,F^{(0,0,1,0)}[z,\ u,\ \text{Conjugate}[z],\ \text{Conjugate}[u]]\right)\right)$

In[ ]:=
```
Coefficient[z31f, Phi[h + 3, 1, 1, 1]] // Simplify
```

Out[ ]= $-2\, i\, \text{tau}^{1-\frac{h}{4}}\, F^{(0,0,1,0)}[z, u, \text{Conjugate}[z], \text{Conjugate}[u]]$

In[ ]:=
```
Coefficient[z31f, Phi[h + 3, 1, -1, 1]] // Simplify
```

Out[ ]= $\text{tau}^{\frac{2-h}{4}} \left(i\, F^{(0,0,0,1)}[z, u, \text{Conjugate}[z], \text{Conjugate}[u]] + 2\, u\, F^{(0,0,1,0)}[z, u, \text{Conjugate}[z], \text{Conjugate}[u]] \right)$

So the antiholomorphic derivatives vanish .