## 5d. Integration of theta functions

Orthogonality in (2.56)

In[ ]:=
```
Clear[int, diff, x, y, r, sg, ph, ps, xi, ell, c, ella, ca, k, ka]
i1 = int[x, 0, 1] × int[y, 0, 1] × int[r, 0, 2 / sg] × Th[ell, c, ph[xi], ns[x, y, r], k]
      Conjugate[Th[ella, ca, ps[xi], ns[x, y, r], ka]] diff[x] × diff[y] × diff[r] //. gensub /.
   {Conjugate[ella] → ella, Conjugate[ca] → ca, Conjugate[x] → x, Conjugate[y] → y,
    Conjugate[r] → r, Conjugate[ka] → ka, Conjugate[sum[ka]] → sum[ka]} // Simplify
```

Out[ ]=

$$e^{2 i \pi ((-c+ca) x+ella (-r+2 ka x+x y)+ell (r-x (2 k+y)))} \text{Conjugate}\left[ps\left[\frac{ca}{2\, ella}+ka+y\right]\right] diff[r] \times diff[x] \times diff[y] \times int\left[r, 0, \frac{2}{sg}\right] \times int[x, 0, 1] \times int[y, 0, 1] \times ph\left[\frac{c}{2\, ell}+k+y\right] \times sum[k] \times sum[ka]$$

Integration over r concerns only the following factor

In[ ]:=
```
i1fr = E^(2 Pi I (ell r - ella r))
```

Out[ ]=

$$e^{2 i \pi (ell\ r - ella\ r)}$$

Since $l$-$l$' is 0 modulo $\sigma/2$ integration over r in [0,2/$\sigma$] yields zero unless $l$=$l$'.

In[ ]:=
```
i2 = i1 /. ella → ell /. int[r, 0, 2/sg] → 2 / sg /. diff[r] → 1 // Simplify
```

Out[ ]=

$$\frac{1}{sg} 2\, e^{-2 i (c-ca+2\, ell\, (k-ka)) \pi x} \text{Conjugate}\left[ps\left[\frac{ca}{2\, ell}+ka+y\right]\right] diff[x] \times diff[y] \times int[x, 0, 1] \times int[y, 0, 1] \times ph\left[\frac{c}{2\, ell}+k+y\right] \times sum[k] \times sum[ka]$$

Integration over x in [0,1] involves the factor

In[ ]:=
```
i2fx = e^(-2 i (c-ca+2 ell (k-ka)) π x)
```

Out[ ]=

$$e^{-2 i (c-ca+2\, ell\, (k-ka)) \pi x}$$

Since c-ca+2ell(k- ka) is integral the integral over x yields zero unless this quantity vanishes. So c-ca should be zero modulo 2ell. We can assume that c and ca are taken in [0,2|ell|-1]. So we get zero unless c=ca and k=ka

In[ ]:=
```
i3 = i2 /. ca → c /. ka → k /. sum[k]^2 → sum[k] /. int[x, 0, 1] → 1 /. diff[x] → 1 // Simplify
```

Out[ ]=

$$\frac{1}{sg} 2\, \text{Conjugate}\left[ps\left[\frac{c}{2\, ell}+k+y\right]\right] diff[y] \times int[y, 0, 1] \times ph\left[\frac{c}{2\, ell}+k+y\right] \times sum[k]$$

Take k + y as new variable, integrating over R. This gives the orthogonality formula.