5g. Operator B

See the discussion of the metaplectic action in §3.3.2.

Action on Hermite functions

Check that B acts on $h_{l,m}$ as multiplication by (i/2)(2m+1) Sign(ell)

First term $(8\,\pi i l)^{-1}\, D[h, \{xi, 2\}]$

In[◦]:=
```
Clear[xi, ell, m]
```

In[◦]:=
```
hermn[ell, m]
dif[%] /. hermrel
term1 = (8 Pi I ell)^(-1) dif[%] /. hermrel // Simplify
```

Out[◦]= hermn[ell, m]

Out[◦]= $2\sqrt{\pi}\,\sqrt{\text{Abs[ell]}}\left(\frac{\sqrt{m}\,\text{hermn[ell, -1 + m]}}{\sqrt{2}} - \frac{\sqrt{1+m}\,\text{hermn[ell, 1 + m]}}{\sqrt{2}}\right)$

Out[◦]= $-\frac{1}{4\,\text{ell}}\, i\,\text{Abs[ell]} \left(\sqrt{-1+m}\,\sqrt{m}\,\text{hermn[ell, -2 + m]} - (1+2\,m)\,\text{hermn[ell, m]} + \sqrt{1+m}\,\sqrt{2+m}\,\text{hermn[ell, 2 + m]}\right)$

In[◦]:=
```
hermn[ell, m]
xi % /. hermrel
term2 = ((2 Pi I ell) xi % // Expand) /. hermrel // Simplify
```

Out[◦]= hermn[ell, m]

Out[◦]= $\frac{\frac{\sqrt{m}\,\text{hermn[ell,-1+m]}}{\sqrt{2}} + \frac{\sqrt{1+m}\,\text{hermn[ell,1+m]}}{\sqrt{2}}}{2\sqrt{\pi}\,\sqrt{\text{Abs[ell]}}}$

Out[◦]= $\frac{1}{4\,\text{Abs[ell]}}\, i\,\text{ell} \left(\sqrt{-1+m}\,\sqrt{m}\,\text{hermn[ell, -2 + m]} + (1+2\,m)\,\text{hermn[ell, m]} + \sqrt{1+m}\,\sqrt{2+m}\,\text{hermn[ell, 2 + m]}\right)$

In[◦]:=
```
term1 + term2 // Simplify
% /. Abs[ell]^2 → ell^2 /. Abs[ell] → eps ell
```

Out[◦]= $\left(i\left(\sqrt{-1+m}\,\sqrt{m}\,(\text{ell}^2 - \text{Abs[ell]}^2)\,\text{hermn[ell, -2 + m]} + (1+2\,m)\,(\text{ell}^2 + \text{Abs[ell]}^2)\,\text{hermn[ell, m]} + \sqrt{1+m}\,\sqrt{2+m}\,(\text{ell}^2 - \text{Abs[ell]}^2)\,\text{hermn[ell, 2 + m]}\right)\right) / (4\,\text{ell Abs[ell]})$

Out[◦]= $\frac{i\,(1+2\,m)\,\text{hermn[ell, m]}}{2\,\text{eps}}$

Commutation relations

```
In[ ]:= Clear[ph, ell, xi, XX0, XX1, XX2, dpi, B]
        dpi[XX0, phh_] := Pi I ell phh
        dpi[XX1, phh_] := -4 Pi I ell xi phh
        dpi[XX2, phh_] := D[phh, xi]
        B[phh_] := (8 Pi I ell)^(-1) D[phh, {xi, 2}] + 2 Pi I ell xi^2 phh;

In[ ]:= B[dpi[XX1, ph[xi]]] - dpi[XX1, B[ph[xi]]] == - dpi[XX2, ph[xi]] // Simplify
Out[ ]= True

In[ ]:= B[dpi[XX2, ph[xi]]] - dpi[XX2, B[ph[xi]]] ==  dpi[XX1, ph[xi]] // Simplify
Out[ ]= True

In[ ]:= B[dpi[XX0, ph[xi]]] - dpi[XX0, B[ph[xi]]] // Simplify
Out[ ]= 0
```

Further computations

```
In[ ]:= Clear[v, z, r, x, y]
        mm[E^(I v)].nm[z, r].mm[E^(-I v)] == nm[E^(3 I v) z, r] // Simplify
        % /. Im[v] → 0 /. Conjugate[v] → v
```

Out[ ]= $e^{6\,\mathrm{Im}[v]}$ Abs[z] == Abs[z] && $e^{3\,i\,v}$ Conjugate[z] == $e^{3\,i\,\mathrm{Conjugate}[v]}$ Conjugate[z]

```
Out[ ]= True

In[ ]:= E^(3 I v) (x + I y) == (Cos[3 v] x - Sin[3 v] y) + I (Sin[3 v] x + Cos[3 v] y) // Simplify
Out[ ]= True

In[ ]:= {Cos[3 v] x - Sin[3 v] y, (Sin[3 v] x + Cos[3 v] y)}
        vxy = D[%, v] /. v → 0

Out[ ]= {x Cos[3 v] - y Sin[3 v], y Cos[3 v] + x Sin[3 v]}
Out[ ]= {-3 y, 3 x}

In[ ]:= vxy /. x → 1 /. y → 0
        vxy /. x → 0 /. y → 1
Out[ ]= {0, 3}
Out[ ]= {-3, 0}
```