7 Shift operators, construction

§3.1

7a. Shift operators in general (**g**,K)-modules

Here we carry out computations supporting the proof of Proposition 3.1 and check Table 3.4.

```
In[ ]:= Clear[h, p, v, w]
```

First upward shift operator

v satisfies the following relations

CKi**v = -I h v

WW0**v = - i p v

CasKZ**v=-p(p+2)v

Z21**v=0

The action and the composition of operators are presented by ** (NonCommutativeMultiply)

We use the relations to establish the desired relation for w=Z31 v

```
In[ ]:=
CKi ** w + I (h + 3) w /. w → Z31 ** v /. CKi ** Z31 ** v → Z31 ** (-I h v) + lb[CKi, Z31] ** v /.
   XX_ ** (h YY_) → h (XX ** YY) // Expand

Out[ ]= 0

In[ ]:= WW0 ** w + I (p + 1) w /. w → Z31 ** v /. WW0 ** Z31 ** v → Z31 ** (-I p v) + lb[WW0, Z31] ** v /.
   XX_ ** (p YY_) → p (XX ** YY) // Expand

Out[ ]= 0

In[ ]:= CasKZ ** w + (p + 1) (p + 3) w //. {WW0 ** w → -I (p + 1) w, XX_ ** ((p + 1) YY_) → (p + 1) (XX ** YY)} /.
     w → Z31 ** v //. {Z21 ** Z31 ** v :→ 0 + lb[Z21, Z31] ** v} /. nul → 0 // Expand

Out[ ]= 0
```

Second upward shift operator

Again we have:

```
In[ ]:= CKi ** v == -I h v
        WW0 ** v == - I p v
        CasKZ ** v == -p (p + 2) v
        Z21 ** v == 0
Out[ ]= CKi ** v == -ⅈ h v
Out[ ]= WW0 ** v == -ⅈ p v
Out[ ]= -2 ⅈ WW0 ** v + WW0 ** WW0 ** v + Z12 ** Z21 ** v == -p (2 + p) v
Out[ ]= Z21 ** v == 0
```

Now w = Z_{23} v

```
In[ ]:= w = X23 ** v
Out[ ]= X23 ** v

In[ ]:= CKi ** w + I (h - 3) w /. w → Z23 ** v /. CKi ** v → -I h v /. XX_ ** (h YY_) :→ h (XX ** YY) // Expand
Out[ ]= 0

In[ ]:= WW0 ** w + I (p + 1) w /. w → Z23 ** v /. WW0 ** v → - I p v /. XX_ ** (p YY_) :→ p (XX ** YY) // Expand
Out[ ]= 0

In[ ]:= CasKZ ** w + (p + 1) (p + 3) w /. w → Z23 ** v // Expand
        % //. {WW0 ** v → -I p v, XX_ ** (p YY_) :→ p (XX ** YY), Z21 ** v → 0} /. nul → 0
Out[ ]= -nul + 4 p Z23 ** v + p^2 Z23 ** v + 2 Z13 ** Z21 ** v -
          4 ⅈ Z23 ** WW0 ** v + Z23 ** WW0 ** WW0 ** v + Z23 ** Z12 ** Z21 ** v
Out[ ]= 0
```

Commutator

```
In[ ]:= lb[Z31, Z23]
Out[ ]= Z21/2
```

This vanishes on highest weight vectors.

First downward shift operator

Given is v with the properties

```
In[ ]:= Clear[u, v]
```

```
In[ ]:= CKi ** v == -I h v
        WW0 ** v == - I p v
        CasKZ ** v == -p (p + 2) v
        Z21 ** v == 0
```

```
Out[ ]= CKi ** v == -ⅈ h v
Out[ ]= WW0 ** v == -ⅈ p v
Out[ ]= -2 ⅈ WW0 ** v + WW0 ** WW0 ** v + Z12 ** Z21 ** v == -p (2 + p) v
Out[ ]= Z21 ** v == 0
```

We take $u = Z_{32}\, v$

```
In[ ]:= CKi ** u + I (h + 3) u /. u → Z32 ** v //.
          {CKi ** Z32 ** v → Z32 ** (-I h v) + lb[CKi, Z32] ** v, XX_ ** (h YY_) → h (XX ** YY)} // Expand
Out[ ]= 0
```

```
In[ ]:= WW0 ** u + I (p - 1) u /. u → Z32 ** v //.
          {WW0 ** Z32 ** v → Z32 ** (-I p v) + lb[WW0, Z32] ** v, XX_ ** (p YY_) → p (XX ** YY)} // Expand
Out[ ]= 0
```

```
In[ ]:= CasKu = CasKZ ** u /. u → Z32 ** v //. {WW0 ** Z32 ** v → -I p Z32 ** v + lb[WW0, Z32] ** v,
             XX_ ** (p YY_) → p (XX ** YY), Z21 ** Z32 ** v → 0 + lb[Z21, Z32] ** v} /. Z32 ** v → u // Expand
Out[ ]= u - p^2 u + 2 Z12 ** Z31 ** v
```

```
In[ ]:= Clear[up, um, CKu]
        eq = {up + um == u, -(p - 1) (p + 1) um - (p + 1) (p + 3) up == CKu}
        sol = Solve[eq, {up, um}][[1]] /. Z32 ** v → u // FullSimplify
Out[ ]= {um + up == u, (1 - p) × (1 + p) um - (1 + p) × (3 + p) up == CKu}
```

Out[ ]= $\left\{up \to -\frac{CKu + \left(-1 + p^2\right) u}{4 \times (1 + p)},\ um \to \frac{CKu + (1 + p) \times (3 + p)\, u}{4 \times (1 + p)}\right\}$

```
In[ ]:= sol /. CKu → CasKu // Simplify
```

Out[ ]= $\left\{up \to -\frac{Z12 ** Z31 ** v}{2 + 2\, p},\ um \to u + \frac{Z12 ** Z31 ** v}{2 + 2\, p}\right\}$

We take $\left(S^3\right)_{-1} v\ =\ u_{p-1}$,
and check the descriptions in (3.5)

```
In[ ]:= {um, up} = {um, up} /. sol /. CKu → CasKu
```

Out[ ]= $\left\{\frac{u - p^2\, u + (1 + p) \times (3 + p)\, u + 2\, Z12 ** Z31 ** v}{4 \times (1 + p)},\ -\frac{u - p^2\, u + \left(-1 + p^2\right) u + 2\, Z12 ** Z31 ** v}{4 \times (1 + p)}\right\}$

```
In[ ]:= {um == Z32 ** v + (2 (p + 1)) ^ (- 1) Z12 ** Z31 ** v,
          um == p (p + 1) ^ (- 1) Z32 ** v + (2 (p + 1)) ^ (- 1) Z31 ** Z12 ** v} /. u → Z32 ** v // Expand // Simplify

Out[ ]= {True, True}
```

Second downward shift operator

Given is v with the properties

```
In[ ]:= CKi ** v == -I h v
        WW0 ** v == - I p v
        CasKZ ** v == -p (p + 2) v
        Z21 ** v == 0

Out[ ]= CKi ** v == -ⅈ h v

Out[ ]= WW0 ** v == -ⅈ p v

Out[ ]= -2 ⅈ WW0 ** v + WW0 ** WW0 ** v + Z12 ** Z21 ** v == -p (2 + p) v

Out[ ]= Z21 ** v == 0
```

We take u as given by $\left(S^{-3}\right)_{-1}$ v with the shift operator indicated in Table 3.4 (first description)

```
In[ ]:= Clear[u]
        usub = {u → Z13 ** v - (2 (p + 1)) ^ (- 1) Z12 ** Z23 ** v}
```

$$\text{Out[ ]= } \left\{u \to Z13 ** v - \frac{2\, Z13 ** v + Z23 ** Z12 ** v}{2 \times (1 + p)}\right\}$$

Check of eigenvalue of C_i and $\boldsymbol{W_0}$

```
In[ ]:= CKi ** u + I (h - 3) u /. usub //. {CKi ** v → -I h v, nul → 0,
          aa_ ** (h bb_) :→ h aa ** bb, aa_ ** (bb_ / (p + 1)) :→ (p + 1) ^ (- 1) (aa ** bb),
          Z23 ** Z12 ** v → Z12 ** Z23 ** v + lb[Z23, Z12] ** v} // Simplify

Out[ ]= 0

In[ ]:= WW0 ** u + I (p - 1) u /. usub //. {YY_ ** ((p + 1) ^ (- 1) XX_) :→ (p + 1) ^ (- 1) (YY ** XX),
          WW0 ** v → -I p v, YY_ ** (p XX_) :→ p (YY ** XX)} // Simplify

Out[ ]= 0
```

Check of K-type

```
In[ ]:= CasKZ ** u + (p - 1) (p + 1) u /. usub //. {Z21 ** v → 0, WW0 ** v → -I p v, nul → 0,
          aa_ ** (p bb_) :→ p aa ** bb, aa_ ** (bb_ / (p + 1)) :→ (p + 1) ^ (- 1) (aa ** bb)} // Simplify

Out[ ]= 0
```

Highest weight?

In[ ]:=
```
Z21 ** u /. usub //. {WW0 ** v → -I p v, Z21 ** v → 0, nul → 0,
    aa_ ** (bb_ / (p + 1)) :→ (p + 1)^(-1) (aa ** bb), aa_ ** (p bb_) :→ p aa ** bb} // Simplify
```

Out[ ]= 0

So indeed u ∈ $V_{h-3,p-1,p-1}$

Alternative description

In[ ]:=
```
u == p (p + 1)^(-1) Z13 ** v - (2 (p + 1))^(-1) Z23 ** Z12 ** v /. usub // Expand // Simplify
```

Out[ ]= True

Commutator of downward shift operators

Take v as above

In[ ]:=
```
CKi ** v == -I h v
WW0 ** v == - I p v
CasKZ ** v == -p (p + 2) v
Z21 ** v == 0
```

Out[ ]= CKi ** v == $-i$ h v

Out[ ]= WW0 ** v == $-i$ p v

Out[ ]= $-2\,i$ WW0 ** v + WW0 ** WW0 ** v + Z12 ** Z21 ** v == -p (2 + p) v

Out[ ]= Z21 ** v == 0

Both downward shift operators decrease p by one.

In[ ]:=
```
p1 = {(Z13 - (2 p)^(-1) Z12 ** Z23) ** (Z32 + (2 (p + 1))^(-1) Z12 ** Z31) ** v,
    (Z32 + (2 p)^(-1) Z12 ** Z31) ** (Z13 - (2 (p + 1))^(-1) Z12 ** Z23) ** v} // Expand
```

Out[ ]= $\left\{\text{Z13} ** \text{Z32} ** \text{v} + \frac{1}{2}\,\text{Z13} ** \frac{\text{Z12} ** \text{Z31}}{1+\text{p}} ** \text{v} - \frac{1}{2}\,\frac{2\,\text{Z13} + \text{Z23} ** \text{Z12}}{\text{p}} ** \text{Z32} ** \text{v} - \frac{1}{4}\,\frac{2\,\text{Z13} + \text{Z23} ** \text{Z12}}{\text{p}} ** \frac{\text{Z12} ** \text{Z31}}{1+\text{p}} ** \text{v},\ -\frac{\text{Z12} ** \text{v}}{2} + \text{Z13} ** \text{Z32} ** \text{v} - \frac{1}{2}\,\text{Z32} ** \frac{2\,\text{Z13} + \text{Z23} ** \text{Z12}}{1+\text{p}} ** \text{v} + \frac{1}{2}\,\frac{\text{Z12} ** \text{Z31}}{\text{p}} ** \text{Z13} ** \text{v} - \frac{1}{4}\,\frac{\text{Z12} ** \text{Z31}}{\text{p}} ** \frac{2\,\text{Z13} + \text{Z23} ** \text{Z12}}{1+\text{p}} ** \text{v}\right\}$

Factors to the front, and then use relations for v.

In[ ]:=
```
p1[[1]] - p1[[2]] //. {XX_ ** (ff_^(-1) YY_) :→ ff^(-1) XX ** YY, (ff_^(-1) XX_) ** YY_ :→ ff^(-1) XX ** YY,
     CKi ** v → -I h v, YY_ ** (h XX_) :→ h (YY ** XX), WW0 ** v → -I p v,
     YY_ ** (p XX_) :→ p (YY ** XX), Z21 ** v → 0} /. nul → 0 // Expand // Simplify
```

Out[ ]= 0