Differentiable manifolds – hand-in sheet 3

Hand in by 21/Nov

Prelude to the exercise

Definition 1. A *graded vector space* is a vector space which decomposes as a direct sum

$$V = \oplus_{n=0}^{\infty} V_n,$$

where each V_n is a vector space and the elements of V_n are said to be the *homogenous* elements of *degree* n.

This means that an element of V does not have a degree associated to it, but it can be written as a sum of homogeneous elements of different degrees.

An example of graded vector space is given by polynomials in n variables with real, complex or even matrix coefficients. In this case, V_n is the set of homogeneous polynomials if degree n.

Definition 2. A *graded algebra* over the real numbers with a product of degree n is a real graded vector space, $A = \oplus A_i$, endowed with a bilinear operation

$$A \times A \xrightarrow{\cdot} A \qquad (X, Y) \mapsto X \cdot Y.$$

Such that for $X \in A_i$ and $Y \in A_j$, $X \cdot Y \in A^{i+j+n}$.

Using polynomial multiplication as algebra operation, polynomials of several variables are an example of a graded algebra with a multiplication of degree zero. Any algebra A is an example of a graded algebra with a product of degree zero by setting $A_0 = A$ and $A_i = \{0\}$ for $i > 0$.

Definition 3. A *graded Lie algebra* over the real numbers with a bracket of degree n is a real graded algebra, $A = \oplus A_i$ where the algebra operation is given by the bracket

$$A \times A \xrightarrow{[\cdot,\cdot]} A \qquad (X, Y) \mapsto [X, Y].$$

Such that for $X \in A_i$, $Y \in A_j$ and $Z \in A_k$

1. $[X, Y] \in A^{i+j+n}$, (degree n),
2. $[X, Y] = (-1)^{(i+n)\cdot(j+n)+1}[Y, X]$ (graded skew),
3. $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{(n+i)(n+j)}[Y, [X, Z]]$ (graded Jacobi).

Exercise

1) For a fixed $m \in \mathbb{N}$, a different algebra structure can be introduced on the set of polynomials in **one variable** with coefficients in $m \times m$ matrices. Namely, we declare that the homogenous polynomials of degree n are the elements of degree $2n$ in the algebra (hence there are no elements of odd degree) and define the algebra operation to be

$$(p, q) \longrightarrow [p, q] = p \cdot q - q \cdot p, \qquad \forall p, q,$$

where $\cdot$ denotes usual polynomial multiplication together with matrix multiplication of the coefficients. Show that with this operation the set of polynomials in one variable with coefficients in $m \times m$ matrices is a graded Lie algebra with a bracket of degree 0.

2) Let $(A, [\cdot, \cdot])$ be a graded Lie algebra with a bracket of degree n and assume that $d \in A^l$, with $n + l = 1$ mod 2, satisfies $[d, d] = 0$. Define a new bracket, $[\cdot, \cdot]_d$ on A by

$$[X, Y]_d = [[X, d], Y]].$$

Show that for $X \in A_i$, $Y \in A_j$ and $Z \in A_k$

- $[X, Y]_d \in A^{i+j+2n+l}$ (degree $2n + l$);
- $[X, [Y, Z]_d]_d = [[X, Y]_d, Z]_d + (-1)^{(2n+l+i)(2n+l+j)}[Y, [X, Z]_d]_d$ (graded Jacobi)

Hint: It may simplify your computations to define first the linear operator

$$D : A \longrightarrow A \qquad D(X) := DX := [X, d],$$

so that

$$[X, Y]_d = [DX, Y]$$

and check that

$$D^2 = 0 \qquad \textit{and} \qquad D[X, Y] = (-1)^{n+j}[DX, Y] + [X, DY].$$

and then apply D to the Jacobi identity

$$[DX, [Y, Z]] = [[DX, Y], Z] + (-1)^{(n+i+1)(n+j)}[Y, [DX, Z]]$$

Remark: The bracket $[\cdot, \cdot]_d$ defined above is not necessarily graded skew, hence it is not in general a graded Lie bracket.