Differentiable manifolds – homework 8

Read the section regarding Frobenius theorem.
Read the section regarding tensor and exterior algebra of vector spaces.

Exercise 1. Let $A : V \longrightarrow W$ be linear. Show that the following map induced by A is also linear:

$$A^* : \otimes^k W^* \longrightarrow \otimes^k V^*; \qquad A \mapsto A^*(\omega),$$

where $A^*\omega(X_1, \cdots X_k) := \omega(AX_1, \cdots, AX_k)$.

Exercise 2. Compute the dimension of $\wedge^k V^*$.

Exercise 3. Show that if $A : V \longrightarrow W$ is linear and $\omega \in \wedge^k W^*$, then $A^*\omega \in \wedge^k V^*$.

Exercise 4. Let V and W be vector spaces and let $B \in V^* \otimes W^*$ be a non degenerate element, i.e., B satisfies the property

$$B(X, Y) = 0 \qquad \text{for all} X \Rightarrow Y = 0$$
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Thinking of B as an element in $\mathrm{Hom}(V, W^*)$, show that B is an isomorphism of vector spaces. Conversely, given an isomorphism $B : V \longrightarrow W^*$, show that the corresponding tensor in $V^* \otimes W^*$ is nondegenerate.

Exercise 5. Let $A \in \otimes^2 V^*$. Show that there are $b \in \wedge^2 V^*$ and $g \in \mathrm{Sym}^2 V^*$ such that $A = g + b$.

Exercise 6.

1. (exterior product) Let $\xi \in V^*$ and $\omega \in \wedge^k V^*$. Show that $\xi \wedge \xi \wedge \omega = 0$.
2. (interior product) Interior product is a map

   $$\iota : V \times \wedge^k V^* \longrightarrow \wedge^{k-1} V^*, \qquad (X, \omega) \mapsto \iota_X \omega,$$

   where

   $$\iota_X \omega(X_2, \cdots, X_k) = \omega(X, X_2, \cdots, X_k).$$

   Show that $\iota_X \iota_X \omega = 0$ for all $X \in V$ and for all $\omega \in \wedge^k V^*$.
3. For $X \in V$, $\xi \in V^*$ and $\omega \in \wedge^k V^*$, define

   $$(X + \xi) \cdot \omega = \iota_X \omega + \xi \wedge \omega \in \wedge V^*$$

   Show that

   $$(X + \xi) \cdot ((X + \xi) \cdot \omega) = \xi(X)\omega.$$