Differentiable manifolds

Hand-in sheet 1

Hand-in by 13:00 on 29/11/2013

Let G be an Abelian group, $K < G$ be a subgroup and $H = G/K$ be the quotient so that we have natural maps

$$\{e\} \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow \{e\}$$

with the property that the image of each map is the kernel of the next.

Let M be a manifold and $\mathfrak{U}$ be a locally finite open cover of M. From previous exercises we know that we get a corresponding sequence of maps

$$\{0\} \longrightarrow \check{C}^k(M;K;\mathfrak{U}) \xrightarrow{\iota} \check{C}^k(M;G;\mathfrak{U}) \xrightarrow{\pi} \check{C}^k(M;H;\mathfrak{U}) \longrightarrow \{0\}$$

for which the kernel and image of consecutive maps agree, that is, ι is an injection, π is a surjection and the kernel of π is the image of ι and these maps commute with Čech differentials and hence give maps in the corresponding cohomologies:

$$\check{H}^k(M;K;\mathfrak{U}) \xrightarrow{\iota^*} \check{H}^k(M;G;\mathfrak{U}) \xrightarrow{\pi^*} \check{H}^k(M;H;\mathfrak{U}).$$

We know that $\ker(\pi^*) = \mathrm{Im}(\iota^*)$.

Exercise 1. The objective of this exercise is to construct a map

$$\check{H}^k(M;H;\mathfrak{U}) \xrightarrow{\partial} \check{H}^{k+1}(M;K;\mathfrak{U})$$

with the property that in the sequence of maps

$$\cdots \longrightarrow \check{H}^k(M;K;\mathfrak{U}) \xrightarrow{\iota^*} \check{H}^k(M;G;\mathfrak{U}) \xrightarrow{\pi^*} \check{H}^k(M;H;\mathfrak{U}) \xrightarrow{\partial} \check{H}^{k+1}(M;K;\mathfrak{U}) \xrightarrow{\iota^*} \check{H}^{k+1}(M;G;\mathfrak{U}) \longrightarrow \cdots$$

the kernel and image of consecutive maps is the same.

To construct the map ∂, let $h \in \check{H}^k(M;H;\mathfrak{U})$. Pick $\tilde{h} \in \check{C}^k(M;H;\mathfrak{U})$ a cochain representing it and let $g \in \check{C}^{k+2}(M;G;\mathfrak{U})$ be a cochain such that $\pi g = \tilde{h}$ (i.e., g is a lift of $\tilde{h}$). Then show that

1. $\delta g \in \mathrm{Im}(\iota)$ and hence we can regard $\delta g \in \check{C}^{k+1}(M;K;\mathfrak{U})$;
2. $\delta(\delta g) = 0 \in \check{C}^k(M;K;\mathfrak{U})$ and hence δg represents a class in $\check{H}^{k+1}(M;K;\mathfrak{U})$;
3. Show that the cohomology class of $[\delta g] \in \check{H}^{k+1}(M;K;\mathfrak{U})$ does not depend on the representative $\tilde{h}$ chosen for the cohomology class h or on the lift g chosen for the cochain $\tilde{h}$, hence we can define a map
$$\partial : \check{H}^k(M;H;\mathfrak{U}) \longrightarrow \check{H}^{k+1}(M;K;\mathfrak{U}), \qquad \partial(h) = [\delta g].$$
4. Show that $\ker(\partial) = \mathrm{Im}(\pi^*)$.
5. Show that $\mathrm{Im}(\partial) = \ker(\iota^*)$.