Differentiable manifolds – Mock exam 2

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.

Some useful definitions and results

- **Definition**. A path on a manifold $\gamma : \mathbb{R} \longrightarrow M$ is *periodic* if there is $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$.
- **Definition**. A *star shaped domain* of $\mathbb{R}^n$ is an open set $U \subset \mathbb{R}^n$ such that there is $p \in U$ with the property that if $q \in U$, then all the points in the segment connecting p and q are also in U, that is, there is p such that
$$(1-t)p + tq \in U; \text{ for all } q \in U \text{ and all } t \in [0,1].$$
The Poincaré Lemma in full generality states

  Theorem 1 (Poincaré Lemma). *If U is (diffeomorphic to) a star shaped domain of $\mathbb{R}^n$ then*
$$H^k(U) = \{0\} \qquad \text{for } k > 0.$$
- **Definition**. An open cover $\mathcal{U}$ of a manifold M is *fine* if any finite intersection of opens sets in $\mathcal{U}$ is either empty or (diffeomorphic to) a disc.

  With this definition, we have proved in the hand-in exercise sheets

  Theorem 2 (Čech to de Rham). *The Čech cohomology with real coefficients of any fine cover of M is isomorphic to the de Rham cohomology of M.*

Questions

1) (2 pt) Let D be a rank k, involutive distribution on a manifold M. Show that if $\alpha \in \Omega^1(M)$ is such that $\alpha(X) = 0$ for all $X \in \Gamma(D)$, then $(d\alpha)(X,Y) = 0$ for all $X, Y \in \Gamma(D)$.

2) (2 pt) Let X be a smooth vector field on a manifold M and let $\gamma : (a,b) \longrightarrow M$ be a maximal integral curve of X. Show that exactly one of the following holds

- γ is the constant path;
- γ is injective;
- γ is defined for all time and is periodic and nonconstant.

3) (2 pt) Consider the form $\rho \in \Omega^2(\mathbb{R}^3 \backslash \{0\})$
$$\rho = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

a) Show that $d\rho = 0$;

b) Compute the integral of ρ over the 2-sphere of radius 2 in $\mathbb{R}^3$ centered at $(0, 0, 1)$.

c) Compute the integral of ρ over the 2-sphere of radius 2 in $\mathbb{R}^3$ centered at $(0, 0, 3)$.

d) Does ρ represent a nontrivial cohomology class in $\mathbb{R}^3 \backslash \{0\}$? Does ρ represent a nontrivial class in

$$\mathbb{R}^3 \backslash \{(0, 0, x) : x \geq 0\}?$$

4) (2 pt) Compute the dimension of the degree 1 de Rham cohomology of $S^1 \times S^1$.

5) (2 pt) Let $E \xrightarrow{\pi} M$ be a vector bundle over a manifold M. Show that $\check{H}^k(E; \mathbb{R}; \mathcal{V}) \cong \check{H}^k(M; \mathbb{R}; \mathcal{U})$ for all k as long as $\mathcal{U}$ and $\mathcal{V}$ are fine covers of M and E respectively. (Hint: use the Čech to de Rham theorem to conclude that it is enough to prove the result for a single pair of fine covers $\mathcal{U}$ and $\mathcal{V}$, then use $\mathcal{U}$ to choose wisely a fine cover $\mathcal{V}$ for E.)