Notes on Čech cohomology

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1 Čech cohomology

1.1 Digression: creating manifolds out of changes of coordinates

Let me start this section with a digression. In lectures our approach to manifolds was that we were God-given one and that it should have certain properties (Hausdorff, second coutable locally Euclidean topological space with a smooth structure). If such a space was given, we could produce coordinate charts that helped us to define smooth functions, tangent vectors and other things. Figure 1 below is a picture that encompasses this information, in this order.

In several cases, one actually reverses this picture (which is often the 'physicists' approach) and is given

- Several domains $V_\alpha \subset \mathbb{R}^n$ (these are to be thought of as the image of a coordinate chart)
- Diffeomorphisms ψ^α_β, called *change of coordinates* identifying some points on V_α with some points on V_β for all α and β in the index set.

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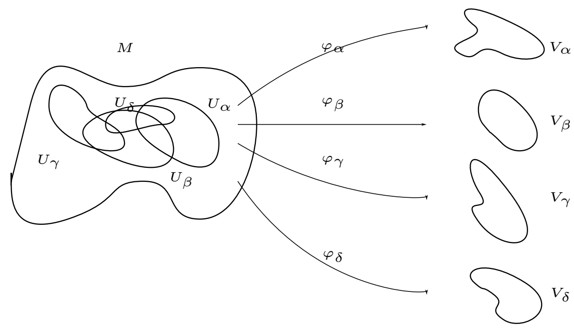

Figure 1: Manifold + charts ⇒ open sets in $\mathbb{R}^n$ + change of coordinates.

Of course, for these ψ^α_β to be what we usually call the change of coordinates, i.e., for them to correspond to

$$\psi^\alpha_\beta = \varphi_\beta \circ \varphi_\alpha^{-1} \tag{1}$$

where φ_α is the coordinate chart whose image is V_α, they must satisfy some compatibility conditions that follow from (1), namely:

$$\begin{aligned} \psi^\alpha_\alpha &= \mathrm{Id}; \\ \psi^\gamma_\alpha \circ \psi^\beta_\gamma \circ \psi^\alpha_\beta &= \mathrm{Id}. \end{aligned} \tag{2}$$

Together, these conditions give some sort of "skew symmetry" of the functions ψ^α_β, indeed, taking $\gamma = \alpha$ and using both conditions together one arrives at

$$(\psi^\alpha_\beta)^{-1} = \psi^\beta_\alpha.$$

Further, if (2) holds and one is given, say, four changes of coordinates cycling back to the original, say ψ^α_β, ψ^β_γ, ψ^γ_δ and ψ^δ_α, we have

$$\psi^\delta_\alpha \circ \psi^\gamma_\delta \circ \psi^\beta_\gamma \circ \psi^\alpha_\beta = (\psi^\delta_\alpha \circ \psi^\gamma_\delta \circ \psi^\alpha_\gamma) \circ (\psi^\gamma_\alpha \circ \psi^\beta_\gamma \circ \psi^\alpha_\beta) = \mathrm{Id} \circ \mathrm{Id} = \mathrm{Id}$$

and the same argument holds for an arbitrary number of change of coordinates cycling back to the original one.

Now to create a manifold for which the ψ^α_β actually correspond to change of coordinates one first takes the disjoint union of all V_α: $\mathcal{V} = \dot\cup V_\alpha$. This means that $x \in \mathcal{V}$ if and only if $x \in V_\alpha$ for some α and we declare that $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$. Then the manifold itself is a quotient space of $\mathcal{V}$ by an equivalence relation $M = \mathcal{V}/\sim$, where $x \in V_\alpha$ is equivalent to $y \in V_\beta$ if and only if $\psi^\alpha_\beta(x) = y$.

Since whenever we cycle back to V_α we always get the identity map, we see that no two points in V_α are identified by this relation therefore we still have $V_\alpha \subset M$. The map

$$\varphi_\alpha : V_\alpha \subset M \longrightarrow V_\alpha \subset \mathbb{R}^n; \qquad \varphi_\alpha(x) = x$$

gives a local Euclidean structure to M and the transition functions for these choices of coordinates are precisely ψ^α_β which are diffeomorphisms. Hence the open sets $\{V_\alpha\}$ together with change of coordinates $\{\psi^\alpha_\beta\}$ satisfying (2) give us a smooth structure on $M = \mathcal{V}/\sim$.

What can potentially go wrong with the procedure above, even if (2) holds, is that the topology of M gets spoiled and M may fail be second countable and/or Hausdorff. If there are only enumerably many V_α, one gets second countability. Unfortunately the Hausdorff property may be lost in the quotient process and one needs more precise knowledge about the ψ^α_β to prove that it holds.

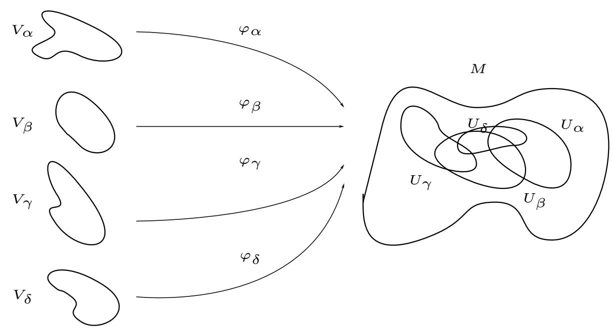

Figure 2: Open sets in $\mathbb{R}^n$ + change of coordinates $\Rightarrow$ Manifold + charts.

This pathology can be illustrated with the following simple example whose details we leave as exercise for the reader to spell out.

Exercise 1.1. Let $U_0 = U_1 = (-1, 1)$ and let $\psi_1^0 : (-1, 0) \cup (0, 1) \subset U_0 \longrightarrow (-1, 0) \cup (0, 1) \subset U_1$ be given by $\varphi_1^0(x) = x$. Then ψ_1^0 is a smooth map and we take ψ_0^1 to be its inverse. With these choices, the collection $\{\psi_1^0, \psi_0^1\}$ forms a family of change of coordinates satisfying conditions (2). Following the proposed procedure, we get that $M = U_0 \dot{\cup} U_1 / \sim$ intuitively corresponds to the the interval $(-1, 1)$ with the point 0 "doubled". This is a standard example of non Hausdorff space in topology textbooks (compare with exercise 1 in the first exercise sheet).

1.2 Čech cochains and differential with real coefficients

Čech cohomology is obtained using an open cover of a topological space and it arises using purely combinatorial data. The idea being that if one has information about the open sets that make up a space as well as how these sets are glued together one can deduce global properties of the space from local data.

Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a connected manifold M. For $\alpha_1, \cdots, \alpha_n \in A$, we denote

$$U_{\alpha_0 \cdots \alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k},$$

or, equivalently, in multi-index notation, if $\mathbf{a} = \{\alpha_0, \cdots, \alpha_k\}$

$$U_{\mathbf{a}} = \bigcap_{\alpha_i \in \mathbf{a}} U_{\alpha_i}.$$

Definition 1.2. A degree k *Čech cochain with real coeficients* for the cover $\mathfrak{U}$ is a collection of functions

$$\check{f} := \{f_{\mathbf{a}} | \mathbf{a} \text{ ordered subset of } A \text{ with } k+1 \text{ elements}\} \tag{3}$$

where each $f_{\mathbf{a}} \in \check{f}$ is a *constant real function* (coefficients in $\mathbb{R}$)

$$f_{\mathbf{a}} : U_{\mathbf{a}} \longrightarrow \mathbb{R}.$$

satisfying

$$f_{\alpha_0 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_k} = -f_{\alpha_0 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_k} \tag{skew symmetry}$$

We denote the set of all degree k Čech cochains with real coefficients obtained from a cover $\mathfrak{U}$ of M by $\check{C}^k(M; \mathbb{R}; \mathfrak{U})$. Note that pointwise addition of real numbers makes $\check{C}^k(M; \mathbb{R}; \mathfrak{U})$ into an Abelian group and scalar multiplication gives it the structure of a real vector space.

Notice that according to this definition, and element of $\check{C}^0(M;\mathbb{R};\mathfrak{U})$ corresponds to the assigment of a constant function to each open set of $\mathfrak{U}$. In particular if the cover $\mathfrak{U}$ is finite, say $\#\mathfrak{U}=n$, then $\check{C}^0(M;\mathbb{R};\mathfrak{U})=\mathbb{R}^n$. Similarly, the elements in $\check{C}^1(M;\mathbb{R};\mathfrak{U})$ correspond to constant functions defined on overlaps of two sets of $\mathfrak{U}$. Let's see this in a concrete example.

Example 1.3. Consider S^1 as the interval $[0,1]$ with the ends identified. We can cover S^1 by the open sets $U_0=(0,2/3)$, $U_1=(1/3,1)$ and $U_2=(2/3,1)\cup(0,1/3)$. Then $U_{0,1}=(1/3,2/3)$, $U_{1,2}=(2/3,1)$ and $U_{2,0}=(0,1/3)$ and $U_{0,1,2}=\emptyset$. That is, for this open decomposition of S^1 there are only Čech cycles of degree zero and one. An element in $\check{C}^0(M;\mathbb{R};\mathfrak{U})$ is given by three constants, hence $\check{C}^0(M;\mathbb{R};\mathfrak{U})=\mathbb{R}^3$. Similarly, since there are only three double overlaps, $\check{C}^1(M;\mathbb{R};\mathfrak{U})=\mathbb{R}^3$.

Definition 1.4. The *Čech differential* is a linear map $\delta^k:\check{C}^k(M;\mathbb{R};\mathfrak{U})\longrightarrow\check{C}^{k+1}(M;\mathbb{R};\mathfrak{U})$,

$$\delta^k(\check{f})_{\alpha_0\cdots\alpha_{k+1}}=\sum_i(-1)^i f_{\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{k+1}}.$$

In what follows we will denote all maps δ^k defined above simply by δ. The main property of δ is given in the following proposition:

Proposition 1.5. *The Čech differential satisfies*

$$\delta^2=0.$$

Proof. Let $\check{f}$ be a k-cochain. Then

$$(\delta\check{f})_{\alpha_0\cdots\alpha_{k+1}}=\sum_{i=0}^{k+1}(-1)^i(\check{f})_{\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\alpha_{k+1}}$$

Hence

$$\begin{aligned}(\delta^2 f)_{\alpha_0,\cdots\alpha_{k+2}}&=\sum_{i=0}^{k+2}(-1)^i(\delta\check{f})_{\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\alpha_{k+2}}\\&=\sum_{j<i}(-1)^{i+j}(\check{f})_{\alpha_0\cdots\alpha_{j-1}\alpha_{j+1}\cdots\alpha_{i-1}\alpha_{i+1}\alpha_{k+2}}\\&+\sum_{i<j}(-1)^{i+j-1}(\check{f})_{\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{j-1}\alpha_{j+1}\cdots\alpha_{k+2}}\\=0&\end{aligned}$$

$\square$

It is standard practice in mathematics that whenever one finds a sequence of linear maps between vector spaces

$$\delta^k:V^k\longrightarrow V^{k+1}$$

with $\delta^k\circ\delta^{k-1}=0$ one defines *cohomology spaces*:

$$H^k:=\frac{\ker(\delta^k)}{\operatorname{Im}(\delta^{k-1})}.$$

In our case, these spaces depend on M and the open cover $\mathfrak{U}$, so we write:

$$\check{H}^k(M;\mathbb{R};\mathfrak{U})=\frac{\ker(\delta:\check{C}^k\longrightarrow\check{C}^{k+1})}{\operatorname{Im}(\delta:\check{C}^{k-1}\longrightarrow\check{C}^k)}.$$

Definition 1.6. We say that an element $\check{f} \in \check{C}^k$ is *closed* or a *cocycle* if $\delta \check{f} = 0$. An element $\check{f} \in \check{C}^k$ is *exact* or a *coboundary* if $\check{f}$ is in the image of δ, i.e., there is $\check{g} \in \check{C}^{k-1}$ for which $\delta \check{g} = \check{f}$.

Example 1.7 (Degree zero Čech cocycles)**.** Let M be a connected manifold and $\mathfrak{U}$ be a locally finite open cover. Next we see that degree zero Čech cohomology is particularly easy to describe. Since $\check{C}^{-1} = \{0\}$, we have

$$\check{H}^0 = \ker(\delta : \check{C}^0 \longrightarrow \check{C}^1)$$

Further, if $\check{f} \in \ker(\delta : \check{C}^0 \longrightarrow \check{C}^1)$, then if U_α intersects U_β we have

$$0 = (\delta \check{f})_{\alpha\beta} = \check{f}_\beta - \check{f}_\alpha,$$

that is $\check{f}_\alpha = \check{f}_\beta$ whenever U_α intersects U_β. Now, for such an $\check{f}$, let $c = \check{f}_\alpha(x)$ for a fixed x in a fixed U_α. Now, if we let $V \subset M$ be the set of points defined by

$$V = \{p \in M : \text{ if } p \in U_\beta \text{ then } \check{f}_\beta(p) = c\}.$$

By the cocycle condition and the choice of c we see that $x \in V$, hence $V \neq \emptyset$. Further V is defined by a closed condition, so it is a closed subset of M. Finally, if $p \in V$, let $U_\beta \in \mathfrak{U}$ be an open set containing p (U_β exists because $\mathfrak{U}$ is a cover). Then $\check{f}_\beta(p) = c$ and hence, again by the cocycle condition $\check{f}_\gamma(p) = c$ whenever $p \in U_\gamma$. Hence V is open (by locally finiteness) and since M is connected, $V = M$. That is for all α, $\check{f}_\alpha = c$ and each $\check{f}_\alpha$ is just the restriction of the globally defined function

$$f : M \longrightarrow \mathbb{R}; \qquad f \equiv c$$

to U_α. Or said another way, $\check{f}$ corresponds to the restriction of a globally defined function to the open sets of the cover $\mathfrak{U}$:

$$\check{H}^0 = \{\text{Globally defined constant functions}\}$$

Exercise 1.8. For the cover of S^1 obtained in Example 1.3, compute $\check{H}^0$ and $\check{H}^1$.

Exercise 1.9 (Euler characteristic)**.** Let $\{V^k : 0 \leq k \leq n\}$ be a family of finite dimensional vector spaces where $n \in \mathbb{N}$ is some fixed number. Whenever necessary, let $V_{-1} = V_{n+1} = \{0\}$. Let $d_k : V^k \longrightarrow V^{k+1}$ be linear maps such that $d_{k+1} \circ d_k = 0$ for all i and define

$$H^k = \frac{\ker(d_k)}{\operatorname{Im}(d_{k-1})}.$$

Show that

$$\sum (-1)^k \dim(V^k) = \sum (-1)^k \dim(H^k).$$

Conclude that if $\mathfrak{U}$ is a finite open cover of a manifold then

$$\sum (-1)^k \dim(\check{C}^k(M;\mathbb{R};\mathfrak{U})) = \sum (-1)^k \dim(\check{H}^k(M;\mathbb{R};\mathfrak{U})).$$

Hint: Use the rank nullity theorem from linear algebra, namely, if $A : V \longrightarrow W$ is a linear map,

$$\dim(V) = \dim(\operatorname{Im}(A)) + \dim(\ker(A)).$$

Definition 1.10. For a cover $\mathfrak{U}$ of M, the *Euler characteristic of M with respect to the cover* $\mathfrak{U}$ is the number

$$\chi(M;\mathfrak{U}) = \sum (-1)^k \dim(\check{H}^k(M;\mathbb{R};\mathfrak{U}))$$

The *Euler characteristic of M*, denoted by $\chi(M)$, is $\chi(M;\mathfrak{U})$ where $\mathfrak{U}$ is any finite good cover of M.

Exercise 1.11. Cover the sphere S^2 with four open sets obtained by slightly enlarging the tetrahedral triagulation of the sphere (see Figure 1). Compute the Euler characteristic of S^2 with respect to this open cover.

Exercise 1.12. Consider S^1 as the interval $[0,1]$ with the ends identified. Cover S^1 by the open sets $U_0 = (0, 2/3)$, $U_1 = (1/3, 1)$ and $U_2 = (2/3, 1) \cup (0, 1/3)$. Compute the Euler characteristic of S^1 from this cover.

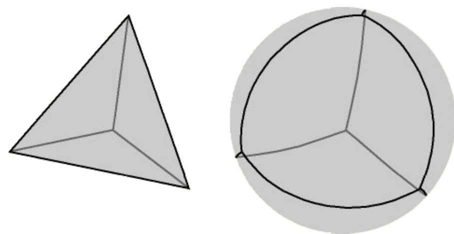

Figure 3: Tetrahedral decomposition of the sphere.

1.3 Čech cochains and differential with other coefficients

1.3.1 Coefficients in other Abelian groups

Now notice that we used very little of the structure of the real numbers and in fact all the argument used above can be carried out for constant functions with values in any Abelian group, such as $\mathbb{Z}$, $\mathbb{Z}_n$, S^1, $\mathbb{R}^*$, $\mathbb{C}^*$, etc. This way we obtain cohomology groups $\check{H}^\bullet(M;G;\mathfrak{U})$ (which are not necessarily vector spaces) for any Abelian group G.

Example 1.13 (Čech cohomology with coefficients in $\mathbb{R}^*$)**.** Here we work out explicitly the changes that happen when considering Čech cohomology with coefficients in $\mathbb{R}^*$ (constant coefficients). The only difference of this case compared to the previous one is not conceptual, but notational. Let me be precise. For real coefficients, the symbol $+$ indicated the group operation in the Abelian group $\mathbb{R}$, the symbol $-$ indicated inversion and 0 was the identity element. Therefore, in the case of $\mathbb{R}^*$, which is an Abelian group with multiplication of real numbers, these operations get replaced by multiplication of real numbers, $\cdot$, and inversion of real numbers, $\cdot^{-1}$, respectively and 0 is replaced by 1.

With these changes in mind, Čech cochains are defined in a completely analogous way, namely, a degree k *Čech cochain with* $\mathbb{R}^*$ *coeficients* for the cover $\mathfrak{U}$ is a collection of functions

$$\check{f} := \{f_{\mathbf{a}} | \mathbf{a} \text{ ordered subset of } A \text{ with } k+1 \text{ elements}\}$$

where each $f_{\mathbf{a}} \in \check{f}$ is a *constant function* (coefficients in $\mathbb{R}^*$)

$$f_{\mathbf{a}} : U_{\mathbf{a}} \longrightarrow \mathbb{R}^*.$$

satisfying

$$f_{\alpha_0\cdots\alpha_i\alpha_{i+1}\cdots\alpha_k} = (f_{\alpha_0\cdots\alpha_{i+1}\alpha_i\cdots\alpha_k})^{-1} \qquad \text{(skew symmetry)}$$

The set of all degree k Čech cochains with $\mathbb{R}^*$ coefficients obtained from a cover $\mathfrak{U}$ of M is denoted by $\check{C}^k(M;\mathbb{R}^*;\mathfrak{U})$. Again, pointwise multiplication of real numbers makes $\check{C}^k(M;\mathbb{R}^*;\mathfrak{U})$ into and Abelian group.

Also the Čech differential is defined in a similar fashion:

$$\delta^k : \check{C}^k(M;\mathbb{R}^*;\mathfrak{U}) \longrightarrow \check{C}^{k+1}(M;\mathbb{R}^*;\mathfrak{U}),$$

$$\delta^k(\check{f})_{\alpha_0\cdots\alpha_{k+1}} = \Pi_i(f_{\alpha_0\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{k+1}})^{(-1)^i}.$$

And again we will omit the superscript k and just think of δ as an operator acting on all Čech cochains.

So, for example if $\check{f} \in \check{C}^0(M;\mathbb{R}^*;\mathfrak{U})$, then

$$(\delta\check{f})_{\alpha\beta} = f_\beta \cdot (f_\alpha)^{-1}.$$

Also, if $\check{g} \in \check{C}^1(M;\mathbb{R}^*;\mathfrak{U})$, then

$$(\delta\check{g})_{\alpha\beta\gamma} = g_{\beta\gamma} \cdot (g_\alpha\gamma)^{-1} \cdot g_{\alpha\beta}.$$

Using skew symmetry, we can re-write this condition in a more mnemonic way:

$$(\delta \check{g})_{\alpha\beta\gamma} = g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha}.$$

Notice that now the cocycle condition for an element $\check{f} \in \check{C}^k(M; \mathbb{R}^*; \mathfrak{U})$ becomes

$$\delta \check{f} = 1.$$

1.3.2 Coefficients on smooth functions

We can also relax the condition that the functions $f_{\mathbf{a}}$ are constant. For example we have

Definition 1.14. A degree k *Čech cochain with coeficients in the smooth functions* for the cover $\mathfrak{U}$ is a collection of functions

$$\check{f} := \{f_{\mathbf{a}} | \mathbf{a} \text{ ordered subset of } A \text{ with } k+1 \text{ elements}\}$$

where each $f_{\mathbf{a}} \in \check{f}$ is a *smooth real function*

$$f_{\mathbf{a}} : U_{\mathbf{a}} \longrightarrow \mathbb{R}.$$

satisfying

$$f_{\alpha_0 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_k} = -f_{\alpha_0 \cdots \alpha_{i+1} \alpha_i \cdots \alpha_k} \qquad \text{(skew symmetry)}$$

We denote the set of all degree k Čech cochains with smooth functions as coefficients obtained from a cover $\mathfrak{U}$ of M by $\check{C}^k(M; C^\infty(M); \mathfrak{U})$. Note that pointwise addition of functions makes $\check{C}^k(M; C^\infty(M); \mathfrak{U})$ into and Abelian group and scalar multiplication gives it the structure of a real vector space.

The Čech differential is defined in the same way as before and the same proof still yields $\delta^2 = 0$ hence we also have Čech cohomology with coefficients in the smooth functions.

Exercise 1.15 (Čech cohomology with coefficients in $C^\infty(M)$)**.** Repeat the argument from Example 1.7 and conclude that $\check{H}^0(M; C^\infty(M); \mathfrak{U})$ can be identified with the space

$$C^\infty(M) = \{f : M \longrightarrow \mathbb{R} : f \text{ is smooth}\}.$$

Differently from the case of real coefficients, when we consider smooth functions, there is no cohomology in degree higher than zero:

Theorem 1.16. *For $k > 0$,*

$$\check{H}^k(M; C^\infty(M); \mathfrak{U}) = \{0\}.$$

Equivalently, every closed Čech cochain is a coboundary.

Proof. This theorem is a consequence of the existence of partitions of unity. Indeed, let $\check{f} \in \check{C}^k(M; C^\infty(M); \mathfrak{U})$ be a cocycle and $(\varphi_\alpha : \alpha \in A)$ be a partition of unity subbordinated to $\mathfrak{U}$. Spelling out the cocycle condition we have

$$0 = (\delta \check{f})_{\alpha_0, \cdots \alpha_{k+1}} = \sum_{i=0}^{k+1} (-1)^i \check{f}_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_{k+1}} \qquad \forall \alpha_i \in A.$$

Equivalently,

$$\check{f}_{\alpha_1 \cdots \alpha_{k+1}} = \sum_{i=1}^{k+1} (-1)^{i+1} \check{f}_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_{k+1}} \qquad \forall \alpha_i \in A. \tag{4}$$

Define $\check{g} \in \check{C}^{k-1}(M; C^\infty(M); \mathfrak{U})$ by

$$\check{g}_{\alpha_1, \cdots \alpha_k} = \sum_{\alpha \in A} \varphi_\alpha f_{\alpha \alpha_1 \cdots \alpha_k}.$$

Notice that even though $f_{\alpha\alpha_1\cdots\alpha_k}$ is only defined on $U_{\alpha\alpha_1\cdots\alpha_k}$, since φ_α has compact support in U_α, $\varphi_\alpha f_{\alpha\alpha_1\cdots\alpha_k}$ can be extended to $U_{\alpha_1\cdots\alpha_k}$ by declaring that it vanishes on $U_{\alpha_1\cdots\alpha_k}\backslash U_{\alpha\alpha_1\cdots\alpha_k}$ so $\check{g}_{\alpha_1\cdots\alpha_k}$ defined above is indeed a smooth function on $U_{\alpha_1\cdots\alpha_k}$.

Now we compute

$$\begin{aligned}(\delta\check{g})_{\alpha_1,\cdots,\alpha_{k+1}} &= \sum_{i=1}^{k+1}(-1)^{i+1}\check{g}_{\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{k+1}}\\ &= \sum_{i=1}^{k+1}(-1)^{i+1}\sum_{\alpha\in A}\varphi_\alpha\check{f}_{\alpha\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{k+1}}\\ &= \sum_{\alpha\in A}\varphi_\alpha\sum_{i=1}^{k+1}(-1)^{i+1}\check{f}_{\alpha\alpha_1\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_{k+1}}\\ &= \sum_{\alpha\in A}\varphi_\alpha\check{f}_{\alpha_1\cdots\alpha_{k+1}}\\ &= (\sum_{\alpha\in A}\varphi_\alpha)\check{f}_{\alpha_1\cdots\alpha_{k+1}}\\ &= \check{f}_{\alpha_1\cdots\alpha_{k+1}},\end{aligned}$$

where in the first equality we wrote the definition of Čech differential, in the second we used the definition of $\check{g}$, in the third we commuted the sums, in the fourth we used equation (4), in the fifth we notice that the term $\check{f}_{\alpha_1\cdots\alpha_{k+1}}$ does not depend on the index of summation, hence can be put in evidence and in the last equation we used again that α_α is a partition of unity. □

Exercise 1.17. If the multi-indices are mind boggling, repeat the argument above in the case $f\in\check{C}^2(M;C^\infty(M);\mathfrak{U})$ to convince yourself that everything is fine.

As a consequence of this theorem, we see that the Čech cohomology $\check{H}^\bullet(M;C^\infty(M);\mathfrak{U})$ are rather simple to describe. Indeed, according to Example 1.7 and Exercise 1.15, $\check{H}^0(M;C^\infty(M);\mathfrak{U})$ corresponds to the vector space of globally defined functions and the remaning groups $\check{H}^k(M;C^\infty(M);\mathfrak{U})$ are all trivial for $k>0$. Note that these equalities hold for any locally finite cover, that is these groups are independent of $\mathfrak{U}$, hence, in this case, it makes sense to write simply $\check{H}^\bullet(M;C^\infty(M))$.

1.3.3 Further coefficients

One can mix the last two generalizations and take coefficients is smooth functions with values in some Abelian group. For discrete Abelian groups, such as $\mathbb{Z}$ and $\mathbb{Z}_n$, smooth automatically means constant and hence this does not change the discussion from Section 1.3.1. For Lie groups (such as S^1, $\mathbb{R}^*$ and $\mathbb{C}^*$), the change from constant coefficients to smooth coefficients can potentially lead to different Čech cohomology space when compared with the case of constant coefficients, much like the passage from constant real coefficients to smooth real functions.

Notice that the argument used in Theorem 1.16 can not be applied to Čech cohomology with coefficients in say smooth functions with values in $\mathbb{R}^*$ because even if $\check{f}$ take values in $\mathbb{R}^*$, and $\{\varphi_\alpha\}$ is a partition of unity then $\varphi_\alpha f_\alpha$ takes values in $\mathbb{R}$ and hence it is no longer a Čech cocycle with coefficients in smooth functions with values in $\mathbb{R}^*$.

While there are many coefficients one can take for Čech cohomology, the main examples that will concern us are coefficients in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $C^\infty(M;\mathbb{R}^*)$, $C^\infty(M;S^1)$ and $C^\infty(M;\mathbb{C}^*)$. What you will find written elsewhere is that Čech cohomology is defined for any *sheaf* of Abelian groups over a manifold. We will not explain the meaning of these words in these notes. But we do notice that there are hidden isomorphisms between Čech cohomology with different coefficients. Later we will prove the following:

Theorem 1.18. *Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a manifold M for which each U_{a} is connected for all multi indices $\mathrm{a} \subset A$. Then, for $k > 0$,*

$$\check{H}^k(M; \mathbb{Z}_2; \mathfrak{U}) \cong \check{H}^k(M; C^\infty(M; \mathbb{R}^*), \mathfrak{U}).$$

Theorem 1.19. *Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a manifold M for which each U_{a} is connected for all multi indices $\mathrm{a} \subset A$. Then, for $k > 0$,*

$$\check{H}^k(M; C^\infty(M; S^1); \mathfrak{U}) \cong \check{H}^k(M; C^\infty(M; \mathbb{C}^*), \mathfrak{U}).$$

Theorem 1.20. *Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of a manifold M for which each U_{a} is connected for all multi indices $\mathrm{a} \subset A$. Then, for $k > 0$,*

$$\check{H}^{k+1}(M; \mathbb{Z}; \mathfrak{U}) \cong \check{H}^k(M; C^\infty(M; \mathbb{C}^*), \mathfrak{U}).$$

1.4 Čech cohomology and maps

A natural question one can ask is how Čech cohomology interacts with smooth maps. Now we let $f : M \longrightarrow N$ be smooth and let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of N. Then it follows from continuity of f that

$$f^{-1}(\mathfrak{U}) = \{f^{-1}(U_\alpha) : \alpha \in A\}$$

is an open cover of M and hence we can define a map

$$f^* : \check{C}^k(N; G; \mathfrak{U}) \longrightarrow \check{C}^k(M; G; f^{-1}(\mathfrak{U})), \qquad (f^*c)_{\mathrm{a}} = c_{\mathrm{a}} \circ f,$$

where $c \in \check{C}^k(N; G; \mathfrak{U})$ and a is an ordered multiindex of size $k+1$, so that $c_{\mathrm{a}} : U_{\mathrm{a}} \longrightarrow G$.

Show that f^* defined above is an isomorphism of Abelian groups for every k and that it commutes with differentials, that is,

$$f^*\delta = \delta f^*.$$

Conclude that the Čech cohomologies of M and N with respect to the covers $\mathfrak{U}$ and $f^{-1}(\mathfrak{U})$ are isomorphic. Conclude further that if f is an diffeomorphism, then M and N have isomorphic Čech cohomologies with respect to any good cover of these manifolds.

Remark: In fact the exercise above shows that if $f : M \longrightarrow N$ is smooth, surjective and $f^{-1}(\mathfrak{U})$ is a good cover of M for some good cover of N, then the cohomologies of M and N are isomorphic. An example where one can use this more general statement is with the map

$$f : \mathbb{C}^* \longrightarrow S^1, \qquad f(z) = \frac{z}{|z|}.$$

2 Vector bundles

2.1 Real vector bundles

The notion of a vector bundle is a direct generalization of the tangent space of a manifold and permeates through all differential geometry. Since the tangent space is covered in Warner, we will jump right into the notion of vector bundle here.

Definition 2.1. A *vector bundle* of *rank* k over a manifold M is a manifold E and a surjective smooth map, the *projection map*,

$$\pi : E \longrightarrow M$$

which satisfies the following local triviality condition: there is an open cover $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ of M and diffeomorphisms

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$$

such that

1. For all $x \in U_\alpha$, $v \in \mathbb{R}^k$, $\pi \circ (\Phi_\alpha)^{-1}(x,v) = x$;
2. If $U_{\alpha\beta} \neq \emptyset$ the function $\Phi_\beta \circ \Phi_\alpha^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \longrightarrow U_{\alpha\beta} \times \mathbb{R}^k$ satisfies

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x,\cdot) : \mathbb{R}^k \longrightarrow \mathbb{R}^k \qquad \text{is a linear map for all } x \in U_{\alpha\beta}.$$

It follows from condition 1 that $\Phi_\beta \circ \Phi_\alpha^{-1} : x \times \mathbb{R}^k \longrightarrow x \times \mathbb{R}^k$ hence we can write this composition as

$$\begin{aligned} \Phi_\beta \circ \Phi_\alpha^{-1} &: U_{\alpha\beta} \times \mathbb{R}^k \longrightarrow U_{\alpha\beta} \times \mathbb{R}^k; \\ \Phi_\beta \circ \Phi_\alpha^{-1}(x,v) &= (x, g_\beta^\alpha(x)v), \qquad g_\beta^\alpha : U_{\alpha\beta} \longrightarrow \mathrm{Gl}(k;\mathbb{R}). \end{aligned} \tag{5}$$

There are several natural notions related to vector bundles.

- Firstly, for $x \in M$ we refer to $\pi^{-1}(x)$ as the *fiber over* x and often denote it by E_x;
- A smooth function $s : M \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}$ is called a *section of* E;
- The pair (U_α, Φ_α) is called a *local trivialisation of* E;
- For a pair of local trivialisations, (U_α, Φ_α) and (U_β, Φ_β) the functions g_β^α defined in (5) are called *transition functions*;
- A vector bundle of rank 1 is also called a *line bundle*.

Example 2.2. [Trivial bundle] The simplest example of a vector bundle of rank k over M is the product $E = M \times \mathbb{R}^k$ where we take π to be the projection onto the first factor: $\pi(x,v) = x$.

The trivial vector bundle comes equipped with k sections, namely,

$$s_i(x) = (x, e_i),$$

where $e_i = (0, \cdots 0, 1, 0, \cdots 0) \in \mathbb{R}^n$ is the i^{th} coordinate vector. For every x, we have that $\{s_1(x), \cdots, s_k(x)\} \subset x \times \mathbb{R}^k$ are linearly independent vectors in $\mathbb{R}^k$. We say that the sections $\{s_i\}$ are (pointwise) linearly independend.

For $k = 1$ we see that a section of E corresponds simply to a function, indeed a section is a map $s : M \longrightarrow M \times \mathbb{R}$ which is the identity on the first factor, hence

$$s(x) = (x, f(x)) \subset M \times \mathbb{R}$$

where for each x, $f(x)$ is a real number, hence f is simply a smooth real valued function. For the rank k case, a section is just a collection of k smooth functions.

Definition 2.3. Let E_1 and E_2 be vector bundles of M, with projection maps π_1 and π_2. A map $\Phi : E_1 \longrightarrow E_2$ is a *bundle map* if the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ M \arrow[r, "\mathrm{Id}"] & M \end{tikzcd}$$

and $\Phi : (x,\cdot) : (E_1)_x \longrightarrow (E_2)_x$ is linear.

We we say that two vector bundles over M are *isomorphic* if there is a bundle map $\Phi : E_1 \longrightarrow E_2$ which is a diffeomorphism.

Exercise 2.4. Show that for a bundle map $\Phi : E_1 \longrightarrow E_2$ to be an isomorphism of vector bundles it is sufficient that $\Phi(x,\cdot) : (E_1)_x \longrightarrow (E_2)_x$ is an isomorphism of vector spaces for all $x \in M$.

Exercise 2.5. Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of M and assume that two vector bundles E_1 and E_2 have trivialisations for this cover:

$$\Phi_\alpha : \pi_1^{-1}(U_\alpha) \subset E_1 \longrightarrow U_\alpha \times \mathbb{R}^k;$$

$$\Psi_\alpha : \pi_2^{-1}(U_\alpha) \subset E_2 \longrightarrow U_\alpha \times \mathbb{R}^k,$$

show that if the transition functions for these trivialisations of E_1 and E_2 agree the bundles E_1 and E_2 are isomorphic.

In light of the definition of bundle isomorphism, we see that the maps Φ_α in the definition of vector bundle correspond to bundle isomorphisms between $E|_{U_\alpha} = \pi^{-1}(U_\alpha)$ and the trivial bundle $U_\alpha \times \mathbb{R}^k$. That is to say that vector bundles are locally indistinguishable from the trivial vector bundle and any non-triviality comes from the way different trivialisations are glued together. In particular, since the bundle is locally trivial, given any point p there is a neighborhood U of p where one can find k (pointwise) linearly independent sections.

Conversely, given k pointwise linearly independent sections $\{s_1, \cdots, s_k\}$ of a rank k vector bundle E defined on an open set U, we can define a map

$$\Phi : U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U);$$

$$\Phi(x, a_1, \cdots a_k) = (a_1 s_1(x) + \cdots a_k s_k(x)).$$

It is clear that for each fixed x this is a linear ismorphism from $\mathbb{R}^k$ into E_x. Further, this map is clearly a smooth bijection, hence, due to Exercise 2.4, it is an isomorphism of vector bundles. That is, a local trivialisation is equivalent to a choice of k linearly independent sections

Example 2.6 (The zero section)**.** Every line bundle comes equipped with a natural section, namely, the one defined in a local trivialisation (U_α, Φ_α) by

$$s(x) = \Phi_\alpha^{-1}(x, 0).$$

Linearity of the transition functions g_β^α means that this definition does not depend on the particular trivialisation chosen:

$$\Phi_\alpha^{-1}(x, 0) = \Phi_\beta^{-1} \circ (\Phi_\beta \circ \Phi_\alpha)(x, 0) = \Phi_\beta^{-1}(x, g_\beta^\alpha 0) = \Phi_\beta^{-1}(x, 0),$$

and hence we conclude that s define above is a globally defined section which is called *the zero section*.

Example 2.7 (Line bundle over the circle)**.** Identify the circle S^1 with the interval $[0, 1]$ with ends identified and let $U_0 = (0, 1)$, $U_1 = (1/2, 1] \cup [0, 1/2)$. Assume that we have a line bundle E over S^1, which is trivial over each U_i, i.e., we have nonvanishing sections s_i over U_i, $i = 0, 1$ which give rise to the local trivialisation maps

$$\Phi_i^{-1} : U_i \times \mathbb{R} \longrightarrow \pi^{-1}(U_i),$$
$$\Phi_i^{-1}(x, \lambda) = \lambda s_i(x).$$

And hence

$$\Phi_1 \circ \Phi_0^{-1}(x, \lambda) = (x, \lambda g_1^0(x)),$$

where $g_1^0(x)$ is the only real number satisfying $s_0(x) = g_1^0(x) s_1(x)$. Indeed, simply computing the composition of functions we have

$$\Phi_1 \circ \Phi_0^{-1}(x, \lambda) = \Phi_1(\lambda s_0(x)) = \Phi_1(\lambda g_1^0(x) s_1(x)) = (x, \lambda g_1^0(x)).$$

Since $U_0 \cap U_1 = (0, 1/2) \cup (1/2, 1)$ has two disconnected components g_1^0 is actually composed of two functions, namely its restrictions to $(0, 1/2)$ and $(1/2, 1)$:

$$g_{01} : (0, 1/2) \longrightarrow \mathbb{R}^*; \qquad g_{10} : (1/2, 1) \longrightarrow \mathbb{R}^*.$$

If both of these functions have the same sign, say, if they are both positive, then the bundle has a globally defined section. Indeed, let φ_i, $i = 0, 1$, be a partition of unity subbordinate to the cover $\{U_0, U_1\}$ and define $s = \phi_0 s_0 + \phi_1 s_1$. Since ϕ_0 has compact support in U_0, the section $\phi_0 s_0$ has support in U_0 and can be extended as zero to the complement of U_0, so that we can regard it as a globally defined section. The same argument can be applied to $\phi_1 s_1$ to make it into a global section. Now notice that in U_1 we have

$$\phi_0 s_0 + \phi_1 s_1 = \phi_0 g_1^0(x) s_1(x) + \phi_1 s_1 = (\phi_0 g_1^0 + \phi_1) s_1$$

and that the coefficient $\phi_0 g_1^0 + \phi_1$ is nowhere vanishing since it is a sum of two non-negative numbers one of which is nonzero. Similarly, s is also a nonvanishing section of E over U_0 and hence s is a nowhere vanising section of E. Therefore if g_{01} and g_{10} have the same sign, the bundle has a (global) nowhere vanishing section and hence is trivial.

Next we show that if g_{01} and g_{10} have opposite signs, say, g_{01} is positive and g_{10} is negative then E does not have a global nowhere vanishing section and hence it is not trivial. Indeed, assume that E has a nowhere vanishing section s, then over U_0 we have $s = f s_0$ for some nonvanishing real function. Since f does not vanish, it has a sign. Say f is positive. Then, over $(0, 1/2)$ we have

$$s = f s_0 = f g_{01} s_1$$

and over $(1/2, 1)$

$$s = f s_0 = f g_{10} s_1.$$

Since f is positive, we see that s is a positive multiple of s_1 on $(0, 1/2)$ and a negative multiple of s_1 on $(1/2, 1)$, hence, by continuity, it must vanish at 1, which is a contradiction.

Exercise 2.8. Show that there are only two line bundles over the circle. In the language of the example above, show that any two bundles for which g_{01} and g_{10} have opposite signs are isomorphic.

2.2 Creating a bundle from transition functions

2.2.1 Vector bundles

The same argument used to create a manifold out of domains of $\mathbb{R}^n$ and changes of coordinates given in Section 1.1 can be repeated for vector bundles. Namely, if instead of $E \xrightarrow{\pi} M$ satisfying the vector bundle properties one is given an open cover U_α of M of domains of coordinate charts $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^n$ together with a collection of transition functions $g^\alpha_\beta : U_{\alpha\beta} \longrightarrow \mathrm{Gl}(k; \mathbb{R})$ satisfying the properties one would have if these functions came from a vector bundle, namely

$$\begin{aligned} g^\alpha_\alpha &= \mathrm{Id}; \\ g^\gamma_\alpha \cdot g^\beta_\gamma \cdot g^\alpha_\beta &= \mathrm{Id}. \end{aligned} \tag{6}$$

One can try and define a vector bundle over M by taking $\mathcal{V} = \dot\cup(U_\alpha \times \mathbb{R}^k)$ and then taking again the quotient by an equivalence relation, namely, declaring that $(x, v) \in U_\alpha \times \mathbb{R}^k$ is equivalent to $(y, w) \in U_\beta \times \mathbb{R}^k$ if $x = y$ and $w = g^\alpha_\beta(x) v$. Just as before, the conditions (6) mean that a point $(x, v) \in U_\alpha \times \mathbb{R}^k$ is not identified with any other point in $U_\alpha \times \mathbb{R}^k$ and hence we have $U_\alpha \times \mathbb{R}^k \subset E$. Then the maps

$$(\varphi_\alpha, \mathrm{Id}) : U_\alpha \times \mathbb{R}^k \subset E \longrightarrow \mathbb{R}^n \times \mathbb{R}^k; \qquad (\varphi_\alpha, \mathrm{Id})(x, v) = (\varphi_\alpha(x), v)$$

provide a local Euclidean structure for E. Differently from the case of constructing manifolds out of change of coordinates, now there is no issue regarding the topology of E.

Proposition 2.9. *The space E constructed above is a manifold.*

Proof. We must prove that the topological space E is second countable and Hausdorff and that the proposed local Euclidean structure in fact gives smooth change of coordinates.

Smooth structure: The maps above giving the local Euclidean structure themselves already give rise to a smooth structure as one can readily see composing different charts appropriately:

$$\Phi^{\alpha}_{\beta} : \varphi_{\alpha}(U_{\alpha} \cap U_{\beta}) \times \mathbb{R}^k \longrightarrow \varphi_{\beta}(U_{\alpha} \cap U_{\beta}) \times \mathbb{R}^k$$

$$\Phi^{\alpha}_{\beta}(x, v) = (\varphi_{\beta} \circ \varphi_{\alpha}^{-1}(x), g^{\alpha}_{\beta}(\varphi_{\alpha}(x))v).$$

Smoothness of the first component follows from the fact that φ_{α} and φ_{β} are coordinates on M and smoothness of the second factor follows from smoothness of the function g^{α}_{β}.

Second countability: Since M is a manifold, its topology has a countable basis $\{A_i : i \in \mathbb{N}\}$ and we can take a sub basis

$$\{A_i : A_i \subset U_{\alpha} \text{ for some } \alpha\}.$$

which we still denote by $\{A_i\}$.Then the topology of E is generated by the open sets

$$\{A_i \times B_r(v) \subset U^{\alpha} \times \mathbb{R}^k : A_i \in U_{\alpha}; r \in \mathbb{Q}_+; v \in \mathbb{Q}^k \subset \mathbb{R}^k\},$$

where $B_r(v)$ is the ball or radius r and center v in $\mathbb{R}^k$. Since this set is countable, the topology of E has a countable basis.

Hausdorff: Given distinct points $(x_1, v_1) \in U_{\alpha} \times \mathbb{R}^k$ and $(x_2, v_2) \in U_{\beta} \times \mathbb{R}^k$, notice that if $x_1 \neq x_2$, then since M is Hausdorff, one can find disjoint open sets A_1 and A_2 with $x_i \in A_i$. In this case $(x_1, v_1) \in A_1 \times R^k$ and $(x_2, v_2) \in A_2 \times \mathbb{R}^k$ and $A_1 \times R^k$ does not intersect $A_2 \times R^k$, showing the Hausdorff property in this case. If $x_1 = x_2$ then, since no two points in $U_{\alpha} \times \mathbb{R}^k$ get identified by the equivalence relation, the we get that $v_1 \neq v_2$. Then, since $\mathbb{R}^k$ is Hausdorff, there are disjoint open sets V_1 and $V_2 \subset \mathbb{R}^k$ such that $v_i \in V_i$. Then $(x_i, v_i) \in U_{\alpha} \times V_i$, and $U_{\alpha} \times V_1$ and $U_{\alpha} \times V_2$ are disjoint open sets in E. Hence we see that E is Hausdorff and hence a manifold. □

Proposition 2.10. *The maps $\pi_{\alpha} : U_{\alpha} \times \mathbb{R}^k \longrightarrow M$, $\pi_{\alpha}(x, v) = x$ satisfy*

$$\pi_{\alpha}|_{U_{\alpha\beta} \times \mathbb{R}^k} = \pi_{\beta}|_{U_{\alpha\beta} \times \mathbb{R}^k}$$

and hence give rise to a globally defined map

$$\pi : E \longrightarrow M.$$

The map π makes E into a rank k vector bundle over M and there are trivialisations of E over each U_{α} for which the transition functions are the g^{α}_{β}.

Proof. The first statement follows immediately from the equivalence relation. Namely, if $(x, v) \in U_{\alpha} \times \mathbb{R}^k$ is equivalent to $(y, w) \in U_{\beta} \times \mathbb{R}^k$ then $x = y$ and hence $\pi_{\alpha}(x, v) = x = y = \pi_{\beta}(y, w)$. Since the maps π_{α} agree on overlaps, they define a global map

$$\pi : E \longrightarrow M; \qquad \pi(x, v) = \pi_{\alpha}(x, v) = x \text{ on } U_{\alpha} \times \mathbb{R}^k.$$

With this, it is clear that identity map from $U_{\alpha} \times \mathbb{R}^k \subset E$ to $U_{\alpha} \times \mathbb{R}^k$ is the bundle trivialisation over U_{α} and the transition functions for these choices are precisely the g^{α}_{β}. □

As a consequence of this construction we see that the family of functions $\check{g} = \{g^{\alpha}_{\beta} : \alpha, \beta \in A\}$ contains enough information to reconstruct the vector bundle E together with trivialisations over the sets U_{α} and is conversely determined by E and trivialisations. In view of our study of Čech cohomology in the first half of these notes it would be natural to thing of the family $\check{g}$ as a Čech cochain, $\check{g} \in \check{C}^1(M; C^{\infty}(M; \mathrm{Gl}(k; \mathbb{R})); \mathfrak{U})$. Conditions (6) are then just the usual skew-symmetry of cochains together with a condition that $\check{g}$ should satisfy some sort of cocycle condition. Note however that for $k > 1$, $\mathrm{Gl}(k; \mathbb{R})$ is not Abelian and hence much of the Čech theory developed in these notes can not be used directly to study vector bundles.

2.3 Classification of line bundles

One way to phrase the computation from previous section is that isomorphism classes of rank k vector bundles with *trivialisations over an open cover* $\{U_\alpha\}$ are in bijective correspondence with families of transition functions $g_\beta^\alpha : U_{\alpha\beta} \longrightarrow \mathrm{Gl}(k;\mathbb{R})$ satisfying conditions (6). Of course one is rarely interested in isomorphism classes of vector bundles with *trivialisations*. A far more useful question in which people are more commonly interested is the classifications of isomorphism classes of vector bundles. Next we argue how to go from what we have developed so far to an answer of the more useful question.

A bundle which is trivial over each open set of a cover $\mathfrak{U}$ can be trivialised in many different ways over each open set of the cover, and for each such trivialisation one gets collections of transition functions $\{g_\beta^\alpha\}$ all of which describe the same bundle. So, in order to classify rank k vector bundles one must study the effect these changes of trivialisations have on the family of transition functions. Given two trivialisations of E over sets of an open cover $\mathfrak{U}$

$$\Phi_\alpha, \Psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$$

we can consider the map

$$\Phi_\alpha \circ \Psi_\alpha^{-1} : U_\alpha \times \mathbb{R}^k \longrightarrow U_\alpha \times \mathbb{R}^k.$$

It follows from the basic property of trivialisations that this map is the identity in the first factor and an invertible linear map on the second, that is, we get

$$\Phi_\alpha \circ \Psi_\alpha^{-1}(x, v) = (x, f_\alpha(x)v),$$

where $f_\alpha : U_\alpha \longrightarrow \mathrm{Gl}(k;\mathbb{R})$.

We can then use the maps f_α to relate the transition functions $\tilde{g}_\beta^\alpha$ for the trivialisation Ψ to the transition functions g_β^α for the trivialisation Φ. Indeed, we have

$$\begin{aligned}
(x, \tilde{g}_\beta^\alpha v) &= \Psi_\beta \circ \Psi_\alpha^{-1}(x, v) = \Psi_\beta \circ (\Phi_\beta^{-1} \circ \Phi_\beta) \circ (\Phi_\alpha^{-1} \circ \Phi_\alpha) \circ \Psi_\alpha^{-1}(x, v) \\
&= (\Psi_\beta \circ \Phi_\beta^{-1})(\Phi_\beta \circ \Phi_\alpha^{-1}) \circ (\Phi_\alpha \circ \Psi_\alpha^{-1})(x, v) \\
&= (\Psi_\beta \circ \Phi_\beta^{-1})(\Phi_\beta \circ \Phi_\alpha^{-1})(x, f_\alpha v) \\
&= (\Psi_\beta \circ \Phi_\beta^{-1})(x, g_\beta^\alpha \circ f_\alpha v) \\
&= (x, f_\beta^{-1} \circ g_\beta^\alpha \circ f_\alpha v)
\end{aligned}$$

Hence, we have that $\tilde{g}_\beta^\alpha = f_\beta^{-1} \circ g_\beta^\alpha \circ f_\alpha$.

Conversely, given a family of trivialisations Ψ_α and a family of functions $f_\alpha : U_\alpha \longrightarrow \mathrm{Gl}(k;\mathbb{R})$ we can form new trivialisations Φ_α by declaring

$$\Phi_\alpha = (\mathrm{Id}, f_\alpha) \circ \Psi_\alpha.$$

Until now, we have assumed that we could trivialise any given bundle over our cover. Technically, every bundle can be trivialised over some cover, but it is at first not clear that there is a cover which trivialises them all. The key fact which removes this dependence on the cover is the following Lemma, which we will not prove in these notes

Lemma 2.11. Every rank k vector bundle over a disc D is isomorphic to the trivial one.

Therefore, as long as we take a cover $\mathfrak{U}$ for which each U_α is diffeomorphic to a disc the hypothesis we have been using of existence of trivialisation over U_α is automatically guaranteed. Adding this argument up, what we have had so far in the following:

Proposition 2.12. *Let $\{U_\alpha : \alpha \in A\}$ be an open cover of M for which each U_α is diffeomorphic to a disc. Then isomorphism classes of rank k vector bundles are in one to one correspondence with equivalence classes of families of functions*

$$\{g^\alpha_\beta : U_{\alpha\beta} \longrightarrow \mathrm{Gl}(\mathrm{k}, \mathbb{R}) : \alpha, \beta \in A\}$$

satisfying

$$g^\alpha_\alpha = \mathrm{Id};$$
$$g^\gamma_\alpha \cdot g^\beta_\gamma \cdot g^\alpha_\beta = \mathrm{Id}.$$

where two families $\{g^\alpha_\beta\}$ and $\{\tilde{g}^\alpha_\beta\}$ are equivalent if there is a family of functions

$$\{f_\alpha : U_\alpha \longrightarrow \mathrm{Gl}(\mathrm{k}, \mathbb{R}) : \alpha \in A\}$$

such that

$$\tilde{g}^\alpha_\beta = f_\beta^{-1} \circ g^\alpha_\beta \circ f_\alpha. \tag{7}$$

For general vector bundles, the previous proposition is the end of the story. Yet, for line bundles, since $\mathrm{Gl}(1, \mathbb{R}) = \mathbb{R}^*$ is commutative, we can use Čech cohomology to go further. Indeed, according to Proposition 2.10 a cocycle $\check{g} \in \check{C}^1(M; C^\infty(M; \mathbb{R}^*); \mathfrak{U})$ fully determines the line bundle and if the cover $\mathfrak{U}$ is made of discs, any line bundle can be described by a cocycle. Further, condition (7) tells us that two cocycles represent the same bundle if and only if

$$\frac{g^\alpha_\beta}{\tilde{g}^\alpha_\beta} = \frac{f_\beta}{f_\alpha} = (\delta f)_{\alpha\beta}.$$

That is, two cocycles represent the same bundle if and only if their "difference" using the group operation is a coboundary. Hence we conclude

Theorem 2.13. *Isomorphim classes of line bundles over a manifold M are in one to one correspondence with elements in $\check{H}^1(M; \mathbb{Z}_2; \mathfrak{U})$ for any cover $\mathfrak{U}$ made up of discs.*

Corollary 2.14. *The group $\check{H}^1(M; \mathbb{Z}_2; \mathfrak{U})$ does not depend on the cover $\mathfrak{U}$ as long as $\mathfrak{U}$ made up of discs.*

We let

$$w_1 : \{\text{isomorphism classes of line bundles}\} \longrightarrow \check{H}^1(M; \mathbb{Z}_2; \mathfrak{U}) \qquad E \mapsto w_1(E) \in \check{H}^1(M; \mathbb{Z}_2; \mathfrak{U})$$

be the isomorphism obtained between these two sets.

The map w_1 can be extended to higher rank bundles:

Definition 2.15. Let E be a rank k-vector bundle over a manifold M. The *first Stiefel-Whitney class of* E is the cohomology class $w_1(E) \in \check{H}^1(M; \mathbb{Z}_2)$ corresponding to the line bundle $\wedge^k E$ under the bijection of Theorem 2.13.

Corollary 2.16. *A vector bundle E over M is orientable if and only if $w_1(E) = 0$.*