Differentiable manifolds – exercise sheet 10

Exercise 1. Let M be a compact manifold. Show that every vector field on M is complete.

Exercise 2. Let X be a vector field on a manifold M and let $p \in M$ be a point where X vanishes (i.e., $X_p = 0$). Show that the path $\gamma : \mathbb{R} \longrightarrow M$ given by $\gamma(t) = p$ is an integral curve of X.

Exercise 3. Check that the Lie bracket of vector fields is indeed a Lie bracket, i.e., that it is skew symmetric, bilinear and satisfies the Jacobi identity.

Exercise 4. Let $\varphi : M \longrightarrow N$ be a embedding whose image is a closed subset of N and let $X, Y \in \mathfrak{X}(M)$ be vector fields. Show that there are vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(N)$ such that for every $p \in M$ $\varphi_*|_p(X_p) = \tilde{X}_{\varphi(p)}$ and similarly for Y and $\tilde{Y}$. The vector fields $\tilde{X}$ and $\tilde{Y}$ are normally referred to as *extensions* of X and Y to N.

Show that for $q \in \mathrm{Im}(\varphi)$, $[\tilde{X}, \tilde{Y}]_q$ only depends on X and Y and not on the particular extensions chosen. Precisely, if $q = \varphi(p)$, show that $[\tilde{X}, \tilde{Y}]|_{\varphi(p)} = \varphi_*|_p([X, Y]|_p)$.

Exercise 5. Let $X, Y \in \mathfrak{X}(M)$ be vector fields on M and let $p \in M$. Let e^{tX} be the time t flow of X so that

$$e^{tX}_* : T_pM \longrightarrow T_{e^{tX}p}M.$$

By letting t vary and keeping p fixed, consider the following path on T_pM:

$$\gamma : I \longrightarrow T_pM; \qquad \gamma(t) = (e^{tX}_*)^{-1} Y_{e^{tX}p}.$$

Show that $\frac{d}{dt}\gamma\big|_{t=0} = [X, Y]_p$.