Differentiable manifolds – exercise sheet 11

Exercise 1. (Try to) Prove that

$$V_1 \otimes (V_2 \otimes V_3) \cong V_1 \otimes V_2 \otimes V_3,$$

$$V_1 \otimes V_2 \cong V_2 \otimes V_1.$$

Exercise 2. In lectures, for $\xi, \eta \in V^*$, we defined $\xi \otimes \eta : V \times V \longrightarrow \mathbb{R}$ by

$$\xi \otimes \eta(v_1, v_2) = \xi(v_1)\eta(v_2).$$

Show that $\xi \otimes \eta \in V^* \otimes V^*$.

Exercise 3. Show that $V \otimes \mathbb{R} \cong V$.

Exercise 4. Let $A : V \longrightarrow W$ be linear. Show that A induces linear maps defined on generators by

$$A : \otimes^k V \longrightarrow \otimes^k W; \qquad A(v_1 \otimes \cdots \otimes v_k) = A(v_1) \otimes \cdots \otimes A(v_k).$$

$$A^* : \otimes^k W^* \longrightarrow \otimes^k V^*; \qquad A^*(\xi_1 \otimes \cdots \otimes \xi_k) = A^*(\xi_1) \otimes \cdots \otimes A^*(\xi_k).$$

Show further that

$$A^* : \mathrm{Sym}^k W^* \longrightarrow \mathrm{Sym}^k V^*,$$

$$A^* : \wedge^k W^* \longrightarrow \wedge^k V^*.$$

Exercise 5. Compute the dimension of $\mathrm{Sym}^k V$.

Exercise 6. Show that $\otimes^2 V = \wedge^2 V \oplus \mathrm{Sym}^2 V$. Is it true in general that $\otimes^k V = \wedge^k V \oplus \mathrm{Sym}^k V$?

Exercise 7. Prove the following identities for $\eta, \xi, \xi_1, \cdots, \xi_k \in V^*$:

1. $\xi \wedge \eta = \frac{1}{2}(\xi \otimes \eta - \eta \otimes \xi)$;
2. $\xi \wedge \xi = 0$;
3. $\xi_1 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_i \wedge \cdots \wedge \xi_k = -\xi_1 \wedge \cdots \wedge \xi_i \wedge \xi_{i-1} \wedge \cdots \wedge \xi_k$;
4. if $\alpha \in \wedge^k V^*$, $\beta \in \wedge^l V^*$ then $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

Exercise 8. Read the section *Abstract tensor products of vector spaces.*