Differentiable manifolds – exercise sheet 14

Exercise 1. Let $\alpha \in \Omega^1(\mathbb{R}^2\backslash\{0\})$ be given by

$$\alpha = \frac{xdy - ydx}{x^2+y^2}.$$

Compute $d\alpha$. Compute the integral of α over

- the unit circle oriented counterclockwise.
- the circle of radius 1 centered at $(0,2)$ oriented counterclockwise.
- the circle of radius 2 centered at $(1,0)$ oriented counterclockwise.

Exercise 2. Let $\varphi : S^2\backslash\{N\} \longrightarrow \mathbb{R}^2$ be the stereographic projection. Show that $\varphi^*(\frac{dx\wedge dy}{1+(x^2+y^2)^2})$ extends to the north pole to give rise to a smooth 2-form on S^2. Compute its integral over S^2.

Exercise 3. Using the Poicanré lemma and integration, show that $H^1(S^2) = \{0\}$.

Exercise 4. A k-form α is *harmonic* if α and $\star\alpha$ are closed. Show that is a harmonic form is not everywhere zero, it represents a nontrivial cohomology class.

Exercise 5. Let $\omega \in \Omega^2(\mathbb{R}^{2n})$ be given by

$$\omega = dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

Let $\varphi : \Sigma^2 \longrightarrow \mathbb{R}^{2n}$ be a smooth map. Compute

$$\int_\Sigma \varphi^*\omega.$$

Convention: in a compact oriented Riemannian manifold for $f \in C^\infty(M)$ we define

$$\int_M f := \int_M \star f.$$

Exercise 6 (The divergent)**.** Let M be an oriented Riemannian manifold and let $X \in \mathfrak{X}(M)$. Define the divergent of X to be

$$\nabla \cdot X = \star^{-1} d \star g(X).$$

Show that if M is $\mathbb{R}^n$ with usual metric and orientation

$$\nabla \cdot (X_i \frac{\partial}{\partial x_i}) = \sum \frac{\partial X_i}{\partial x_i}.$$

Exercise 7. Let $X \in \mathfrak{X}(M)$ be a vector field on an oriented compact Riemannian manifold with boundary. Let N be the unit outward pointing normal vector to boundary. Show that

$$\int_{\partial M} g(X,N) = \int_M \nabla \cdot X.$$

Exercise 8. Let M be a compact manifold and $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover such that for each multi index $\mathrm{a} \subset A$ the corresponding U_{a} is either empty of diffeomorphic to a disc. Show that $\check{H}^k(M;\mathfrak{U};\mathbb{R}) = H^k(M)$ for all k.

A possible route to prove this result relies on two facts.

Fact 1 (Uses the same argument as Theorem 1.16 from the notes on Cech cohomology): For any vector bundle E,

$$\check{H}^k(M;\mathfrak{U};\Gamma(E)) = \{0\} \text{ if } k > 0 \text{ and}$$
$$\check{H}^0(M;\mathfrak{U};\Gamma(E)) \cong \Gamma(E).$$

Fact 2 (Poincaré Lemma) If U is diffeomorphic to an open ball in $\mathbb{R}^n$, then $H^i(U) = \{0\}$ for $i > 0$ and $H^0(U) = \mathbb{R}$.

And the proof involves introducing another cohomology space, H_D^k below, and then prove that

$$\check{H}^k(M;\mathfrak{U};\mathbb{R}) \cong H_D^k \cong H^k(M).$$

Roadmap to the solution
Consider the $\mathbb{N}\times\mathbb{N}$ graded vector space

$$E^{\bullet,\bullet} = \oplus_{p,q\in\mathbb{N}} E^{p,q}; \qquad E^{p,q} = \check{C}^p(M;\mathfrak{U};\Omega^q(M)).$$

Endow $E^{\bullet,\bullet}$ with the operator

$$D : E^{p,q} \longrightarrow E^{p+1,q} \oplus E^{p,q+1}; \qquad D|_{E^{p,q}} = \check{\delta} + (-1)^p d,$$

If we use a coarser grading for E, namely, if we define

$$\mathcal{E}^k = \oplus_{p+q=k} E^{p,q}$$

then $D : \mathcal{E}^k \longrightarrow \mathcal{E}^{k+1}$.

1. Show that $D^2 = 0$ and hence we can define its cohomology. Since $D : \mathcal{E}^k \longrightarrow \mathcal{E}^{k+1}$ we get an $\mathbb{N}$-grading for the D-cohomology:

$$H_D^k = \frac{\ker(D : \mathcal{E}^k \longrightarrow \mathcal{E}^{k+1})}{\operatorname{Im}(D : \mathcal{E}^{k-1} \longrightarrow \mathcal{E}^k)}$$

2. Using the inclusion of constant functions into smooth functions, we have an inclusion

$$\iota : \check{C}^k(M;\mathfrak{U};\mathbb{R}) \longrightarrow \check{C}^k(M;\mathfrak{U};\Omega^0(M)) = E^{k,0} \subset \mathcal{E}^k.$$

Show that $\iota\circ\delta = D\circ\iota$ and hence conclude that ι induces a map in cohomology

$$\iota^* : \check{H}^k(M;\mathfrak{U};\mathbb{R}) \longrightarrow H_D^k.$$

3. Show that ι^* is a surjection: given an element $\alpha\in\mathcal{E}^k$ with $D\alpha = 0$, decompose it into its (p,q) components:

$$\alpha = \sum_{p+q=k} \alpha^{p,q}, \qquad \alpha^{p,q} \in E^{p,q}.$$

Let q_0 be the biggest integer for which α^{k-q_0,q_0} is not zero. Using the Poincaré Lemma, show that if $q_0 > 0$, there is a another representative $\tilde{\alpha}$ for the D-cohomology class $[\alpha]$ such that $\tilde{\alpha}^{p,q} = 0$ for $q \geq q_0$. Conclude by induction that any cohomology class in H_D^k can be represented by an element in $\check{C}^k(M;\mathfrak{U};\Omega^0(M))$. Conclude that this representative is in the image of the map ι and hence ι^* is surjective.

4. Use a similar argument to the one above to show that ι^* is injective and hence conclude that

$$\check{H}^k(M;\mathfrak{U};\mathbb{R}) \cong H_D^k.$$

5. Using the inclusion of $\kappa : \Omega^k(M) \longrightarrow \check{C}^0(M;\mathfrak{U};\Omega^k)$ obtained by restricting a globally defined section to each open set of the cover $\mathfrak{U}$, show that

$$\kappa \circ d = D \circ \kappa$$

and hence we have a map in cohomology

$$\kappa^* : H^k(M) \longrightarrow H_D^k.$$

6. Use Fact 1 and a similar argument to item 3 to prove that κ^* is surjective.

7. Use Fact 1 and a similar argument to item 4 to prove that κ^* is injective.