Differentiable manifolds – exercise sheet 4

Whenever necessary, you can assume that the Čech cohomology $\check{H}^k(M;G;\mathfrak{U})$ is independent of $\mathfrak{U}$ as long as $\mathfrak{U}$ is a good cover of M.

Exercise 1. Let $f : M \longrightarrow N$ be smooth and surjective and let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of N. Show that

$$f^{-1}(\mathfrak{U}) = \{f^{-1}(U_\alpha) : \alpha \in A\}$$

is an open cover of M and define a map

$$f^* : \check{C}^k(N;G;\mathfrak{U}) \longrightarrow \check{C}^k(M;G;f^{-1}(\mathfrak{U})), \qquad (f^*c)_{\mathrm{a}} = c_{\mathrm{a}} \circ f,$$

where $c \in \check{C}^k(N;G;\mathfrak{U})$ and a is an ordered multiindex of size $k+1$, so that $c_{\mathrm{a}} : U_{\mathrm{a}} \longrightarrow G$.

Show that f^* defined above is an isomorphism of Abelian groups for every k and that it commutes with differentials, that is,

$$f^*\delta = \delta f^*.$$

Conclude that the Čech cohomologies of M and N with respect to the covers $\mathfrak{U}$ and $f^{-1}(\mathfrak{U})$ are isomorphic. Conclude further that if f is an diffeomorphism, then M and N have isomorphic Čech cohomologies with respect to any good cover of these manifolds.

Remark: In fact the exercise above shows that if $f : M \longrightarrow N$ is smooth, surjective and $f^{-1}(\mathfrak{U})$ is a good cover of M for some good cover of N, then the cohomologies of M and N are isomorphic. An example where one can use this more general statement is with the map

$$f : \mathbb{C}^* \longrightarrow S^1, \qquad f(z) = \frac{z}{|z|}.$$

Exercise 2. Show that there are natural identifications

$$\check{H}^0(M;C^\infty(M);\mathfrak{U}) = \{\text{globally defined, smooth functions on } M\}.$$

$$\check{H}^0(M;C^\infty(M;S^1);\mathfrak{U}) = \{\text{globally defined, smooth functions on } M \text{ with values in } S^1\}.$$

Exercise 3. Let V be a vector space. The dual of V is the space

$$V^* = \{f : V \longrightarrow \mathbb{R} : f \text{ linear}\}.$$

1. Show that the map

$$V \longrightarrow V^{**}; \qquad v \mapsto v^{**} : V^* \longrightarrow \mathbb{R}, \qquad v^{**}(f) = f(v).$$

   is linear and that if V is finite dimensional, it is an isomorphism.

2. Given a linear map $A : V \longrightarrow W$, show that

$$A^* : W^* \longrightarrow V^*; \qquad A^*w^* : V \longrightarrow \mathbb{R}; \qquad A^*w^*(v) = w^*(Av).$$

   is a linear map from W^* to V^*.

3. Show that if V and W are finite dimensional $A^{**} = A$.
4. Show that if $A : V \longrightarrow W$ is an injection A^* is a surjection.
5. Show that if $A : V \longrightarrow W$ is a surjection A^* is an injection.

Definition 4. A *Lie group* is a manifold G endowed with a group structure for which group multiplication and inversion

$$G \times G \longrightarrow G; \qquad (g,h) \mapsto g \cdot h;$$
$$G \longrightarrow G; \qquad g \mapsto g^{-1};$$

are smooth maps.

Exercise 5. Show that $S^1 \subset \mathbb{C}$ is a Lie group if we endow it with multiplication of complex numbers as group operation.

Exercise 6. Show that $GL(n;\mathbb{R})$, the space of all invertible $n \times n$ real matrices, is a Lie group.