Examples and counter-examples of log-symplectic manifolds

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A Poisson structure is $\pi \in \Gamma(\wedge^2 TM)$ such that

$[\pi, \pi] = 0$. 
Example (Symplectic manifolds)

If $\omega \in \Omega^2(M)$ is symplectic

$$\omega : TM \xrightarrow{\cong} T^*M \quad \iff \quad \pi := \omega^{-1} : T^*M \xrightarrow{\cong} TM.$$ 

$$d\omega = 0 \quad \iff \quad [\pi, \pi] = 0.$$ 

Example (Trivial Poisson structure)

On any manifold, take $\pi = 0$. 

Log-symplectic Cavalcanti
Poisson geometry

Problem

Find a large class of Poisson manifolds which share important features of symplectic manifolds.
Poisson geometry

**Definition**

A *log-symplectic structure* on $M^{2n}$ is a Poisson structure $\pi$ such that $\pi^n$ only has nondegenerate zeros.

**Aim**

*Find many examples of log-symplectic manifolds and topological obstructions to their existence.*
Outline of Topics

1. Poisson geometry
2. Local forms and invariants
3. Topology
4. Constructing by deformations
5. Surgery
6. Achiral Lefschetz Fibrations
7. Reversing the surgery
Poisson structures are also Dirac structures

\( TM \oplus T^*M \) is endowed with

- the natural pairing

\[
\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y))
\]

- the Courant bracket on \( \Gamma(TM \oplus T^*M) \):

\[
\mathbb{L}[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi.
\]
Poisson structures are also Dirac structures

(Definition 2) A Poisson structure is a maximal isotropic involutive subbundle \( L \subset TM \oplus T^*M \) such that \( L \cap TM = \{0\} \):

\[
L = \{ \pi(\xi) + \xi : \xi \in T^*M \}.
\]
Poisson structures are also Dirac structures

(Definition 3) A Poisson structure is a line $K \subset \wedge^\bullet T^*M$ generated (locally) by a form $\rho \in \Gamma(K)$ pointwise of the form

$$e^\omega \wedge \theta,$$

such that

1. $\theta$ is a decomposable form;
2. $\omega \in \Omega^2(M)$;
3. $(e^\omega \wedge \theta)_{\text{top}} = \omega^{\frac{n-k}{2}} \wedge \theta \neq 0$;
4. There is $X \in \mathfrak{X}(M)$ such that

$$d\rho = \iota_X \rho.$$
Log-symplectic structures

\( K^* \) has a natural section

\[
s : K \rightarrow \mathbb{R} \times M; \quad s(\rho) = \rho_0 \in \Omega^0(M).
\]

\( s \) has nondegenerate zeros \( \Leftrightarrow \pi^n \) has nondegenerate zeros.

**Definition**

- A *log-symplectic structure* on \( M^{2n} \) is a Poisson structure such that \( s \) only has nondegenerate zeros.
- \( Z = [s = 0] \) is the *singular locus* of \( \pi \).
- \( M \setminus Z \) is the *symplectic locus* of \( \pi \).
- A *bona fide* log-symplectic structure is one with \( Z \neq \emptyset \).
Log-symplectic structures

Example
Every surface admits a log-symplectic structure.
Local form 1

Lemma (Guillemin–Miranda–Pires)

Let \( p \in \mathbb{Z} \), then in a neighbourhood \( U \) of \( p \), \( \pi \) is diffeomorphic to the Poisson structure

\[
x_1 \partial x_1 \land \partial y_1 + \partial x_2 \land \partial y_2 + \cdots + \partial x_n \land \partial y_n,
\]

i.e., \( \pi \) is the product of a log-symplectic structure in \( \mathbb{R}^2 \) with a symplectic structure \( \sigma \in \Omega^2(\mathbb{R}^{2n-2}) \).

In \( U \), \( \pi \) is equivalent to the line bundle

\[
e^\sigma (x + dx \land dy) = e^{\sigma + d \log |x|} \land dy.
\]
Theorem (Guillemin–Miranda–Pires)

Let \((M^{2n}, \pi)\) be a log-symplectic manifold and \(Z\) be the zero locus of \(s\). Then

1. \(N_Z\) is isomorphic to \(K_M\rvert_Z \cong \wedge^{2n} T^* M\rvert_Z\),
2. the Poisson structure on \(M\) induces a Poisson structure on \(Z\) and singles out a nowhere vanishing closed section of \(K_Z\)

\[
\rho_Z = e^\sigma \wedge \theta,
\]

where \(\theta\) is a degree 1 form, \(\sigma\) is closed and well defined up to a term of the form \(df \wedge \theta\).

Conversely, if two log-symplectic structures have diffeomorphic compact singular loci, \(Z\), and induce the same data then they are diffeomorphic in a neighbourhood of \(Z\).
Theorem (Rephrased)

Let \((M, \pi)\) be log-symplectic, \(Z\) a compact component of the singular locus. Then, in a neighbourhood of \(Z\), \(\pi\) determines and is determined by:

1. \(N_Z\) as a vector bundle, i.e., \(w_1 \in H^1(Z, \mathbb{Z}_2)\).
2. A closed 1-form \(\theta \in \Omega^1(Z)\).
3. A closed 2-form \(\sigma \in \Omega^2(Z)\) defined up to the addition of \(df \wedge \theta\) such that
   \[
   \theta \wedge \sigma^{n-1} \neq 0.
   \]
Log-symplectic geometry — local forms

Definition

A cosymplectic structure on \( Z^{2n-1} \) is a pair of closed forms \( \theta \in \Omega^1(Z) \) and \( \sigma \in \Omega^2(Z) \) such that

\[
\theta \wedge \sigma^{n-1} \neq 0.
\]
Corollary (Guillemin–Miranda–Pires (2012))

Any log-symplectic structure inducing the cosymplectic structure $(\sigma, \theta)$ on $Z$ is equivalent to a neighbourhood of the zero section of $N_Z$ endowed with the structure

$$d \log |x| \wedge \theta + \sigma,$$

where $|x|$ is the distance to the zero section measured with respect to a fixed fiberwise linear metric on $N_Z$. 
Log-symplectic geometry — local forms

**Definition**

A component, $Z$, of the singular locus is *proper* if it is compact and has a compact symplectic leaf.

\[ Z = \mathbb{R} \times F/\mathbb{Z} \quad (t, p) \sim (t + \lambda, \varphi(p)), \]

for a period $\lambda \in \mathbb{R}$ and a symplectomorphism $\varphi : F \to F$. 

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Log-symplectic geometry — local forms

Theorem (Marcut–Osorno-Torres (2013))

A log-symplectic structure with compact singular locus $Z$ can be deformed into a proper one.
Theorem (Marcut–Osorno-Torres (2013))

If the singular locus of a log-symplectic manifold $M^{2n}$ has a compact component, then there is $a \in H^2(M)$ such that

$$a^{n-1} \neq 0.$$
Proof.

Let \( \psi : [0, \delta] \longrightarrow [0, 1] \) be a bump function. For each component of the singular locus modify the log-symplectic form to

\[
d(\psi(|x|) \log |x|) \wedge \theta + \sigma
\]

This gives a globally defined closed 2-form \( \tilde{\omega} \) and

\[
\theta \wedge \tilde{\omega}^{n-1}|_Z = \theta \wedge \sigma^{n-1} \neq 0 \in H^{2n-1}(Z).
\]

Let \( a = [\tilde{\omega}] \).
Corollary (Marcut–Osorno-Torres (2013))

Nonabelian compact connected Lie groups have no log-symplectic structure.

Corollary (Marcut–Osorno-Torres (2013))

If $n_1 > 2$, $S^{n_1} \times \cdots \times S^{n_k}$ has no log-symplectic structure.
Topology

Theorem (Cavalcanti (2013))

If $M^{2n}$ is a compact oriented bona fide log-symplectic manifold, then there is $b \in H^2(M)$ such that

$$b^2 = 0 \quad \text{and} \quad b \cup a^{n-1} \neq 0.$$ 

Proof.

Deform the structure into a proper one. Then each fiber $F$ of $Z \rightarrow S^1$ is homologically nontrivial in $M$ since

$$\langle a^{n-1}, [F] \rangle \neq 0,$$

hence $b = PD(F) \neq 0$. Within $Z$, $F$ has no self intersection so $b^2 = 0$. Finally, $\langle b \cup a^{n-1}, [M] \rangle = \langle a^{n-1}, [F] \rangle \neq 0$.  

Log-symplectic
Corollary (Cavalcanti (2013))

In a compact orientable bona fide log-symplectic manifold $M^{2n}$, $b_{2i}(M) \geq 2$ for $0 < i < n$.

Corollary (Cavalcanti (2013))

For $n > 1$, $\mathbb{C}P^n$ has no bona fide log-symplectic structure.

Corollary (Cavalcanti (2013))

The blow-up of $\mathbb{C}P^n$ along a symplectic submanifold of codimension greater than 4 has no bona fide log-symplectic structure (McDuff examples).
Corollary (Cavalcanti (2013))

A compact, orientable four manifold with definite intersection form has no bona fide log-symplectic structure.

Corollary (Cavalcanti (2013))

$k\mathbb{C}P^n$ has no log-symplectic structure if $n > 1$. 
Deformations

**Theorem (Cavalcanti (2013))**

Let \((M^{2n}, \omega)\) be a symplectic manifold and \(F^{2n-2} \subset M\) a compact symplectic submanifold with trivial normal bundle. Then \(M\) has log symplectic structures with an arbitrary number of singular loci.
Deformations

Proof.

Symp nhood thm ⇒ $\mathcal{N}_F = D^2 \times F$ with product structure. Deform the Poisson structure on $D^2$

$$\pi_D = \partial_{r^2} \wedge \partial_\theta \quad \sim \quad \tilde{\pi}_D = f(r)\partial_{r^2} \wedge \partial_\theta,$$

where the graph of $f$ is

Set $\omega = \tilde{\pi}_D^{-1} + \omega_F$. 

Log-symplectic Cavalcanti
Deformations

Corollary (Cavalcanti (2013))

The blow-up of \( \mathbb{CP}^2 \) at a point has a bona fide log-symplectic structure.

Corollary (Cavalcanti (2013))

For \( m, n > 0 \), \( \#m\mathbb{CP}^2 \#n\overline{\mathbb{CP}^2} \) has a log-symplectic structure.
# Deformations

<table>
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<th>\textit{bona fide} log-symplectic</th>
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Log-symplectic Cavalcanti
**Surgery**

**Building block:** Let \((Z, \sigma, \theta)\) be a co-symplectic manifold, let

\[
\mathcal{N} = (-1, 1) \times Z; \quad \omega_{\mathcal{N}} = d \log |x| \wedge \theta + \sigma.
\]

\(Z\) embeds in the symplectic locus of \(\mathcal{N}\) as \(\{ -\varepsilon \} \times Z\) and \(\omega_{\mathcal{N}}|_Z = \sigma\).

Given a function \(f : Z \rightarrow \mathbb{R}\) we have a second disjoint embedding of \(Z\) in the symplectic locus of \(\mathcal{N}\):

\[
p \in \mathbb{Z} \mapsto (e^{\varepsilon f(p)}, p).
\]

For this embedding \(\omega|_Z = \sigma + df \wedge \theta\).
Surgery

Building block

\[ Z \]

(singular locus)

\( Z \)
(coisotropic submanifold)

\( Z \)
(coisotropic submanifold)

Log-symplectic Cavalcanti
Surgery

Ingredients: A log-symplectic manifold \((M^{2n}, \pi)\) together with two embeddings of a compact, connected, cosymplectic manifold \((Z, \sigma, \theta), \iota_i : Z^{2n-1} \hookrightarrow M\), such that

1. Each \(\iota_i(Z)\) is a separating submanifold of \(M\) lying in the symplectic locus of \(\pi\) and \(\iota_1(Z) \cap \iota_2(Z) = \emptyset\);

2. There is \(f \in \mathcal{C}^\infty(Z)\) such that \(\iota_1^*\omega = \iota_2^*\omega - df \wedge \theta = \sigma\), where \(\omega\) is the symplectic form on the symplectic locus of \(M\).
Surgery

Ingredient

\((Z, \sigma)\)
(coisotropic submanifold)
Surgery

Ingredient:

\[(Z, \sigma)\] (coisotropic submanifold)

\[
\begin{array}{c}
\text{Interior} \\
\downarrow \\
\text{Exterior}
\end{array}
\]
Surgery

The Surgery:

\((Z, \sigma)\) (coisotropic submanifold)

\(Z\) (singular locus) \(Z\) (coisotropic submanifold)

Interior

\(Z\) (coisotropic submanifold) \(Z\) (coisotropic submanifold)

Interior
Surgery

The Surgery:

$(Z, \sigma)$
(coisotropic submanifold)

$Z$
(singular locus)
$Z$
(coisotropic submanifold)

Interior

Log-symplectic Cavalcanti
Surgery

Ingredient 2

\[(Z, \sigma + df\theta) \quad \text{(coisotropic submanifold)}\]

Interior

Exterior

Log-symplectic Cavalcanti
Surgery

Ingredient 2

\[(Z, \sigma + df\theta) \text{ (coisotropic submanifold)}\]
The surgery

\[(Z, \sigma) \quad (\text{coisotropic submanifold})\]

\[(\theta + \sigma, \Sigma) \quad (\text{coisotropic submanifold})\]

Interior

\[Z \quad (\text{singular locus})\]

Interior

Log-symplectic Cavalcanti
Theorem (Cavalcanti 2013)

The manifold

$$M = M_1^I \cup_Z M_2^I,$$

has a log-symplectic structure whose singular locus contains the identified boundaries.
Example: \( n \# S^2 \times S^2 \)

Let \( E(n) \to \mathbb{C}P^1 \) be the elliptic surface with \( 12n \) nodal fibers.
\[
E(1) = \mathbb{C}P^2 \# 9\mathbb{C}P^2;
\]
\[
E(2) = K3.
\]

In \( p: E(2n) \to \mathbb{C}P^1 \), take an embedded circle \( \gamma: S^1 \to \mathbb{C}P^1 \) and let \( \Sigma^I \) be the interior region of \( \mathbb{C}P^1 \) determined by \( \gamma \).
Example: $n \# S^2 \times S^2$

Then $Z = p^{-1}(\gamma(S^1))$ is a proper cosymplectic manifold.

$$p^{-1}(\Sigma^I) \cup \partial p^{-1}(\Sigma^I)$$

has a log-symplectic structure.

Four-manifold theory tells us that this is $n \# S^2 \times S^2$ where $n + 1$ is the number of singular values in $\Sigma^I$.

**Corollary**

*For all $n > 0$, $n \# S^2 \times S^2$ has a log-symplectic structure.*
Achiral Lefschetz Fibrations

Definition

An achiral Lefschetz fibration is a smooth map $p : M^{2n} \to \Sigma^2$ between compact manifolds such that

- the pre-image of any critical value has only one critical point and
- for any such pair of critical value and critical point there are complex coordinates which render

$$p(z_1, \cdots, z_n) = z_1^2 + \cdots + z_n^2.$$
Achiral Lefschetz Fibrations

Theorem (Cavalcanti (2013))

Let $M^4$ and $\Sigma^2$ be compact manifolds and $p : M \rightarrow \Sigma$ be an achiral Lefschetz fibration for which the fibers are orientable and represent a nontrivial real homology class. Then $M$ has a bona fide log-symplectic structure.

Proof.

Adapt Gompf’s construction of symplectic four manifolds from Lefschetz fibrations.
Achiral Lefschetz Fibrations

Remarks
‘The fibers are orientable and represent a nontrivial real homology class’

1. If $p : M \to \Sigma$ is an achiral Lefschetz fibration and $M$ and $\Sigma$ are orientable, then the fibers are orientable.

2. If the generic fiber has genus different from 1, it represents a nontrivial real homology class.

3. If $\Sigma$ is orientable and the fibration has a section, then the fiber represents a nontrivial homology class.

4. $S^4$ admits an achiral Lefschetz fibration over $S^2$ whose generic fibers are tori.
Theorem (Etnyre–Fuller (2006))

Let $M^4$ be a compact oriented 4-manifold. Then there is a framed circle in $M$ such that the manifold obtained by surgery along that circle admits an achiral Lefschetz fibration with base $\mathbb{CP}^1$. Moreover, this fibration admits sections.

Theorem (Etnyre–Fuller (2006))

Let $M^4$ be compact and simply connected. Then $M^4 \# \overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$ and $M^4 \# (S^2 \times S^2)$ admit an achiral Lefschetz fibration with a section.
Corollary (Cavalcanti (2013))

Let $M^4$ be a compact oriented 4-manifold. Then there is a framed circle in $M$ such that the manifold obtained by surgery along that circle admits a bona fide log-symplectic structure.

Corollary (Cavalcanti (2013))

Let $M^4$ be compact and simply connected. Then $M^4 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and $M^4 \# (S^2 \times S^2)$ admits bona fide log-symplectic structures.
Reversing the Surgery

Question

Does every compact compact oriented log-symplectic manifold arise from the surgery?

The answer is yes if every co-symplectic manifold appears as a separating submanifold of a compact symplectic manifold.
Reversing the Surgery

In four dimensions this is the case:

**Theorem (Kronheimer–Mrowka)**

Every compact co-symplectic three-manifold $Z$ appears as a separating submanifold of a compact symplectic manifold $M^4$.

**Theorem (Cavalcanti (2013))**

Let $(M^4, \pi)$ be a compact, orientable, log-symplectic manifold. Then each unoriented component of $M \setminus Z$ can be compactified as a symplectic manifold.