Geometry and Topology – Exam 1

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt)**.** In each list of spaces below, decide which spaces are homotopy equivalent to each other (remember to justify your answer)

a)
$$S^1 \times S^1 \backslash \{p\}, \qquad S^1 \vee S^1, \qquad \mathbb{K}\backslash\{p\}, \qquad S^2\backslash\{p_1, p_2, p_3\},$$
where $\mathbb{K}$ denotes the Klein bottle.

b)
$$\mathbb{R}\backslash\{0\}, \qquad \mathbb{R}^2\backslash\{0\}, \qquad \mathbb{R}^3\backslash\{0\}$$

Exercise 2 (2.0 pt)**.**

a) Show that a retract of a contractible space is contractible.

b) Let $f : X \longrightarrow Y$ be continuous. Show that if there are maps $g, h : Y \longrightarrow X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $h \circ f \simeq \mathrm{Id}_X$ then f is a homotopy equivalence.

c) Show that if $X = X_1 \cup X_2$ is a CW complex, X_1, X_2 are subcomplexes and X_1, X_2 and $X_1 \cap X_2$ are contractible then X is contractible.

Exercise 3 (2.0 pt)**.** Show that for a space X, the following three statements are equivalent:

a) Every map $S^1 \longrightarrow X$ is homotopic to a constant map.

b) Every map $S^1 \longrightarrow X$ extends to a map $D^2 \longrightarrow X$.

c) $\pi_1(X, x_0) = \{0\}$ for all $x_0 \in X$.

Exercise 4 (3.0 pt)**.**

a) Show that if a space X is obtained from a path connected space X_0 by attaching n-cells with $n > 2$ then the inclusion $\iota : X_0 \hookrightarrow X$ induces an isomorphism of fundamental groups: $\iota_* : \pi_1(X_0, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$.

b) Let $X \in \mathbb{R}^n$ be the union of convex open sets $\{X_1, \cdots, X_m\}$ such that $X_i \cup X_j \cup X_k \neq \emptyset$ for all i, j, k. Show that X is simply connected.

Exercise 5 (1.0 pt)**.** Show that if X is path connected and locally path connected space and $\pi_1(X)$ is finite then every map $f : X \longrightarrow S^1$ is null homotopic.