Group theory – Hand in sheet 4

When marking, each item is worth a point.

(1)

a) *If $H < G$ is a normal subgroup and $K < G$ is a subgroup, show that $H \cap K$ is a normal subgroup of K.*

b) *Conclude that A_n is the only normal subgroup of S_n for $n > 5$.*

c) *Show that S_4 has a normal subgroup different of A_4.*

(2) *Let $n > 4$ and $m < n$. Show that*

a) *If S_n acts on a set with m elements, then the size of any the orbits is either 1 or 2.*

b) *The only action of A_n on a set with m elements is the trivial one.*

(3) *Let $\mathbb{H}$ denote the space $\mathbb{R}^4$ endowed with its quaternionic structure, i. e., we see the standard basis of $\mathbb{R}^4$ as $1, i, j, k$ with (associative) multiplication given by the rules*

$$i^2 = j^2 = k^2 = ijk = -1,$$

which means that for real $a, b, c, d, \alpha, \beta, \gamma, \delta$,

$$\begin{aligned}(a + bi + cj + dk)(\alpha + \beta i + \gamma j + \delta k) = a\alpha - b\beta - c\gamma - d\delta + (a\beta + b\alpha + c\delta - d\gamma)i + \\ + (a\gamma + c\alpha - b\delta + \beta d)j + (a\delta + d\alpha + b\gamma - c\beta)k\end{aligned} \tag{1}$$

Define the conjugate operation by

$$\overline{a + bi + cj + dk} = a - bi - cj - dk.$$

a) *Show that if $p = a + bi + cj + dk$, then*

$$p\overline{p} = a^2 + b^2 + c^2 + d^2 = |p|^2.$$

Also show that $\overline{pq} = \overline{q}\ \overline{p}$.

b) *Show that the unit sphere inside $\mathbb{R}^4$, given by*

$$S^3 = \{a + bi + cj + dk : a^2 + b^2 + c^2 + d^2 = 1\}$$

is a subgroup of $\mathbb{H} \backslash \{0\}$, endowed with quaternionic multiplication.

c) *Show that the map*

$$(a + bi + cj + dk) \mapsto \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}$$

is an isomorphism between S^3 *and* $SU(2)$.

(4) *Let* G *be a finite group and* $n \geq 0$ *a natural number. A* representation of G *is a group homomorphism*

$$\varphi : G \longrightarrow U(n).$$

A representation is irreducible *if there is no subspace* $V \subset \mathbb{C}^n$, *with* $V \neq \{0\}, V \neq \mathbb{C}^n$, *such that*

$$\varphi(g)(V) = V, \qquad \forall g \in G.$$

a) *Show that given two representations* $\varphi_1 : G \longrightarrow U(n_1)$ *and* $\varphi_1 : G \longrightarrow U(n_2)$, *we can construct a new representation* $(\varphi_1, \varphi_2) : G \longrightarrow U(n_1 + n_2)$. *Is* (φ_1, φ_2) *irreducible?*

b) *Show that if* G *is a finite Abelian group and* $\varphi : G \longrightarrow U(n)$ *is an irreducible representation, then* $n = 1$ *(hint: you can use without proof that given a set of commuting matrices in* $U(n)$, *there is a basis which diagonalizes all of them simultaneously.)*